Logaritmik tekil integro diferansiyel denklemlerin nümerik çözümleri

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The first-kind Chebyshev polynomial, collocation methods, logarithmic integral equation, Symm's integral equation, approximation method.

ABSTRACT

The main purpose of this article is to present an approximation method for logarithmic singular (Symm's integral equation [1] integro-differential equations in the most general form under the mixed conditions in terms of the first kind Chebyshev polynomials. This method is based on the first-kind Chebyshev polynomials. The solution is obtained in terms of the first-kind Chebyshev polynomials. This scheme is based on taking the truncated the first-kind Chebyshev expansion of the function in the Logaritmic singular integro-differential equations. Hence, the result matrix equation can be solved and the unknown the first-kind Chebyshev polynomial coefficients can be found approximately. The error analysis and convergence for the proposed method is also introduced.

Numerical solution of Logaritmic singular integro differential equations

Anahtar Kelimeler:
Birinci tip Chebyshev polinomları, sıralama yöntemi, logaritmik tekil integral, Symm's integral denklemi, yaklaşım yöntemleri.

ÖZET

Bu çalışmanın temel amacı karışık koşullar ile en genel halde verilen logaritmik tekil integro diferasiyel denklemlerin nümerik çözümleri için birinci tip Chebyshev polinomları yardımıyla bir yöntem geliştirmektir. Bu yöntem denklem çözümünün birinci tip Chebyshev polinomları cinsinden seriye açılma esasına dayanır. Logaritmik tekil integro diferansiyel denklemde bulunan fonksiyonların birinci tip Chebyshev serisine açılması ile elde edilen matris denklemi çözülerek Chebyshev katsayıları bulunmuştur. Önerilen yöntem için hata analizi ve yakınsaklık incelemesi yapılmıştır.

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1. Introduction

Many problems of mathematical physics, engineering and contact problems in the theory of elasticity lead to singular integro differential equations. Logarithmic singular integro differential equation are found in many applications such elasticity and potential theory[1-2], asacoustic scattering [3] and fluid mechanics[4-7]. Several methods for the solution of singular equations have been presented, which are the H and H-R method [8], Spline Galerkin method[9] and others [10-11]. In recent years the Chebyshev polynomials have been used to find the approximate solutions ofdifferential equation[12], integro-differential-difference equations[13], Abel equation[14], Volterra differential equation[15], singular-perturbation equation[16], pantograph equations[17]. In this paper, we consider the logarithmic integro-differential equation

$$\sum_{k=0}^{m} P_k(x) y^{(k)}(x) = f(x) + \lambda \int_{-1}^{1} \frac{\log(t-x)}{\sqrt{1-t^2}} y(t) dt \qquad (1)$$

with the mixed conditions

$$\sum_{k=0}^{m-1} \sum_{j=0}^{s} c^k_{kj} y^{(k)}(c_{kj}) = \mu_k \qquad (2)$$

and the solution is expressed in terms of the the first kind Chebyshev functions as follows:

$$y_N(x) = \sum_{r=0}^{N} a_r T_r(x), \; 0 \le i \le N \qquad (3)$$

where a_i , i=0,1,...,N are the coefficients to be determined. Here $P_k(x)$ and $f(x)$ are continous functions on $[-1,1]$ and c^k_{kj}, c_{kj} and λ are appropriate constants. Note that this kernel has a weak logarithmic singularity at $x = t$. Equation (1) is usually referred to as Symm's integral equation which is of importance in potential theory [1]. Symm's integral equation has a unique solution [18].

2. Preliminaries and notations

In this section, we state some basic results about polynomial approximations. These important properties will enable us to solve the singular integro differential equations. Polynomials are the only functions that the computer can evaluate exactly, so we make approximate functions $R \rightarrow R$ by polynomials. We consider real-valued functions on the compact interval $[-1,1]$:

$$f : [-1,1] \rightarrow R$$

and we denote the set all real-valued polynomials on $[-1,1]$ by P, that is

$$\forall p \in \text{P}, \; \forall x \in [-1,1], \; p(x) = \sum_{i=0}^{N} a_i x^i$$

and

Definition 2.1

For a given continuous function $f \in C[a,b]$, a best approximation polynomial of degree N is a polynomial $p_N^*(f) \in P_N$ such that

$$\left\| f - p_N^*(f) \right\|_{\infty} = \min\{ \left\| f - p \right\|_{\infty} : p \in P_N \}$$

where the uniform norm is defined by $\left\| f \right\|_{\infty} = \max_{x \in [-1,1]} \left| f(x) \right|$.

