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On an Interior Compactness of One Homogeneous Boundary Value Problem

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ABSTRACT
In the paper the conditions are obtained providing existence and uniqueness of the regular solution of the boundary problem for class of the second order homogeneous operator-differential equation with singular coefficients. High term of the equation contains the normal operator the spectrum of which is contained in the certain sectors. Further, it is proved the theorem of internal compactness of space of regular solutions of the considered problem.

Bir Homojen Sınır-Değer Probleminin İç Kompaktlığı Üzerine

Anahtar Kelimeler
Normal operator
discontinuous coefficients
regular solvability
Hilbert space

ÖZET
Makalede ikinci mertebe katsayıları singüler (tekil) olan homojen operator diferansiyel denklemler sınıfı için bir sınır değer probleminde regüler (düzgün) çözümlerin varlığı ve tekliğini garanti eden koşullar bulunmuştur. Denklemin yüksek terimi, spektri belli bir sektörde olan normal operator içermektedir. Daha sonra, ele alınan problemin regüler çözümleri uzayının iç kompakt olduğu kanıtlanmıştır.

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1 INTRODUCTION

We consider the boundary problem for homogeneous operator-differential equation in separable Hilbert space H

$$-\frac{d^2u}{dt^2}+\rho(t)A^2u+A_0\frac{d^2u}{dt^2}+A_1\frac{du}{dt}+A_2u=0, \quad (1)$$

$$u(0)=\varphi, \quad (2)$$

where $\varphi\in H_{3/2}$, $u(t)\in W_2^2(R_+;H)$, $\rho(t)$ is the form of

$$\rho(t)=\begin{cases}\alpha^2, t\in(0;1),\\ \beta^2, t\in(1;\infty),\end{cases}$$

moreover $\alpha>0,\beta>0$, operator coefficients A and A_j $(j=0,1,2)$ satisfy the following conditions

1) A is normal, with quite continuous inverse A^{-1} operator, spectrum of which is contained in a corner sector,

$$S_\varepsilon=\{\lambda:|\arg\lambda|\le\varepsilon\},\ 0\le\varepsilon<\pi/2;$$

2) Operators $B_j=A_jA^{-j}$ $(j=0,1,2,)$ are bounded in H.

Denote by $L_2(R_+;H)$ a Hilbert space of the vector-functions $f(t)$ with values from H, measurable and integrable by square-law with norm

$$\|f\|_{L_2(R_+;H)}=\left(\int_0^{+\infty}\|f(t)\|^2dt\right)^{1/2}.$$

Further we introduce the space e.g. [1]

$$W_2^2(R_+;H)=\left\{u(t):u'',A^2u\in L_2(R_+;H),\|u\|_{W_2^2(R_+;H)}=\left(\|u''\|^2_{L_2(R_+;H)}+\|A^2u\|^2_{L_{2,}(R_+;H)}\right)^{1/2}\right\}.$$

Then from the trace theorem it results that

$$\overset{o}{W_2^2}(R_+;H;\{0\})=\{u\,|\,u\in W_2^2(R_+;H),u(0)=0\}.$$

Definition. If for any $\varphi\in H_{3/2}$ there exists the vector-function $u(t)$ which satisfies (1), and boundary condition (2) in the sense

$$\lim_{t\to+0}\|u(t)-\varphi\|_{3/2}=0$$

also the estimation takes place

$$\|u\|_{W_2^2(R_+;H)}\le const\|\varphi\|_{3/2},$$

then $u(t)$ is called a regular solution of the problem (1),(2), and the problem (1),(2) is called regular solvable.

We shall note that when the equations are not homogeneous and $A_0=1,\alpha=\beta=1$ this problem is investigated in [2], when $A=A^*\ge cE, c>0$, at $A_0=1,\alpha\ne\beta$ in [3]. When the equation is non homogeneous boundary problem (1),(2) is investigated [4] and resolvability of the equation (1) on all axis it is considered in [5].

