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İntegral koşullu hiperbolik tip kısmi diferansiyel denklemlerin taylor polinom çözümleri

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Anahtar Kelimeler: Taylor matris metodu, İki değişkenli Taylor serileri, Hiperbolik kısmi diferansiyel denklemler, Polinom yaklaşımı.

ÖZET

Bu makalede, bir boyutlu hiperbolik kısmi diferansiyel denklemlerin verilen başlangıç ve integral sınır koşulları altında çözümü ele alınmıştır. Önerilen yöntem, verilen denklem ve koşulları matris denklemine dönüştürerek bilinmeyenleri Taylor katsayıları olan lineer cebirsel denklem sistemi elde eder. Bu matris denklemi çözülerek Taylor katsayıları ve polinom yaklaşımı elde edilir. Ayrıca elde edilen sonuçlar bilinen değerlerle karşılaştırılmış; yöntemin doğruluğu ve hata analizi yapılmıştır.

Taylor polynomial solution of hyperbolic type partial differential equation with an integral condition

Key Words: Taylor matrix method, Double Taylor series, Hyperbolic partial differential equations, Polynomial approximation

ABSTRACT

In this paper, the problem of solving the one-dimensional hyperbolic partial differential equation, subject to given initial and nonlocal boundary conditions, is considered. The proposed method converts the equation and conditions to matrix equation, which corresponds to system of linear algebraic equations with unknown Taylor coefficients. Thus by solving the matrix equation, Taylor coefficients and polynomial approach are obtained. Also, the obtained results are compared by the known results; the accuracy of solutions and the error analysis are performed.

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1. Introduction

The development of numerical techniques for solving partial differential equations in physics subject to nonlocal conditions is a subject of considerable interest. Certain problems of modern physics and technology can be effectively described in terms of nonlocal problems for partial differential equations. These nonlocal conditions arise mainly when the data on the boundary cannot be measured directly. Hyperbolic initial boundary value problems in one dimension that involve nonlocal boundary conditions have been studied by several authors [1–15].

The presence of an integral term in boundary condition can greatly complicate the application of standard numerical techniques such as finite difference procedures, finite element methods, spectral techniques, boundary integral equation schemes, etc. It is therefore important to be able to convert nonlocal boundary value problems to a more desirable form, to make them more applicable to problems of practical interest. In many cases this is a hard task. The purpose of this study is to give a Taylor polynomial approximation for the solution of Hyperbolic Type Partial Differential Equation with an integral Condition. The technique used is an improved Taylor matrix method, which has been given for solving ordinary differential, integral and integro-differential equations [16-26]. In this article, the following hyperbolic problem is considered with a nonlocal constraint replacing the standard boundary condition: with initial conditions.

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=G(x,t), \quad 0<x<l, \quad 0<t\le T, \qquad (1)$$

with initial conditions

$$u(x,0)=f(x), \quad 0\le x\le l, \qquad (2)$$ and

Dirichlet boundary condition

$$u(0,t)=h(t), \quad 0<t\le T; \qquad (4)$$

and the nonlocal condition

$$\int_0^1 u(x,t)dx=k(t), \quad 0<t\le 1 \qquad (5)$$

where G, f, g, h and k are known functions. We assume that the solution is expressed in the form

$$u(x,t)=\sum_{r=0}^{N}\sum_{s=0}^{N}a_{r,s}(x-c_0)^r(t-c_1)^s, \quad a_{r,s}=\frac{1}{r!s!}u^{(r,s)}(c_0,c_1), \quad (c_0,c_1)\in[a,b]x[0,T] \qquad (6)$$

so that the Taylor coefficients to be determined are

$a_{r,s}$ (r,s=0,…,N).

