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ON A REGULAR SOLVABILITY OF ONE BOUNDARY-VALUE PROBLEM FOR THE SECOND ORDER OPERATOR -DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper the solvability conditions of one class of elliptic type second order operator-differential equations. In this class the principal part of the operator whose spectrum is in some sector, is a normal operator. The obtained conditions are expressed by the properties of the coefficients.
Keywords: Normal operator, discontinuous coefficients, regular solvability, Hilbert space.

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1. INTRODUCTION

In the paper we study a regular solvability of a boundary-value problem for the second order operator-differential equation.

Studying of the operational-differential equations with operators in boundary conditions represents mathematical interest not only by virtue of that they in themselves contain corresponding boundary problems in a case when coefficients in boundary conditions only complex numbers but also for the reason that such problems are applicable to more wide field of problems for the differential equations in partial derivatives. Among these problems we note, for example, not local problems.

Namely, in a separable Hilbert space H we consider the boundary-value problem

$$Pu = -\frac{d^2u}{dt^2} + \rho(t)A^2u + A_0\frac{d^2u}{dt^2} + A_1\frac{du}{dt} + A_2u = f(t), \quad t \in R_+, \tag{1}$$

$$u'(0) = 0, \tag{2}$$

where $f(t) \in L_2(R_+;H), \quad u(t) \in W_2^2(R_+;H)$.

Denote by $L_2(R_+;H)$ a Hilbert space of the vector-functions $f(t)$ with values from H, measurable and integrable by square-law with norm

$$\|f\|_{L_2} = \left(\int_0^{+\infty} \|f(t)\|^2 dt\right)^{1/2}.$$

Further we introduce space e.g. [1]

$$W_2^2(R_+;H)=\left\{u(t):\ u'',A^2u\in L_2(R;H)\ ,\|u\|_{W_2^2}=\left(\|u''\|_{L_2}^2+\|A^2u\|_{L_2,}^2\right)^{1/2}\right\}.$$

The sub-space $\overset{o}{W}{}_2^2(R_+;H;1)$ is a sub-space of the Hilbert space $W_2^2(R_+;H)$, moreover $\overset{o}{W}{}_2^2(R_+;H;1)=\{u\ |u\in W_2^2(R_+;H),u'(0)=0\}$, the member function $\rho(t)=\alpha^2$ for $t\in(0,1)$ and $\rho(t)=\beta^2$ for $t\in(1,\infty)$, $\alpha>0$, $\beta>0$, but operator coefficients A and A_j $(j=0,1,2)$ satisfy the following conditions:

i) A - is normal invertible operator spectrum of which is contained in a corner sector,

$$S_\varepsilon=\{\lambda:|\arg\lambda|\le\varepsilon\}\ ,\ 0\le\varepsilon<\pi/2;$$

ii) Operators $B_j=A_jA^{-j}$ $(j=0,1,2,)$ are bounded in H.

Definition 1. If the vector-function $u(t)\in W_2^2(R_+;H)$ satisfies equation (1) almost everywhere in $R_+=(0,\infty)$ and boundary condition (2) in the sense of

$$\lim_{t\to+0}\|u'(t)\|_{1/2}=0,$$

then it said to be a regular solution of problem (1),(2).

Definition 2. If for any $f(t)\in L_2(R_+;H)$ there exists a regular solution of problem (1), (2) and it holds the inequality

$$\|u\|_{W_2^2(R_+;H)}\le \text{const}\|f\|_{L_2(R_+;H)}\ ,$$

then problem (1),(2) is said to be regular solvable.

In the present paper the sufficient conditions of regular resolvability of boundary problems (1), (2) expressed only by the operational coefficients are obtained. Corresponding questions for the boundary problems (1), (2) in a case of boundary condition $u(0)=0$ are investigated in work [3]-[4]. We shall note that for $\alpha=\beta=1$ this problem is investigated in work [6], for $A=A^{*}\geq cE,\ c>0,\quad \alpha\neq\beta$ is investigated in [2]. The solvability of the equation (1) on the whole axis is considered in [5].