Theorem 2.1 [19-22] Let $f \in C[a,b]$. Then for any $\varepsilon > 0$, there exist a polynomial p for which

$$\left\| f - p \right\|_{\infty} \le \varepsilon$$

The theorem states that any continuous function f can be approximated uniformly by polynomials, no matter how badly behaved f may be on $[a,b]$. For phrasing; for any continuous function on $[-1,1]$, f, there exist a sequence of polynomial $(p_N)_{N \in N}$ which converges uniformly towards f such that

$$\lim_{N \to \infty} \left\| f - p_N \right\| = 0.$$

Theorem 2.2[19-22] For any $f \in [-1,1]$ and $N \ge 0$ the best approximation polynomial $p_N^*(f)$ exists and is unique.

Definition 2.2

Given an integer $N \ge 1$ then $X = (x_i)_{0 \le i \le N}$ is a grid points of $N+1$ points in $[-1,1]$ such that $-1 \le x_0 < x_1 < \cdots < x_N \le 1$. Then points $(x_i)_{0 \le i \le N}$ are called the nodes of the grid.

Theorem 2.3 [19-22] Given a function $f \in C[-1,1]$ and a grid of $N+1$ nodes $X = (x_i)_{0 \le i \le N}$ there exist a unique polynomial $I_N^X(f)$ of degree N such that

$$I_N^X(f)(x_i) = f(x_i), \; 0 \le i \le N$$

$I_N^X(f)$ is called the interpolant of f through the grid X.

The interpolant $I_N^X(f)$ can be express in the Lagrange form as

$$I_N^X(f) = \sum_{i=0}^{N} f(x_i) \ell_i^X(x)$$

where $\ell_i^X(x)$ is the i-th Lagrange cardinal polynomial associated with the grid X :

$$\ell_i^X(x) = \prod_{j=0, i \ne j}^{N} \frac{x - x_i}{x_i - x_j}, \; 0 \le i \le N.$$

The Lagrange cardinal polynomials are

$$\ell_i^X(x) = \delta_{ij}, \; 0 \le i, j, \le N.$$

The best approximation polynomials $p_N^*(f)$ is also an interpolant of f at $N+1$ nodes and the error in given by :

$$\left\| f - I_N^X(f) \right\|_{\infty} \le (1 + \Lambda_N(X)) \left\| f - p_N^*(f) \right\|_{\infty}$$

where $\Lambda_N(X)$ is the Lebesque constant relative to the grid X

$$P_N = \{ p(x) : \deg(p(x)) \le N, \ N \in Z^+ \}$$

The uniform norm (or maximum norm) is defined by

$$\| f \|_\infty = \max_{x \in [-1,1]} | f(x) |$$

Theorem 2.4 [19,21] For any choice of the grid X, there exist a constant $C > 0$ such that

$$\Lambda_N(X) > \frac{2}{\pi} \ln(N+1) - C$$

Corollary 2.1 Let $\Lambda_N(X)$ be Lebesque constant relative to the grid X, then

$\Lambda_N(X) \to \infty$ as $n \to \infty$.

In a similar way, by a uniform grid,

$$\Lambda_N(X) \sim \frac{2^{N+1}}{eN \ln N} \text{ as } N \to \infty.$$

This means that for any choice of type sampling of $[-1,1]$, there exists a continuous function $f \in C[-1,1]$ such that $I_N^X(f)$ does not convergence uniformly towards f. Let assume that the function f is sufficiently smooth to have derivatives at least up to order $N+1$, with $f^{(N+1)}$ continuous i.e. $f \in C^{N+1}[a,b]$.