2. Fundamental Relations

To obtain the numerical solution of the problem (1)-(5) with presented method, we first evaluate the Taylor coefficients of the unknown function. For convenience, the solution function (6) can be written in the matrix form

$$u(x,t)=\mathbf{X}(x,t)\mathbf{A} \qquad (7)$$

The matrix $\mathbf{X}(x,t)$ can be expressed as

$$\mathbf{X}(x,t)=\left[X_{0,0}(x,t)\ X_{0,1}(x,t)\ldots X_{0,N}(x,t)\ X_{1,0}(x,t)\ X_{1,1}(x,t)\ldots X_{1,N}(x,t)\ \ldots\ X_{N,0}(x,t)\,X_{N,1}(x,t)\ldots X_{N,N}(x,t)\right]$$ where

$$X_{m,n}(x,t)=(x-c_0)^m(t-c_1)^n, \quad m,n=0,1,\ldots,N$$ and

$$\mathbf{A}=\left[a_{0,0}\ a_{0,1}\ \ldots\ a_{0,N}\ \ a_{1,0}\ a_{1,1}\ \ldots\ a_{1,N}\ \ldots\ a_{N,0}\ \ a_{N,1}\ \ldots\ a_{N,N}\right]^T.$$

On the other hand, the relation between the matrix $\mathbf{X}(x,t)$ and its derivative is

$$\mathbf{X}^{(m,n)}(x,t)=\mathbf{X}(x,t)(\bar{\mathbf{B}})^m(\tilde{\mathbf{B}})^n \qquad (8)$$

where

$$\mathbf{B}=\begin{bmatrix}0&1&0&\cdots&0\\0&0&2&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&N\\0&0&0&\cdots&0\end{bmatrix}_{(N+1)x(N+1)}, \quad \bar{\mathbf{B}}=\begin{bmatrix}0&\mathbf{I}&0&\cdots&0\\0&0&2\mathbf{I}&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&N\mathbf{I}\\0&0&0&\cdots&0\end{bmatrix}_{(N+1)^2x(N+1)^2}, \quad \tilde{\mathbf{B}}=\begin{bmatrix}\mathbf{B}&0&\cdots&0\\0&\mathbf{B}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\mathbf{B}\end{bmatrix}_{(N+1)^2x(N+1)^2};$$

here I is the (N+1) x (N+1) identity matrix. By using the relations (7) and (8) we have

$$u^{(m,n)}(x,t)=\mathbf{X}^{(m,n)}(x,t)\mathbf{A}=\mathbf{X}(x,t)(\bar{\mathbf{B}})^m(\tilde{\mathbf{B}})^n\mathbf{A},\qquad m,n=0,1,2.,\tag{9}$$

3. Method of Solution

Our purpose is to investigate the approximate solution of the problem (1)-(5), in the series form (6) or in the matrix form $u(x,t)=\mathbf{X}(x,t)\mathbf{A}$. To obtain the solution, we first reduce the terms of Eq. (1) to matrix forms as follows

$$u_{xx}(x,t)=\mathbf{X}(x,t)(\bar{\mathbf{B}})^2\mathbf{A}\tag{10}$$

and

$$u_{tt}(x,t)=\mathbf{X}(x,t)(\tilde{\mathbf{B}})^2\mathbf{A}\tag{11}$$

We can also expand the function $G(x,t)$ to Taylor series

$$G(x,t)=\sum_{r=0}^{N}\sum_{s=0}^{N}g_{r,s}(x-c_0)^r(t-c_1)^s,\qquad g_{r,s}=\frac{G^{(r,s)}(c_0,c_1)}{r!s!}\tag{12}$$

or from (12) we get the matrix form

$$G(x,t)=\mathbf{X}(x,t)\mathbf{G}\tag{13}$$

where

$$\mathbf{G}=\left[g_{0,0}\;\; g_{0,1}\ldots g_{0,N}\;\; g_{1,0}\;\; g_{1,1}\ldots g_{1,N}\;\;\ldots\;\; g_{N,0}\;\; g_{N,1}\ldots g_{N,N}\right]^T.$$

Substituting the expressions (10)-(13) into Eq. (1) and simplifying the result, we have the matrix equation

$$\left\{(\tilde{\mathbf{B}})^2-(\bar{\mathbf{B}})^2\right\}\mathbf{A}=\mathbf{G}.$$

(14)

Briefly, we can write Eq. (14) in the form

$$\mathbf{WA}=\mathbf{G}\tag{15}$$

where $\mathbf{W}=\left[w_{i,j}\right],\quad i,j=1,\ldots,(N+1)^2$.