Obviously that we can write the boundary-value problem in the from of operator equation

$$Pu\equiv P_0u+P_1u=f\ , \tag{3}$$

where, $f\in L_2(R_+;H)$, $u\in \overset{o}{W}{}_2^2(R_+;H;1)$,

$$P_0u=-\frac{d^2u}{dt^2}+\rho(t)A^2u,\qquad u\in \overset{o}{W}{}_2^2(R_+;H;1) \tag{4}$$

and

$$P_1u=A_0\frac{d^2u}{dt^2}+A_1\frac{du}{dt}+A_2u,\qquad u\in \overset{o}{W}{}_2^2(R_+;H;1)\ . \tag{5}$$

First we investigate solvability of the equation

$$P_0u=f\ , \tag{6}$$

where, $f\in L_2(R_+;H)$, $u\in \overset{o}{W}{}_2^2(R_+;H;1)$.

We proof next theorem.

Theorem 1. Let the operator A satisfies the condition i). Then operator P_0 isomorphically maps the space $W_2^2(R_+;H)$ on space $L_2(R_+;H)$.

Proof. First of all let's show that equation (6) has a solution for any $f \in L_2(R_+;H)$ from the space $\overset{o}{W}{}_2^2(R_+;H;1)$.

It is easy to check

$$u_1(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\zeta^2 E+\alpha^2 A^2\right)^{-1}\left(\int_0^{+\infty} f(s)e^{i(t-s)\zeta}ds\right)d\zeta$$

and

$$u_2(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\zeta^2 E+\beta^2 A^2\right)^{-1}\left(\int_0^{+\infty} f(s)e^{i(t-s)\zeta}ds\right)d\zeta$$

satisfy the equations

$$-\frac{d^2u}{dt^2}+\alpha^2A^2u=f(t) \qquad \text{and} \qquad -\frac{d^2u}{dt^2}+\beta^2A^2u=f(t)$$

correspondingly in R and belong to the space $W_2^2(R_+;H)$.We shall denote by $\psi_1(t)$ and $\psi_2(t)$ their narrowing in $(0;1)$ and $(1;\infty)$ accordingly. Then obviously that, $\psi_1(t)\in W_2^2((0;1);H)$ and $\psi_2(t)\in W_2^2((1;\infty);H)$.

Applying the trace theorem we get that there exist boundary values $\psi_1(0)$, $\psi_1'(0)$, $\psi_1(1),\psi_1'(1)$, $\psi_2(1),\psi_2'(1)$, moreover $\psi_1(0)\in H_{3/2}$, $\psi_1(1)\in H_{3/2}$, $\psi_2(1)\in H_{3/2}$ and $\psi_1'(0)\in H_{1/2}, \psi_1'(1)\in H_{1/2}$, $\psi_2'(1)\in H_{1/2}$.

Let's construct the vector –function

$$u(t)=\begin{cases}\theta_1(t)=\psi_1(t)+e^{-\alpha tA}\varphi_1+e^{-\alpha(1-t)A}\varphi_2\ , & t\in(0;1),\\ \theta_2(t)=\psi_2(t)+e^{\beta A(1-t)}\varphi_3\ , & t\in(1;\infty),\end{cases}$$

where e^{-At} is a semi group of bounded operators generated by operator $(-A)$, vectors $\varphi_1, \varphi_2, \varphi_3$ are unknown and $\varphi_1, \varphi_2, \varphi_3 \in H_{3/2}$.

Then obviously that

$$e^{-\alpha At}\varphi_1 + e^{-\alpha(1-t)A}\varphi_2 \in W_2^2((0;1);H), \qquad e^{-\beta A(t-1)}\varphi_3 \in W_2^2((1,\infty);H).$$

To belong to the space $\overset{o}{W}{}_2^2(R_+;H;1)$, $u(t)$ must satisfy the condition:

$$\begin{cases} u'(0) = 0, \\ \theta_1(1) = \theta_2(1), \\ \theta_1'(1) = \theta_2'(1). \end{cases}$$