2 Determination of the reguler solution

First we shall consider the problem

$$P_0u=-\frac{d^2u}{dt^2}+\rho(t)A^2u=0, \quad (3)$$

$$u(0)=\varphi. \quad (4)$$

Let's seek the regular solution of the problem (3), (4) in the form

$$u_0(t)=\begin{cases}e^{-\alpha tA}c_1+e^{-\alpha(1-t)A}c_2, & t\in(0;1),\\ e^{\beta A(1-t)}c_3, & t\in(1;\infty),\end{cases}$$

where c_1,c_2,c_3 - are unknown elements from $H_{3/2}$. From the condition (4) and inclusion $u_0(t)\in W_2^2(R_+;H)$ it is obtained the

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following system of the equations relatively c_1, c_2 and c_3 :

$$\begin{cases} c_1 + e^{-\alpha A} c_2 = \varphi, \\ e^{-\alpha A} c_1 + c_2 = c_3, \\ -\alpha A e^{-\alpha A} c_1 + \alpha A c_2 = -\beta A c_3 , \end{cases}$$

or

$$\begin{cases} c_1 + e^{-\alpha A} c_2 + 0 \cdot c_3 = \varphi, \\ e^{-\alpha A} c_1 + c_2 - c_3 = 0, \\ -\alpha e^{-\alpha A} c_1 + \alpha c_2 + \beta c_3 = 0, \end{cases}$$

or in an operational

$$\Delta_0(A) c = \tilde{\varphi},$$

where

$$\Delta_0(A) = \begin{vmatrix} E & e^{-\alpha A} & 0 \\ -e^{-\alpha A} & E & -E \\ -\alpha e^{-\alpha A} & \alpha E & \beta E \end{vmatrix},$$

$$c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tilde{\varphi} = \begin{pmatrix} \varphi \\ 0 \\ 0 \end{pmatrix}.$$

As we have shown $\Delta_0(A)$ that it is $H^3 = H \times H \times H$ (see . [4]), therefore, we shall unequivocally define c_1, c_2 and c_3. They belong $H_{3/2}$, as $\varphi \in H_{3/2}$. It is obvious, that

$$\|u_0(t)\|_{W_2^2(R_+;H)} \le const \|\varphi\|_{3/2}.$$

Now we consider the boundary problem (1),(2). For this purpose we take

$$u(t) = \vartheta(t) + u_0(t).$$

Then we get the following equation

$$-\frac{d^2\vartheta(t)}{dt^2} + \rho(t) A^2 \vartheta(t) + A_0 \frac{d^2\vartheta(t)}{dt^2} + A_1 \frac{d\vartheta(t)}{dt} + A_2 \vartheta(t) -$$

$$-\frac{d^2 u_0(t)}{dt^2} + \rho(t) A^2 u_0(t) + A_0 \frac{d^2 u_0(t)}{dt^2} + A_1 \frac{du_0(t)}{dt} + A_2 u_0(t) = 0,$$

$$\vartheta(0) = 0.$$

As

$$-\frac{d^2 u_0(t)}{dt^2} + \rho(t) A^2 u_0(t) = 0 ,$$

$$-\frac{d^2\vartheta(t)}{dt^2} + \rho(t) A^2 \vartheta(t) + A_0 \frac{d^2\vartheta(t)}{dt^2} + A_1 \frac{d\vartheta(t)}{dt} + A_2 \vartheta(t) = g(t) \quad ,$$

$$\vartheta(0) = 0 \quad ,$$

where

$$g(t) = -A_0 \frac{d^2 u_0(t)}{dt^2} - A_1 \frac{du_0(t)}{dt} - A_2 u_0(t).$$

As operators $A_j A^{-j}\ (j=\overline{0,2})$ are bounded

$$\|g(t)\|_{L_2(R_+;H)} \le \|A_0\| \left\| \frac{d^2 u_0}{dt^2} \right\|_{L_2(R_+;H)} +$$

$$+\|A_1 A^{-1}\| \left\| A \frac{du_0}{dt} \right\|_{L_2(R_+;H)} + \|A_2 A^{-2}\| \|A^2 u_0\|_{L_2(R_+;H)}.$$

Applying , the theorem of intermediate derivatives, [1] we have

$$\|g(t)\|_{L_2(R_+;H)} \le const \|u_0\|_{W_2^2(R_+;H)} \le const \|\varphi\|_{3/2}.$$

Thus, we have reduced a boundary problem (1),(2) to the non homogeneous boundary problem with a zero boundary condition. Thus following theorem is valid.