We now present the alternative forms for $u(x,t)$ which are important to simplify matrix forms of the conditions. The simplification in conditions is done only with respect to the variable x or t. Therefore we must use different forms for initial and boundary conditions. For the initial conditions (2) and (3)

$$u(x,t)=\mathbf{X}(x)\mathbf{Q}(t)\mathbf{A}\tag{16}$$

and

$$u_t(x,t) = \mathbf{X}(x)\mathbf{Q}(t)\tilde{\mathbf{B}}\mathbf{A}\,; \tag{17}$$

for the boundary condition (3)

$$u(x,t) = \mathbf{T}(t)\mathbf{J}(x)\mathbf{A}\,. \tag{18}$$

Using (18), we can create the matrix form of the nonlocal condition (4) as follows

$$\int_0^x u(x,t)dx = \int_0^x \mathbf{T}(t)\mathbf{J}(x)\mathbf{A}dx = \mathbf{T}(t)\int_0^x \mathbf{J}(x)\mathbf{A}dx = \mathbf{T}(t)\tilde{\mathbf{J}}(x)\mathbf{A} \tag{19}$$

where

$$\mathbf{X}(x) = \begin{bmatrix} 1 & (x-c_0) & (x-c_0)^2 & \ldots & (x-c_0)^N \end{bmatrix}$$

$$\mathbf{T}(t) = \begin{bmatrix} 1 & (t-c_1) & (t-c_1)^2 & \ldots & (t-c_1)^N \end{bmatrix}$$

$$\tilde{\mathbf{J}}(x) = \begin{bmatrix} (x-c_0)\mathbf{I} & \dfrac{(x-c_0)^2}{2}\mathbf{I} & \dfrac{(x-c_0)^3}{3}\mathbf{I} & \ldots & \dfrac{(x-c_0)^{N+1}}{N+1}\mathbf{I} \end{bmatrix}$$

$$\mathbf{J}(x) = \begin{bmatrix} \mathbf{I} & (x-c_0)\mathbf{I} & (x-c_0)^2\mathbf{I} & \ldots & (x-c_0)^N\mathbf{I} \end{bmatrix}$$

$$\mathbf{Q}(t) = \begin{bmatrix} \mathbf{T}(t) & 0 & \cdots & 0 \\ 0 & \mathbf{T}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{T}(t) \end{bmatrix}.$$

Note that the matrices $\mathbf{J}(\mathrm{x})$ and $\mathbf{Q}(\mathrm{t})$ are of dimensions $(\mathrm{N}+1)\mathrm{x}(\mathrm{N}+1)^2$. On the other hand the matrix representations of non-homogeneous terms of Eqs. (2)-(5) can be written in the forms

$$f(x) = \mathbf{X}(x)\mathbf{F}, \qquad \mathbf{F} = \begin{bmatrix} f_0 \; f_1 \ldots \; f_N \end{bmatrix}^T, \qquad f_n = \frac{f^{(n)}(c_0)}{n!}, \tag{20}$$

$$m(x) = \mathbf{X}(x)\mathbf{M}, \qquad \mathbf{M} = \begin{bmatrix} m_0 \; m_1 \ldots \; m_N \end{bmatrix}^T, \qquad m_n = \frac{m^{(n)}(c_0)}{n!}, \tag{21}$$

$$h(t) = \mathbf{T}(t)\mathbf{H}, \qquad \mathbf{H} = \begin{bmatrix} h_0 \; h_1 \ldots \; h_N \end{bmatrix}^T, \qquad h_n = \frac{h^{(n)}(c_1)}{n!}, \tag{22}$$

$$k(t) = \mathbf{T}(t)\mathbf{K}, \qquad \mathbf{K} = \begin{bmatrix} k_0 \; k_1 \ldots \; k_N \end{bmatrix}^T, \qquad k_n = \frac{k^{(n)}(c_1)}{n!}\mathrm{k} \tag{23}$$