Hence it follows:

$$\begin{cases} \psi_1'(0) - \alpha A\varphi_1 + \alpha A e^{-\alpha A}\varphi_2 = 0, \\ \psi_1(1) + e^{-\alpha A}\varphi_1 + \varphi_2 = \psi_2(1) + \varphi_3, \\ \psi'_1(1) - \alpha A e^{-\alpha A}\varphi_1 + \alpha A\varphi_2 = \psi'_2(1) - \beta A\varphi_3. \end{cases}$$

Hence comparatively φ_1, φ_2 and φ_3 we obtain the following system of the equations

$$\begin{cases} \alpha A\varphi_1 - \alpha A e^{-\alpha A}\varphi_2 + 0\cdot\varphi_3 = \psi_1'(0), \\ e^{-\alpha A}\varphi_1 + \varphi_2 - \varphi_3 = \psi_2(1) - \psi_1(1), \\ -\alpha e^{-\alpha A}\varphi_1 + \alpha\varphi_2 + \beta\varphi_3 = A^{-1}(\psi_2'(1) - \psi_1'(1)). \end{cases}$$

This system has the following main determinant of matrix:

$$\Delta_0(A)= \begin{pmatrix} \alpha A & -\alpha A e^{-\alpha A} & 0 \\ e^{-\alpha A} & E & -E \\ -\alpha e^{-\alpha A} & \alpha E & \beta E \end{pmatrix}.$$

Now we'll show that $\Delta_0(A)$ is an invertible in the space $H^3 = H \times H \times H$.

Let's denote by

$$\Delta_0(\sigma)= \begin{pmatrix} \alpha\sigma & -\alpha\sigma e^{-\alpha\sigma} & 0 \\ e^{-\alpha\sigma} & 1 & -1 \\ -\alpha\, e^{-\alpha\sigma} & \alpha & \beta \end{pmatrix},$$

where, $\sigma \in S_\varepsilon = \{\lambda : |\arg\lambda| \le \varepsilon\}$, $0 \le \varepsilon < \pi/2$, $\operatorname{Re}\lambda \ge \mu_0 > 0$.

Obviously that for $\sigma \to \infty$ $(\sigma \in S_\varepsilon$, $\operatorname{Re}\sigma \ge \mu_0 > 0)$

$$\Delta_0(\sigma)= \begin{pmatrix} \alpha\sigma & 0 & 0 \\ 0 & 1 & -1 \\ 0 & \alpha & \beta \end{pmatrix} + o(\sigma),\ \sigma \to \infty .$$

Then for $\sigma \to \infty$

$$|\det \Delta_0(\sigma)| = \left|\alpha\sigma \det\begin{pmatrix} 1 & -1 \\ \alpha & \beta \end{pmatrix}\right| + o(\sigma) = \alpha\sigma|\alpha+\beta| + o(\sigma),$$

i.e. for $\sigma \to \infty$ $\Delta_0(\sigma)$ is invertible $\left(|\det \Delta_0(\sigma)| \ne 0\right)$.

We shall show that for $\sigma \in S_\varepsilon$ and $\operatorname{Re}\sigma \ge \mu_0 > 0)$ we also have $|\det \Delta_0(\sigma)| \ne 0$. As

$$\det \Delta_0(\sigma) = \begin{vmatrix} \alpha\sigma & -\alpha\sigma\, e^{-\alpha\sigma} & 0 \\ e^{-\alpha\sigma} & 1 & -1 \\ -\alpha\ e^{-\alpha\sigma} & \alpha & \beta \end{vmatrix} = \alpha\sigma \begin{vmatrix} 1 & -1 \\ \alpha & \beta \end{vmatrix} + \alpha\sigma e^{-\alpha\sigma} \begin{vmatrix} e^{-\alpha\sigma} & -1 \\ -\alpha\, e^{-\alpha\sigma} & \beta \end{vmatrix} =$$

$$= \alpha\sigma(\alpha+\beta) - \alpha\sigma e^{-2\alpha\sigma}(\alpha-\beta) = \alpha\sigma\left[(\alpha+\beta) - e^{-2\alpha\sigma}(\alpha-\beta)\right].$$

Then

$$\operatorname{Re}\left(\det \Delta_0(\sigma)\right) = \alpha\sigma\left[(\alpha+\beta) - e^{-2\alpha \operatorname{Re}\sigma}(\alpha-\beta)\right] \ge \alpha\mu_0\left[(\alpha+\beta) - e^{-2\alpha\mu_0}(\alpha-\beta)\right] > const > 0,$$

where $\sigma \in S_\varepsilon$, $\operatorname{Re}\sigma \ge \mu_0$.