Definition 2.3

The nodal polynomial associated with the grid is the unique polynomial of degree $(N+1)$ and leading coefficient 1 whose zeroes are the $N+1$ nodes of X:

$$w_{N+1}^X(x) = \prod_{i=0}^{N} (x - x_i)$$

Theorem 2.5 [19-22] If $f \in C^{N+1}[-1,1]$, then for any grid X of $N+1$ nodes and for any $x \in [-1,1]$, the interpolation error is

$$f(x) - I_N^X(f)(x) = \frac{f^{(N+1)}(\varsigma)}{(N+1)!} w_{N+1}^X(x)$$

where $\varsigma = \varsigma(x) \in [-1,1]$ and $w_{N+1}^X(x)$ nodal polynomial associated with the grid X.

Definition 2.4

The Chebyshev polynomials $T_n(x)$ of the first kind are the polynomials in x of degree n, defined by relation[22]

$T_n(x) = \cos n\theta$, when $x = \cos \theta$

If the range of the variable x is the interval $[-1,1]$, the range the corresponding variables θ can be taken $[0,\pi]$. These polynomials have the following properties [19-20]:

i) $T_{n+1}(x)$ has exactly $n+1$ real zeroes on the interval

$$\Lambda_N(X) := \max_{x \in [-1,1]} \sum_{i=0}^{N} | \ell_i^X(x) |$$

The Lebesque constant contains all the information on the effects of the choice of X on $\| f - I_N^X(f) \|_\infty$.

ii) $T_n(x)$ is ortogonal on $[-1,1]$ with respect to the weight function $w(x) = (1 - x^2)^{-\frac{1}{2}}$.

iii) It is well known that [22] the relation between the powers x^n and the Chebyshev polynomials $T_n(x)$ is

Theorem 2.6 [22]Let $T_n(x)$ be a first kind Chebyshev polynomials then

$$x^{2n} = 2^{-2n+1} \sum_{j=0}^{n} \binom{n}{n-j} T_{2j}(x) \tag{5}$$

$$x^{2n+1} = 2^{-2n} \sum_{j=0}^{n} \binom{2n+1}{n-j} T_{2j+1}(x) \tag{6}$$

$$-\frac{\pi}{n} T_n(x) = \int_{-1}^{1} \frac{\log(t-x)}{\sqrt{1-t^2}} T_n(t) dt, \ n = 1,2,\ldots$$

and for $n = 0$,

$$\int_{-1}^{1} \frac{\log(t-x)}{\sqrt{1-t^2}} T_0(t) dt = -\pi \log 2 \tag{7}$$

Corollary 2.2

If $y(t) \approx \sum_{r=0}^{\infty} a_r T_r(t)$ then

$$-\pi \log(2) a_0 T_0(x) + \sum_{r=1}^{\infty} (-\frac{\pi}{r}) a_r T_r(x) \approx \int_{-1}^{1} \frac{\log(t-x)}{\sqrt{1-t^2}} y(t) dt \tag{8}$$

Definition 2.5

The grid points $X = (x_i)_{0 \le i \le N}$ such that the x_i's are the $N+1$ zeroes of the Chebyshev polynomials of degree $(N+1)$ is called the Chebyshev-Gauss (CG) grid.

Theorem 2.7[20-22] The polynomials of degree $(N+1)$ and leading coefficient 1, the unique polynomial which has the smallest uniform norm on $[-1,1]$ is the $(n+1)$th Chebyshev polynomial divided by 2^N.

3. Fundamental relations

Let us consider Eq. (1) and find the matrix forms of the equation. First we can convert the solution $y_N(x)$ defined by a truncated Chebyshev series (3) and its derivative $y_N^{(k)}(t)$ to matrix forms

$$y_N(x) = \mathbf{T}(x)\mathbf{A}, \ y_N^{(k)}(x) = \mathbf{T}^{(k)}(x)\mathbf{A} \ k = 0,1,\cdots,N \tag{9}$$

where

$$\mathbf{T}(x) = [T_0(x) \, T_1(x) \ldots T_N(x)]$$

$[-1,1]$. The i-th zero $x_{n,i}$ of $T_{n+1}(x)$ is located at

$$x_{n,i} = \cos\frac{(2(n-i)+1)\pi}{2(n+1)} \qquad (4)$$

and for odd N,

$$D = \begin{bmatrix} \frac{1}{2}\binom{0}{0}2^1 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0}2^0 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{2}{1}2^{-1} & 0 & \binom{2}{0}2^{-1} & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \binom{N}{(N-1)/2} & 0 & \cdots & \binom{N}{0}2^{1-N} \end{bmatrix}$$

for even N,

$$D = \begin{bmatrix} \frac{1}{2}\binom{0}{0}2^1 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0}2^0 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{2}{1}2^{-1} & 0 & \binom{2}{0}2^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\binom{N}{N/2}2^{1-N} & 0 & \binom{N}{(N-2)/2}2^{1-N} & \cdots & \binom{N}{0}2^{1-N} \end{bmatrix}$$