By substituting the relations (16)-(23) into (2)-(5) and then simplifying the result, we get the matrix forms of conditions, respectively, as

$$\mathbf{K_1A} = \mathbf{Q}(0)\mathbf{A} = \mathbf{F} \tag{24}$$

$$\mathbf{K_2A} = \mathbf{Q}(0)\tilde{\mathbf{B}}\mathbf{A} = \mathbf{M} \tag{25}$$

$$\mathbf{K}_3\mathbf{A} = \mathbf{J}(0)\mathbf{A}=\mathbf{H} \tag{26}$$

$$\mathbf{K}_4\mathbf{A} = \tilde{\mathbf{J}}(1)\mathbf{A}=\mathbf{K} \tag{27}$$

To obtain solution of Eq. (1) under the conditions (2)-(5), the augmented matrix is formed as follows

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right] = \begin{bmatrix} \mathbf{K}_1 & ; & \mathbf{F} \\ \mathbf{K}_2 & ; & \mathbf{M} \\ \mathbf{K}_3 & ; & \mathbf{H} \\ \mathbf{K}_4 & ; & \mathbf{K} \\ \bar{\mathbf{W}} & ; & \bar{\mathbf{G}} \end{bmatrix} \tag{28}$$

The unknown Taylor coefficients are obtained as

$$\mathbf{A} = \left(\tilde{\tilde{\mathbf{W}}}\right)^{-1} \tilde{\tilde{\mathbf{G}}}$$

where $\left[\tilde{\tilde{\mathbf{W}}};\tilde{\tilde{\mathbf{G}}}\right]$ is generated by using the Gauss elimination method and then removing zero rows of gauss eliminated matrix. Here $\overline{\mathbf{W}}$ and $\overline{\mathbf{G}}$ are obtained by throwing away maximum number of row vectors from W and G, respectively, so that the rank of system defined (28) cannot be smaller than $(N+1)^2$. This process provides higher accuracy because of decreasing truncation error.

4. Accuracy of the solution and error analysis

We can easily check the accuracy of the method. Since the truncated Taylor series (4) is an approximate solution of Eq. (1), when the function $u_{N,N}(x,t)$ and its derivatives are substituted in Eq.(1), the resulting equation must be satisfied approximately; that is, for $x = x_p$, $t = t_q \in [a,b]x[0,T]$, p,q=0,1,2,.

$$E_{N,N}(x_p,t_q) = \left|u_{xx}(x_p,t_q) - u_{tt}(x_p,t_q) - G(x_p,t_q)\right| \cong 0$$

and $E_{N,N}(x_p,t_q) \le 10^{-k_{pq}}$ (k_{pq} positive integer). If $\max 10^{-k_{pq}} = 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E_{N,N}(x_p,t_q)$ at each of the points becomes smaller than the prescribed 10^{-k}.

5. Numerical examples

This section is devoted to computational results. We applied the method presented in this paper and solved two examples. We illustrate it by the following examples. Numerical computations have been done using MAPLE 9.