Thereby $\Delta_0(\sigma)$ is invertible for all $\sigma \in S_\varepsilon$. If we shall define the inverse matrix, then

$$\Delta_0^{-1}(\sigma) = \begin{pmatrix} a_{11}(\sigma) & a_{12}(\sigma) & a_{13}(\sigma) \\ a_{21}(\sigma) & a_{22}(\sigma) & a_{23}(\sigma) \\ a_{31}(\sigma) & a_{32}(\sigma) & a_{33}(\sigma) \end{pmatrix} \frac{1}{\det \Delta_0(\sigma)},$$

where $a_{ij}(\sigma)$ are minor matrices corresponding to element in j th row and i th column . Since each element $\Delta_0(\sigma)$ is bounded in S_ε but $\left|\det \Delta_0(\sigma)\right| \ge const$ then considering that $\sigma \in S_\varepsilon \cap \sigma(A)$ from spectral theory normal operators $\Delta_0^{-1}(\sigma)$ we get that operator exists and is bounded in H^3, since

$$\Delta_0^{-1}(A) = \begin{pmatrix} a_{11}(A)(\det \Delta_0(A))^{-1} & a_{12}(A)(\det \Delta_0(A))^{-1} & a_{13}(A)(\det \Delta_0(A))^{-1} \\ a_{21}(A)(\det \Delta_0(A))^{-1} & a_{22}(A)(\det \Delta_0(A))^{-1} & a_{23}(A)(\det \Delta_0(A))^{-1} \\ a_{31}(A)(\det \Delta_0(A))^{-1} & a_{32}(A)(\det \Delta_0(A))^{-1} & a_{33}(A)(\det \Delta_0(A))^{-1} \end{pmatrix}.$$

Here $\left(\det \Delta_0(A)\right)^{-1} = \left(\alpha A\left[(\alpha+\beta) - e^{-2\alpha A}(\alpha-\beta)\right]\right)^{-1}$.

Thereby we can define the vectors φ_1, φ_2 , φ_3. As it can be seen from expression $\Delta_0^{-1}(A)$, for determination φ_j $(j=1,2,3)$ in j^{th} row of $\Delta_0^{-1}(\sigma)$ we shall write the vectors $-\psi_1(0)\in H_{3/2}, (\psi_2(1)-\psi_1(1))\in H_{3/2}$, but $A^{-1}(\psi_2'(1)-\psi_1'(1))\in H_{3/2}$ and formally we'll find the determinant, then we'll change σ by the operator A and shall get the expressions for ψ_j.

Is it clear that φ_j will be a linear combination of the vectors $-\psi_1(0)\in H_{3/2}$, $(\psi_2(1)-\psi_1(1))\in H_{3/2}$ and $A^{-1}(\psi_2'(1)-\psi_1'(1))\in H_{3/2}$ after acting by boundary limited operators. So $\varphi_j \in H_{3/2}$ $(j=1,2,3)$. Thereby $u \in \overset{o}{W}{}_2^2(R_+;H;1)$.