Then, by taking into account (10) we obtained

$$T(x) = X(x)(D^T)^{-1} \qquad (11)$$

and

$$\mathbf{T}^{(k)}(x) = \mathbf{X}^{(k)}(x)(\mathbf{D}^{-1})^T \ , \ k = 0,1,...,N$$

To obtain the matrix $\mathbf{X}^{(k)}(t)$ in terms of the matrix $\mathbf{X}(t)$, we can use the following relation:

$$\mathbf{X}^{(1)}(x) = \mathbf{X}(x)\mathbf{B}^T$$
$$\mathbf{X}^{(2)}(x) = \mathbf{X}^{(1)}(x)\mathbf{B}^T = \mathbf{X}(x)(\mathbf{B}^T)^2$$
$$\mathbf{X}^{(k)}(x) = \mathbf{X}^{(k)}(x)\mathbf{B}^T = \mathbf{X}(x)(\mathbf{B}^T)^k \ (12)$$

$$\mathbf{A} = [a_0 \ a_1 ... a_N]^T$$

By using the expression (5) and (6), taking n=0,1,…,N we find the corresponding matrix relation as follows

$$\mathbf{X}^T(x) = \mathbf{D}\mathbf{T^T}(x) \quad and \quad \mathbf{X}(x) = \mathbf{T}(x)\mathbf{D}^T \qquad (10)$$

$\mathbf{X}(t) = [1 \ x \dots x^N]$ and for odd N, where $\mathbf{X}(t) = [1 \ x \dots x^N]$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & N & 0 \end{bmatrix}$$

Consequently, by substituting the matrix forms (11) and (12) into (9) we have the matrix relation

$$y_N^{(k)}(x) = \mathbf{X}(x)(\mathbf{B^T})^k(\mathbf{D}^T)^{-1}\mathbf{A}, \ k = 0,1,...,N \qquad (13)$$

The similar way in the above procedure, for the logarithmic integral part (8), we obtained the matrix form as

$$-\pi\log 2a_0 + \sum_{r=0}^{N}(-\frac{\pi}{r})a_r T_r(x) = [-\pi\log 2T_0(x) \quad -\pi T_1(x) \quad \cdots \quad -\frac{\pi}{N}T_N(x)]\mathbf{A}$$

$$= [-\pi\log 2 \quad -\pi x \quad \cdots \quad -\frac{\pi}{N}x^N](\mathbf{D}^T)^{-1}\mathbf{A}$$

$$= \mathbf{X}^*(x)(\mathbf{D}^T)^{-1}\mathbf{A} \qquad (14)$$

4. Method of solution

We are ready to construct the fundamental matrix equation corresponding to Eq. (1). For this purpose, first substituting the matrix relations (13) and (14) into Eq. (1) then we obtain

$$\left(\sum_{k=0}^{m}\mathbf{P}_k(x)\mathbf{X}(x)(\mathbf{B}^T)^k(\mathbf{D}^T)^{-1} - \lambda\mathbf{X}^*(x)(\mathbf{D}^T)^{-1}\right)\mathbf{A} = f(x) \qquad (16)$$

For computing the Chebyshev coefficient matrix $\mathbf{A}$ numerically, the zeroes of the first kind Chebyshev points defined by (4) are putting the above relation (16). We obtained

$$\left(\sum_{k=0}^{m}\mathbf{P}_k(x_i)\mathbf{X}(x_i)(\mathbf{B}^T)^k(\mathbf{D}^T)^{-1} - \lambda\mathbf{X}^*(x_i)(\mathbf{D}^T)^{-1}\right)\mathbf{A} = f(x_i) \ (17)$$