Example 1. Consider the problem (1)-(5) with $T = 0.5$

$$f(x) = 0, \quad 0 < x < 1,$$

$$G(x,t) = 0, \quad 0 < x < 1, \quad 0 < t < 0.5,$$

$$h(t) = \sin(\pi t), \quad 0 < t < 0.5,$$

$$m(x) = \pi\cos(\pi x), \quad 0 < t < 0.5.$$

The exact solution of the problem is $u(x,t) = \cos(\pi x)\sin(\pi t)$ [14, 15

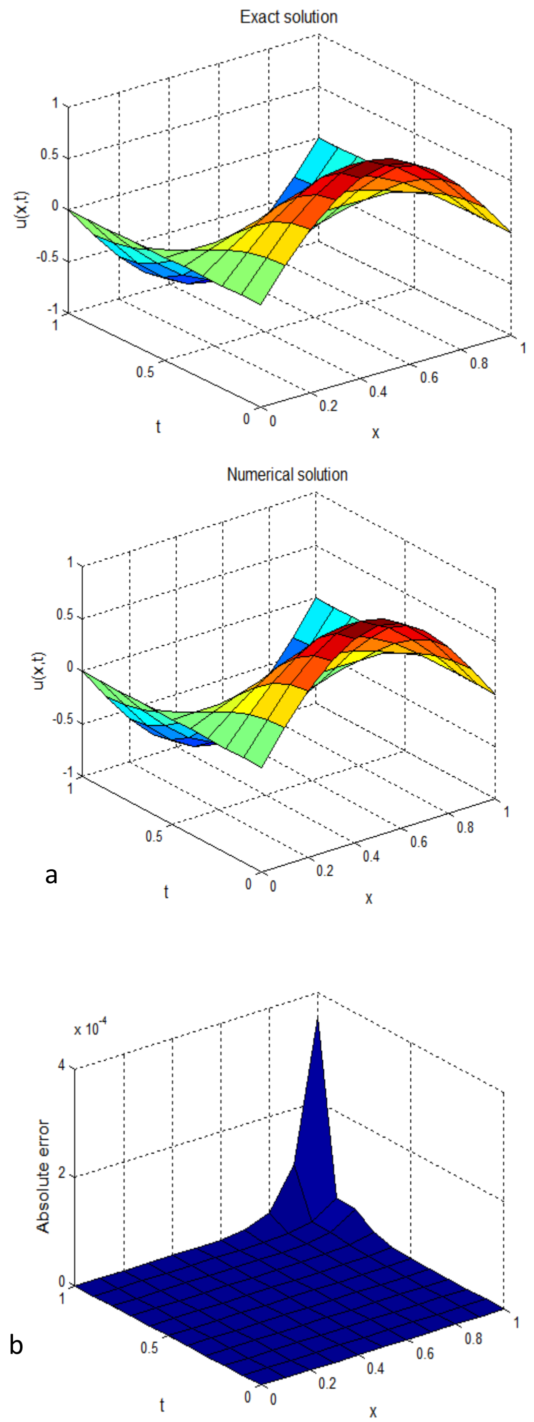

Fig.1. a) Comparison of exact solution with the numerical solution **b)** Absolute error of Example 1 for N=20.

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Table 1. Error analysis of Example 1.

	Absolute error at t=0.5		Maximum error $\left\|u(x,0.5)-\tilde{u}(x_i,0.5)\right\|_{x_i,\infty}$		
t	Present method(N=20)	Legendre-Tau (m=9)[16]	Present method(N=20)	MOL I	MOL II
0.1	5.8E-15	1.5E-6	9.0E-20	1.9E-5	1.0E-7
0.2	1.4E-13	1.2E-6	5.6E-15	2.9E-5	1.0E-7
0.3	2.2E-12	8.5E-6	1.2E-11	2.7E-5	9.5E-8
0.4	2.2E-11	5.3E-6	6.4E-10	1.6E-5	6.2E-8
0.5	1.5E-10	0	1.0E-8	1.1E-14	2.1E-14
0.6	7.8E-10	1.5E-6	8.9E-8	1.6E-5	6.2E-8
0.7	3.0E-9	1.2E-6	5.5E-7	2.7E-5	9.5E-8
0.8	8.7E-9	8.5E-5	2.6E-6	2.9E-5	1.0E-7
0.9	1.0E-8	5.3E-5	1.1E-5	1.9E-5	1.0E-7
1	8.6E-8	1.7E-5	3.9E-5	5.9E-13	2.6E-13

Fig.1 (a) shows the comparison of the exact solution versus the numerical solution obtained using the proposed Taylor method. It seems that the solutions almost identical. We also show the graphic of absolute error for N=20 in Fig. 1 (b). Table 1 shows that the numerical result of the example are better than the results in [15, 16].

Example 2. In the second example, we solve (1)-(5) with T=1 and

$f(x) = \cos(\pi x), \quad 0 < x < 1,$

$m(x) = 0, \quad 0 < x < 1,$

$G(x,t) = 0, \quad 0 < t < 1, \quad 0 < x < 1,$

$h(t) = \cos(\pi t), \quad 0 < t < 1,$

$k(t) = 0, \quad 0 < t < 1.$

The exact solution of this problem is

$u(x,t) = \frac{1}{2}\left[\cos(\pi(x+t)) + \cos(\pi(x-t))\right]$ [4, 15].