Now let's show that equations $P_0 u = 0$ have only zero solution from $\overset{o}{W}{}_2^2(R_+;H;1)$. Really this equation is amounting to with boundary problem

$$-\frac{d^2u}{dt^2}+\rho(t)A^2u=0 \quad , \tag{7}$$

$$u'(0)=0 \quad , \tag{8}$$

where, $u(t)\in W_2^2(R_+;H)$. General solution of the equation (7) from $W_2^2(R_+;H)$ is presented in the from

$$u_0(t)=\begin{cases} e^{-\alpha tA}\varphi_1+e^{-\alpha(1-t)A}\varphi_2 \ , & t\in(0;1) \ , \\ e^{\beta A(1-t)}\varphi_3 \ , & t\in(1;\infty), \end{cases}$$

where, $\varphi_1, \varphi_2, \varphi_3 \in H_{3/2}$. From condition (9) it follows that

$$-\alpha A\varphi_1+\alpha Ae^{-\alpha A}\varphi_2=0 \quad . \tag{9}$$

From condition $u(t) \in W_2^2(R_+;H)$ it follows that $u(1+0) = u(1-0),\ u'(1+0) = u'(1-0)$, i.e.

$$\begin{cases} e^{-\alpha A}\varphi_1 + \varphi_2 = \varphi_3 \ , \\ -\alpha A e^{-\alpha A}\varphi_1 + \alpha A\varphi_2 = -\beta A\varphi_3 \ . \end{cases} \tag{10}$$

From (9) and (10) we get the following system of equations :

$$\begin{cases} -\alpha A\varphi_1 + \alpha A e^{-\alpha A}\varphi_2 + 0\cdot\varphi_3 = 0 \ , \\ e^{-\alpha A}\varphi_1 + \varphi_2 - \varphi_3 = 0 \ , \\ -\alpha e^{-\alpha A}\varphi_1 + \alpha\varphi_2 + \beta\varphi_3 = 0 \ . \end{cases}$$

Since main determinant of matrix

$$\Delta_0(A) = \begin{bmatrix} -\alpha A & \alpha A e^{-\alpha A} & 0 \\ e^{-\alpha A} & E & -E \\ -\alpha e^{-\alpha A} & \alpha E & \beta E \end{bmatrix}$$

is invertible in the space H^3, then $\tilde{\varphi} = (\varphi_1, \varphi_2, \varphi_3) = 0$, consequently, $u_0(t) = 0$.

On the other hand for $u \in \overset{o}{W}{}_2^2(R_+;H;1)$

$$\|P_0 u\|_{L_2}^2 = \left\| -\frac{d^2u}{dt^2} + \rho(t)A^2u \right\|_{L_2}^2 \le 2\left(\left\| \frac{d^2u}{dt^2} \right\|_{L_2}^2 + \|\rho(t)A^2u\|_{L_2}^2 \right) \le$$

$$\le const\left(\left\| \frac{d^2u}{dt^2} \right\|_{L_2}^2 + \|A^2u\|_{L_2}^2 \right) \le const\|u\|_{W_2^2}^2 \ .$$

Then using Banach theorem on inverse operator, we get that operator $P_0^{-1} : L_2(R_+;H) \to \overset{o}{W}{}_2^2(R_+;H;0)$ bounded by operator P_0 is an isomorphism.

The theorem is proved.

To prove the main theorem first of all let's prove the following lemma.

Lemma 1. Let the condition i) be satisfied. Then the inequality exists for any $u \in \overset{o}{W}{}_2^2(R_+;H;1)$:

$$\left\|\rho^{-1/2}P_0u\right\|_{L_2}^2 \geq \left\|\rho^{1/2}A^2u\right\|_{L_2}^2 + \left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|_{L_2}^2 + 2\cos 2\varepsilon\left\|A\frac{du}{dt}\right\|_{L_2}^2 . \tag{11}$$

Proof. Assume that

$$P_0u = -\frac{d^2u}{dt^2} + \rho(t)A^2u, \qquad u \in \overset{o}{W}{}_2^2(R_+;H;1) \qquad (u'(0)=0) \quad .$$

Multiplying equation $P_0u = f$ by function $\rho^{-1/2}(t)$, we get:

$$\left\|\rho^{-1/2}P_0u\right\|_{L_2}^2 = \left\|-\rho^{-1/2}\frac{d^2u}{dt^2} + \rho^{1/2}A^2u\right\|_{L_2}^2 = \left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|_{L_2}^2 + \left\|\rho^{1/2}A^2u\right\|_{L_2}^2 -$$
$$-2\operatorname{Re}\left(\rho^{-1/2}\frac{d^2u}{dt^2}, \rho^{1/2}A^2u\right)_{L_2} = \left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|_{L_2}^2 + \left\|\rho^{1/2}A^2u\right\|_{L_2}^2 - 2\operatorname{Re}\left(\frac{d^2u}{dt^2}, A^2u\right)_{L_2} . \tag{12}$$