So, the fundamental matrix equation is gained

$$\left(\sum_{k=0}^{m}\mathbf{P}_k\mathbf{X}(\mathbf{B}^T)^k(\mathbf{D}^T)^{-1} - \lambda\mathbf{X}^*(\mathbf{D}^T)^{-1}\right)\mathbf{A} = \mathbf{F} \ (18) \text{ where}$$

$$\mathbf{P}_k = \begin{bmatrix} P_k(x_0) & 0 & 0 & \cdots & 0 \\ 0 & P_k(x_1) & 0 & \cdots & 0 \\ 0 & 0 & P_k(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_k(x_N) \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ 1 & x_2 & x_2^2 & \cdots & x_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix}, \mathbf{F} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix}$$

$$\mathbf{X}^* = \begin{bmatrix} -\pi \log 2 & -\pi x_0 & -\dfrac{\pi}{2} x_0^2 & \cdots & -\dfrac{\pi}{N} x_0^N \\ -\pi \log 2 & -\pi x_1 & -\dfrac{\pi}{2} x_1^2 & \cdots & -\dfrac{\pi}{N} x_1^N \\ -\pi \log 2 & -\pi x_2 & -\dfrac{\pi}{2} x_2^2 & \cdots & -\dfrac{\pi}{N} x_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\pi \log 2 & -\pi x_N & -\dfrac{\pi}{2} x_N^2 & \cdots & -\dfrac{\pi}{N} x_N^N \end{bmatrix}$$

The fundamental matrix equation (18) for Eq.(1) corresponds to a system of (N+1) algebraic equations for the (N+1) unknown coefficients a_0, a_1,...,a_N. Briefly we can write Eq.(18) as

$\mathbf{WA}=\mathbf{F}$ or $[\,\mathbf{W}\ \mathbf{F}\,]$ 　　　　　(19)

so that

$\mathbf{W}=[w_{pq}]=\sum_{k=0}^{m}\mathbf{P}_k\mathbf{X}(\mathbf{B^T})^k(\mathbf{D}^T)^{-1}-\mathbf{X}^*(\mathbf{D}^T)^{-1}$ p,q=0,1,...,N (20)

We can obtain the matrix form for the mixed conditions (2), by means of Eq.(19), briefly, as

$\mathbf{U}_i\mathbf{A}=[\ \lambda_i\ ]$; or $[\mathbf{U}_i;\lambda_i\,]$ i=0,1,...,m-1 　(21)

where

$\mathbf{U}_i=\sum_{k=0}^{m-1}c^k\,_{kj}\mathbf{X}(c_j)\mathbf{B}^k(\mathbf{D}^T)^{-1}\equiv[\mathrm{u}_{i0}\ \mathrm{u}_{i0}\ \dots\ \mathrm{u}_{iN}]$

To obtain the solution of Eq.(1) under the conditions (2), by replacing the rows matrices (21) by the last m rows of the matrix (19) we have the required augmented matrix

$$[\mathbf{W}^*;\mathbf{G}^*]=\begin{bmatrix} w_{00} & w_{01} & . & . & . & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & . & . & . & w_{1N} & ; & g(x_1) \\ \dots & \dots & & & & \dots & ; & \dots \\ w_{N-m,0} & w_{N-m,1} & . & . & . & w_{N-m,N} & ; & g(x_{N-m}) \\ u_{00} & u_{01} & . & . & . & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & . & . & . & u_{1N} & ; & \lambda_1 \\ \dots & \dots & & & & \dots & ; & \dots \\ u_{m-1,0} & u_{m-1,1} & . & . & . & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix}\ (22)$$

or the corresponding matrix equation
$\mathbf{W}^*\mathbf{A}=\mathbf{F}$
If rank $(\mathbf{W}^*)$=rank $[\mathbf{W}^*;\mathbf{F}^*]$=N+1, then we can write
$\mathbf{A}=(\mathbf{W}^*)^{-1}\mathbf{F}^*$ (23)
Thus the coefficients a_n;n=0,1,...,N are uniquely determined by Eq.(23).