Theorem 1. Let the operator A satisfies the condition 1), but operators $B_j = A_j A^{-j}\ (j=0,1)$ satisfy condition 2) , moreover

where numbers $c_j(\varepsilon;\alpha;\beta)$ are defined as

$$c_0(\varepsilon;\alpha;\beta) = \frac{1}{\min(\alpha^2;\beta^2)} \begin{cases} 1, & 0 \le \varepsilon < \pi/4, \\ \dfrac{1}{\sqrt{2}\cos\varepsilon}, & \pi/4 \le \varepsilon < \pi/2. \end{cases}$$

$$c_1(\varepsilon;\alpha;\beta) = \frac{1}{2\cos\varepsilon \min(\alpha;\beta)}, \quad 0 \le \varepsilon < \pi/2,$$

$$c_2(\varepsilon;\alpha;\beta) = \frac{\max(\alpha;\beta)}{\min(\alpha^2;\beta^2)} \begin{cases} 1, & 0 \le \varepsilon \le \pi/4, \\ \dfrac{1}{\sqrt{2}\cos\varepsilon}, & \pi/4 \le \varepsilon < \pi/2. \end{cases}$$

Then boundary problem (1), (2) is regularly solvable.

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3 MAIN RESULT

Now we shall study one property of homogeneous regular solutions. Let numbers a, a_1, b_1, b be such, that

$$0 < a < a_1 < b_1 < b < \infty.$$

Denote by $N(P)$ the space of regular solutions of the boundary problem (1), (2). It is obvious, that $N(P)$ - linear full subspace in $W_2^2(R_+;H)$. Really, if $u_n(t) \to u(t)$ in $W_2^2(R_+;H)$ and $P(d/dt)u_n(t) = 0$ $(u_n(t) \in N(P))$,

$$\|P(d/dt)(u(t) - u_n(t))\| \le const\|u(t) - u_n(t)\| \to 0.$$

Then $\|P(d/dt)u(t)\| = 0,$ i.e. $P(d/dt)u(t) = 0,$ hence $u(t) \in N(P)$.

It is obvious, that $N(P) \subset W_2^1(R_+;H)$.

Definition 2. If a, a_1, b_1, b are such, as $0 < a < a_1 < b_1 < b < \infty,$ $M > 0$, the set $\left\{u \mid u \in N(P), \|u\|_{W_2^1((a,b);H)} \le M\right\}$ is compact on norm $\|u\|_{W_2^1((a_1,b_1);H)}$ we say speak, that space of regular solutions of the problem (1), (2) is internally compact.

We note, that definition of internal compactness for the first time has entered P.D.Laks [6]. At different situations of interior compactness of solutions the considered works [7, 8]. Following P.D.Laks's [6] work, we have entered concept of interior compactness of the solutions of the homogeneous equations.

Theorem 2. A condition of the theorem 1 let satisfied. Then the space of regular solutions of a problem (1), (2) is internally compact.

Proof. Let $0 < a < a_1 < b_1 < b < \infty$ and scalar function $\varphi(t) \in C_0^\infty(a,b)$, is such, that

$$\varphi(t) = \begin{cases} 1, & t \in (a_1, b_1), \\ 0, & t \ge b, t \le a. \end{cases}$$

Then it is obvious, that for vector functions $\varphi(t)u(t) \in \overset{o}{W}{}_2^2(R_+;H;0)$ and $u(t) = 0$ at $t \le a$ and $t \ge b$.

As we have proved, that at performance of conditions of the theorem the following inequality takes place:

$$\|P(d/dt)\varphi u\|_{L_2(R_+;H)} \ge const\|\varphi u\|_{W_2^2(R_+;H)}$$

For all $u \in \overset{o}{W}{}_2^2(R_+;H;0)$ (see [4], the theorem 2).

From here we have:

$$\left\| -\frac{d^2\varphi(t)u(t)}{dt^2} + \rho(t)\varphi(t)A^2u(t) + A_0\frac{d^2\varphi(t)u(t)}{dt^2} + A_1\frac{d\varphi(t)u(t)}{dt} + A_2\varphi(t)u(t) \right\|_{L_2(R_+;H)} \ge$$

$$\ge const\|\varphi u\|_{W_2^2(R_+;H)}.$$

As

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$$P(d/dt)\varphi(t)u(t) = -\frac{d^2\varphi(t)u(t)}{dt^2} + \rho(t)\varphi(t)A^2u(t) + A_0\frac{d^2\varphi(t)u(t)}{dt^2} +$$

$$+A_1\frac{d\varphi(t)u(t)}{dt} + A_2\varphi(t)u(t) = -\frac{d^2\varphi(t)}{dt^2}u(t) - 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt} - \varphi(t)\frac{d^2u(t)}{dt^2} +$$