Table 2. Absolute errors of Example 2 for $N = 20$.

	Absolute error at $u(x,1)$		Absolute error at $u(x,0.25)$	
t	Present method(N=20)	Cubic B-Spline method(M=4)	Present method(N=20)	Legendre Tau(M=9)
0.1	2.2E-11	1.9E-5	3.6E-22	3.0E-7
0.2	2.5E-10	2.4E-5	7.1E-20	2.6E-7
0.3	2.2E-9	1.8E-5	8.4E-17	1.9E-7
0.4	1.4E-8	1.1E-6	5.9E-16	8.9E-8
0.5	7.9E-8	0	1.8E-15	0
0.6	3.6E-7	1.1E-5	1.0E-14	8.9E-8
0.7	1.4E-6	1.8E-5	1.7E-13	1.9E-7
0.8	4.5E-6	2.4E-6	1.1E-13	2.6E-7
0.9	1.0E-6	1.9E-5	2.7E-12	3.0E-7
1	5.0E-5	4.3E-5	1.8E-11	3.1E-7

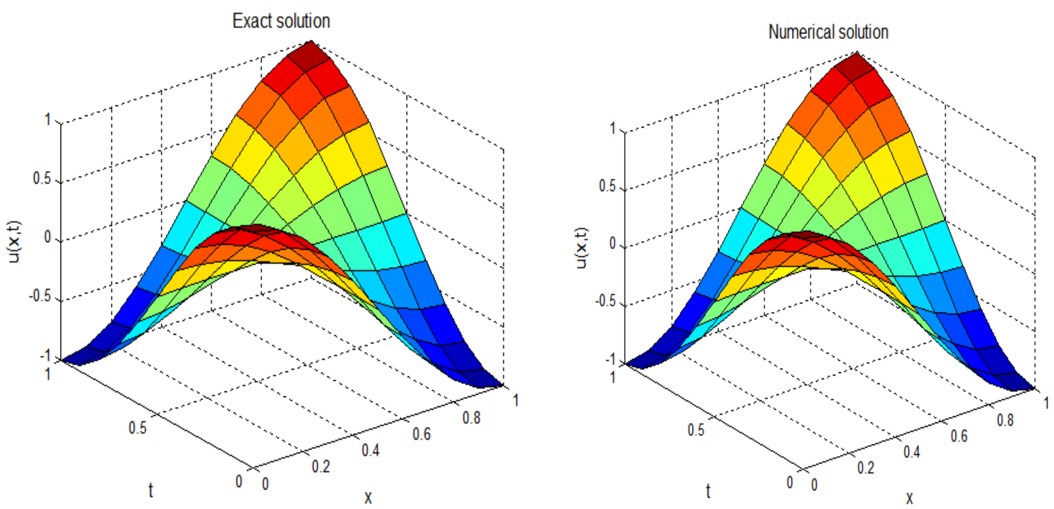

Fig. 2. Comparison between the exact solution and approximate solution for N = 20.

We solve the problem for N =20 and show the results in Table 2 and in Fig. 2. It is seen from Table 2 that the numerical results are better than the results in [4, 16].

6. Conclusion

In this paper, a very simple but effective Taylor matrix method was proposed for the numerical solution of the second-order wave equation with given initial conditions and a boundary condition and an integral condition in place of the classical boundary condition. One of the advantages of this method that the solution is expressed as a truncated Taylor series then $u(x,t)$ can be easily evaluated for arbitrary values of x and t by using the computer program without any computational effort. From the given illustrative examples, it can be seen that the Taylor series approach can obtain very accurate and satisfactory results. Although the approach is only illustrated here for one dimensional problem, its extension to similar two-dimensional problems is straightforward and requires only minor programming. We believe that our fundamental techniques can be applied or extended to a much larger class of problems.

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