4.1 Error analysis and convergence

Since, $\|T_{N+1}\|_\infty=1$, we conclude that if we choose the grid nodes $(x_i)_{0\le i\le N}$ to be zero the (N+1) zeroes of the Chebyshev polynomials T_{N+1}, we have

$$\left\|w_{N+1}^X\right\|=\frac{1}{2^N}$$

and this is the smallest possible value. In particular, from Theorem 2.10, for any $y\in C^{N+1}[-1,1]$ we have

$$\|y-y_N\|_\infty\le\frac{1}{2^N(N+1)!}\|f^{N+1}\|_\infty$$

If $y^{(N+1)}$ is uniformly bounded, the convergence of the interpolation y_N towards y when $N\to\infty$ is then extremly fast. Also the Lebesgue constant associated with the Chebyshev-Gauss grid is small

$$\Lambda_N(X)\sim\frac{2}{\pi}\ln(N+1)\text{ as }N\to\infty$$

This is much beter than uniform grids and close to the optimal value.

5. Illustra tive example

In this section, a numerical example is given to illustrate the accuracy and effectiveness properties of the method and all of them were performed on the computer using a program written in Maple 9.

Example 5.1

Consider the following logarithmic singular integro-differential equation

$$y'+y=(\pi+1)x+2+\pi\log 2+\int_{-1}^{1}\frac{\log(t-x)}{\sqrt{1+t^2}}y(t)dt$$

with $y(0)=1$. We seek the solution $y_N(x)$ as a truncated first-kind Chebyshev polynomial

$y_N(x)=\sum_{r=0}^{N}a_rT_r(x)\cdot$

So that, $f(x)=(\pi+1)x+2\pi\log 2$, $P_0(x)=1$, $P_0(x)=1$, $\lambda=1$. Then, for $N=5$ the zeroes of $T_6(x)$

$x_0=-\cos\left(\frac{\pi}{12}\right)$, $x_1=-\frac{\sqrt{2}}{2}$, $x_2=-\cos\left(\frac{5\pi}{12}\right)$, $x_3=\cos\left(\frac{5\pi}{12}\right)$, $x_4=\frac{\sqrt{2}}{2}$, $x_5=\cos\left(\frac{\pi}{12}\right)$

and the fundamental matrix equation of the problem is defined by

$\left(\mathbf{P}_0\mathbf{X}(\mathbf{B}^T)^0(\mathbf{D}^T)^{-1}+\mathbf{P}_1\mathbf{X}(\mathbf{B}^T)^1(\mathbf{D}^T)^{-1}-\lambda\mathbf{X}^*(x)(\mathbf{D}^T)^{-1}\right)\mathbf{A}=\mathbf{F}$ · (24)
And matrices for condition are

$\mathbf{X}(0)\,(\mathbf{M}^T)^{-1}=[1\ \ 0\ \ -1\ \ 0\ \ 1\ \ 0\,]\mathbf{A}=[1]$,　　(25)

where $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{B}$, $\mathbf{F}$, $\mathbf{D}$ are matrices of order (6x6) defined by

$$P_0=\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}\ P_1=\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}$$

$$B=\begin{bmatrix} 0&1&0&0&0&0 \\ 0&0&2&0&0&0 \\ 0&0&0&3&0&0 \\ 0&0&0&0&4&0 \\ 0&0&0&0&0&5 \\ 0&0&0&0&0&0 \end{bmatrix}\ F=\begin{bmatrix} -1.054757 \\ 0.017165 \\ 1.873790 \\ 4.017636 \\ 5.874261 \\ 6.946187 \end{bmatrix}$$

$$D=\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ \frac{1}{2}&0&\frac{1}{2}&0&0&0 \\ 0&\frac{3}{4}&0&\frac{1}{4}&0&0 \\ \frac{3}{8}&0&\frac{1}{2}&0&\frac{1}{8}&0 \\ 0&\frac{5}{8}&0&\frac{5}{16}&0&\frac{1}{16} \end{bmatrix}$$

If these matrices are substituted (24), we have the linear algebraic system:

$$[\mathbf{W}^*;\mathbf{F}^*]=\begin{bmatrix} 1.945713 & -3.000471 & -1.012245 & 12.817656 & -18.193538 & 13.650684 & ; & -1.054757 \\ 1.945713 & -1.928548 & -2.203344 & 8.890470 & -4.766675 & -9.772448 & ; & 0.017165 \\ 1.945713 & -0.071923 & -2.636568 & 0.877642 & 4.218422 & -3.340257 & ; & 1.873790 \\ 1.945713 & 2.071923 & -0.566016 & -5.269947 & -2.954180 & 6.019749 & ; & 4.017636 \\ 1.945713 & 3.928548 & 3.453509 & -2.890470 & -4.766775 & -0.227551 & ; & 5.874261 \\ 0 & 1 & 0 & -3 & 0 & 5 & ; & 1 \end{bmatrix}$$