$$+\rho(t)\varphi(t)A^2u(t) + A_0\left(\frac{d^2\varphi(t)}{dt^2}u(t) + 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt} + \frac{d^2u(t)}{dt^2}\varphi(t)\right) +$$

$$+A_1\left(\frac{d\varphi(t)}{dt}u(t) + \frac{du(t)}{dt}\varphi(t)\right) + A_2\varphi(t)u(t) =$$

$$=\varphi(t)\left(-\frac{d^2u(t)}{dt^2} + \rho(t)A^2u(t) + A_0\frac{d^2u(t)}{dt^2} + A_1\frac{du(t)}{dt} + A_2u(t)\right) +$$

$$+\left(-\frac{d^2\varphi(t)}{dt^2}u(t) - 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt} + A_0\left(\frac{d^2\varphi(t)}{dt^2}u(t) + 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt}\right) + A_1\frac{d\varphi(t)}{dt}u(t)\right).$$

As $\varphi(t) = 0$ at $t \geq b$, $t \leq a$ and $u(t)$ - the regular decision,

$$P(d/dt)u(t) = -\frac{d^2u(t)}{dt^2} + \rho(t)A^2u(t) + A_0\frac{d^2u(t)}{dt^2} + A_1\frac{du(t)}{dt} + A_2u(t) = 0$$

and

$$\|P(d/dt)u(t)\|_{L_2(R_+;H)} = \left\| -\frac{d^2\varphi(t)}{dt^2}u(t) - 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt} + \right.$$

$$\left. + A_0\left(\frac{d^2\varphi(t)}{dt^2}u(t) + 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt}\right) + A_1\frac{d\varphi(t)}{dt}u(t)\right\|_{L_2(R_+;H)} \geq$$

$$\geq const\|\varphi u\|_{W_2^2(R_+;H)} = const\|\varphi u\|_{W_2^2((a;b);H)} \geq$$

$$\geq const\|\varphi u\|_{W_2^2((a_1;b_1);H)} = const\|u\|_{W_2^2((a_1;b_1);H)}.$$

Thus,

$$\|u\|_{W_2^2((a_1;b_1);H)} \le \Bigg\| -\frac{d^2\varphi(t)}{dt^2}u(t) - 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt} +$$

$$+ A_0\left(\frac{d^2\varphi(t)}{dt^2}u(t) + 2\frac{d\varphi(t)}{dt}\frac{du(t)}{dt}\right) + A_1\frac{d\varphi(t)}{dt}u(t)\Bigg\|_{L_2(R_+;H)} \le$$

$$\le const\|u(t)\|_{L_2((a;b);H)} + const\left\|\frac{du(t)}{dt}\right\|_{L_2((a;b);H)} + const\|A_0u(t)\|_{L_2((a;b);H)} +$$

$$+ const\left\|A_0\frac{du(t)}{dt}\right\|_{L_2((a;b);H)} + const\|A_1u(t)\|_{L_2((a;b);H)} \le$$

$$\le const\Bigg(\|u\|_{L_2((a;b);H)} + \left\|\frac{du(t)}{dt}\right\|_{L_2((a;b);H)} + \|A_0\|\|u\|_{L_2((a;b);H)} +$$

$$+ \|A_0\|\left\|\frac{du(t)}{dt}\right\|_{L_2^2((a;b);H)} + \|A_1A^{-1}\|\|Au\|_{L_2((a;b);H)}\Bigg) \le const\|u\|_{W_2^1((a;b);H)} \le M.$$

Hence, for anyone $0 < a < a_1 < b_1 < b$ we have:

$$\|u\|_{W_2^2((a_1;b_1);H)} \le const\|u\|_{W_2^1((a;b);H)} \le M.$$

As $u \in N(P)$, the set $\left\{u \mid \|u\|_{W_2^2((a_1;b_1);H)}, u \in N(P)\right\}$ is limited. From quite continuity of the operator A^{-1} follows, that the space $W_2^2((a_1,b_1);H)$ is enclosed in space $W_2^1((a_1,b_1);H)$ compactly, i.e.

$$W_2^2((a_1,b_1);H) \subset W_2^1((a_1,b_1);H)$$

compactly. Hence, $N(P)$ - compact set of century $W_2^1((a_1,b_1);H)$. Thus, we have proved internal compactness of decisions of a problem (1), (2). The theorem is proved.

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