Then, when this system is solved, we obtain the Chebyshev coefficient matrix **A** as:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

When the Chebyshev coefficient matrix **A** is subsituting in Eq.(3), we obtained the approximate solution of the problem for $N = 5$

$$y_5(x) = \sum_{r=0}^{5} a_r T_r(x) = 1 + x$$

which is the exact solution of this problem.

Example 5.2

Consider the following logarithmic singular integro-differential equation

$$y''-xy'+3y = \frac{x^3}{3} + 6x + \frac{1}{\pi}\int_{-1}^{1} \frac{\log(t-x)}{\sqrt{1+t^2}}\, y(t)dt$$

with the conditions $y(0) = 0$, $y'(0) = 0$. The exact solution of this problem is $y = x^3$. We obtained the approximate solution of the problem for N=4,5,6 which are tabulated and graphed. For numerical results, see Table 1. We display a plot of absolute difference exact and approximate solutions in Fig.1 and error functions for various N is shown in Fig.2.

Table 1 Error analysis of Example 5.2 for the x value

| x	Exact Sol.	Present Method					
		N=4	N_e=4	N=5	N_e=5	N=6	N_e=6
-1.0	-1.000	-0.999999	0.700E-6	-0.999999	0.300E-7	-0.999999	0.500E-8
-0.8	-0.512	-0.511999	0.500E-6	-0.51199	0.200E-7	-0.511999	0.100E-8
-0.6	-0.216	-0.215999	0.200E-6	-0.215999	0.100E-7	-0.216000	0.000E-0
-0.4	-0.064	-0.063999	0.700E-7	-0.063999	0.600E-8	-0.064000	0.300E-9
-0.2	-0.008	-0.008000	0.300E-8	-0.007999	0.190E-8	-0.008000	0.600E-10
0.0	0.000	0.000000	0.000E-0	0.000000	0.000E-0	0.000000	0.000E-0
0.2	0.008	0.007999	0.100E-8	0.007999	0.250E-8	0.008000	0.800E-10
0.4	0.064	0.063999	0.150E-6	0.063999	0.600E-8	0.064000	0.500E-9
0.6	0.216	0.215999	0.600E-6	0.215999	0.200E-7	0.216000	0.000E-0
0.8	0.512	0.511998	0.170E-5	0.511999	0.100E-7	0.511999	0.300E-8
1.0	1.000	0.999996	0.390E-5	0.999999	0.100E-7	0.999999	0.150E-7
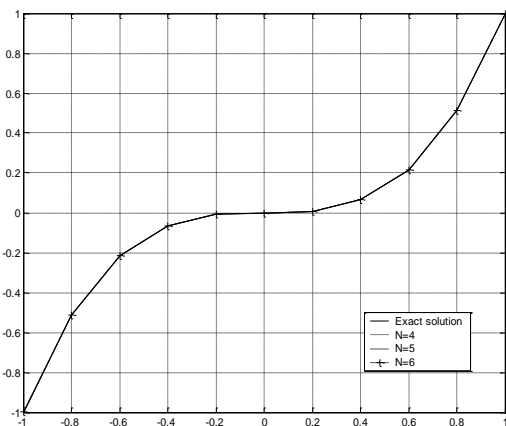
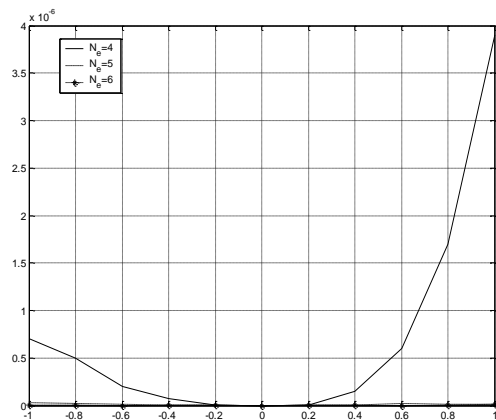Figure 1: Numerical and exact solution of the Example 5.2 for N=4,5,6.

Figure 2: Error function of Example 5.2for varios N.

Example 5.3

Let us consider the following logarithmic singular integro-differential equation

$$y''(x) = f(x) + \frac{1}{\pi}\int_{-1}^{1}\frac{\log(t-x)}{\sqrt{1+t^2}}y(t)dt, \quad y(0)=0, \quad y(1)=e$$

The exact solution of this problem is $y(x) = \exp(x)$.
We approximately solve this problem by our method. Then, we give comparison of exact solution and approximate solutions in Table 2. Also, Fig.3 display numerical results and exact solution for various N and Fig.4 is given absolute errors.

Table 2 Error analysis of Example 5.3 for the x value

| x	Exact Sol.	N=4	N_e=4	N=6	N_e=6	N=8	N_e=8
0.0	1.000000	0.999999	0.100E-9	1.000000	0.000E-0	1.000000	0.000E-0
0.1	1.105170	1.099438	0.573E-2	1.105128	0.420E-4	1.105175	0.286E-5
0.2	1.221402	1.197128	0.242E-1	1.221603	0.201E-3	1.221405	0.326E-5
0.3	1.349858	1.292297	0.576E-1	1.349363	0.495E-3	1.349854	0.423E-5
0.4	1.491182	1.384398	0.107E-0	1.491752	0.570E-3	1.491188	0.601E-5
0.5	1.648721	1.473106	0.175E-0	1.648050	0.670E-3	1.648729	0.815E-5
0.6	1.822118	1.558321	0.263E-0	1.821158	0.960E-3	1.822127	0.944E-5
0.7	2.013752	1.640164	0.373E-0	2.014863	0.111E-2	2.013742	0.100E-4
0.8	2.225540	1.718983	0.506E-0	2.223250	0.229E-2	2.225555	0.155E-4
0.9	2.459603	1.795334	0.664E-0	2.455881	0.372E-2	2.459156	0.236E-4
1.0	2.718281	1.870050	0.848E-0	2.724769	0.648E-2	2.718258	0.536E-4
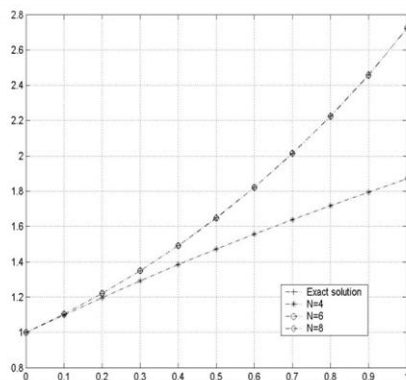
Figure 3: Numerical and exact solution of the Example 5.3 for N=4,6,8
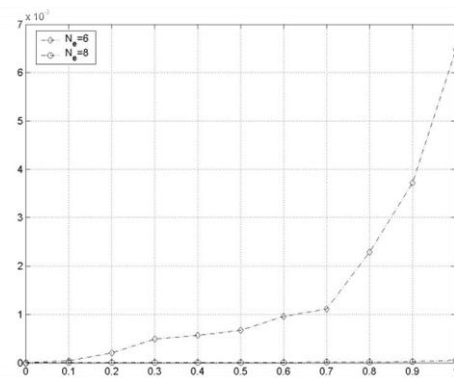

Figure 4: Error function of Example 5.3 for varios N.

5. Conclusion

A new method based on the truncated Chebyshev series of the first kind is developed to numerical solve logarithmic singular integro-differential equations with mixed conditions on Chebyshev-Gauss grid. Logarithmic integro-differential equations and logarithmic singular equations are usually difficult to solve analytically. In many cases, it is required to obtained the approximate solution. For this propose, the present method can be proposed. In this paper, the first kind Chebyshev polynomial approach has been used for the approximate solution of logarithmic singular integro-differential equations. For the suggested method, we show error analysis and converge. Thus the proposed method is suggested as an efficient. Examples with the satisfactory results are used to demonstrate the application of this method. Suggested approximations make this method very attractive and contributed to the good agreement between approximate and exact values in the numerical example.

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