Since, for $u \in \overset{o}{W}{}_2^2(R_+;H;1)$ $u'(0)=0$ therefore, after integration by parts we have:

$$-2\operatorname{Re}\left(\frac{d^2u}{dt^2}, A^2u\right)_{L_2} = 2\operatorname{Re}\left(A^*\frac{du}{dt}, A\frac{du}{dt}\right)_{L_2} \geq 2\cos 2\varepsilon\left(A\frac{du}{dt}, A\frac{du}{dt}\right)_{L_2} = 2\cos 2\varepsilon\left\|A\frac{du}{dt}\right\|_{L_2}^2 .$$

Allowing for this inequality in (12), we get:

$$\left\|\rho^{-1/2}P_0u\right\|_{L_2}^2 \geq \left\|\rho^{1/2}A^2u\right\|_{L_2}^2 + \left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|_{L_2}^2 + 2\cos 2\varepsilon\left\|A\frac{du}{dt}\right\|_{L_2}^2 .$$

Lemma is proved.

Lemma 2. Let condition i) be satisfied executed. Then for any $u \in \overset{o}{W}{}_2^2(R_+;H;1)$ the following estimations are true:

$$\left\| A^2 u \right\|_{L_2(R_+;H)} \le c_0(\varepsilon;\alpha;\beta) \left\| P_0 u \right\|_{L_2(R_+;H)}, \tag{13}$$

$$\left\| A \frac{du}{dt} \right\|_{L_2(R_+;H)} \le c_1(\varepsilon;\alpha;\beta) \left\| P_0 u \right\|_{L_2(R_+;H)}, \tag{14}$$

$$\left\| \frac{d^2 u}{dt^2} \right\|_{L_2(R_+;H)} \le c_2(\varepsilon;\alpha;\beta) \left\| P_0 u \right\|_{L_2(R_+;H)}, \tag{15}$$

where

$$c_0(\varepsilon;\alpha;\beta) = \frac{1}{\min(\alpha^2;\beta^2)} \begin{cases} 1 \quad , & 0 \le \varepsilon \le \pi/4 \ , \\ \dfrac{1}{\sqrt{2}\cos\varepsilon}, & \pi/4 \le \varepsilon < \pi/2 \ , \end{cases} \tag{16}$$

$$c_1(\varepsilon;\alpha;\beta) = \frac{1}{2\cos\varepsilon \, \min(\alpha;\beta)}, \qquad 0 \le \varepsilon < \pi/2, \tag{17}$$

$$c_2(\varepsilon;\alpha;\beta) = \frac{\max(\alpha;\beta)}{\min(\alpha;\beta)} \begin{cases} 1 \quad , & 0 \le \varepsilon \le \pi/4 \ , \\ \dfrac{1}{\sqrt{2}\cos\varepsilon}, & \pi/4 \le \varepsilon < \pi/2 . \end{cases} \tag{18}$$

Proof. Since for $u \in \overset{o}{W}{}_2^2(R_+;H;1)$, $u'(0) = 0$ and as A is a normal operator,

$$\left\| A\frac{du}{dt} \right\|_{L_2}^2 = \left\| C\frac{du}{dt} \right\|_{L_2}^2 \quad . \tag{19}$$

Then after integration on a parts we obtain :

$$\left\| C\frac{du}{dt} \right\|_{L_2}^2 = \int_0^{+\infty} \left(C\frac{du}{dt}, C\frac{du}{dt} \right) dt = -\int_0^{+\infty} \left(C^2 u, \frac{d^2 u}{dt^2} \right) dt =$$

$$= -\int_0^{+\infty} \left(\rho^{1/2} C^2 u, \rho^{-1/2} \frac{d^2 u}{dt^2} \right) dt = -\left(\rho^{1/2} C^2 u, \rho^{-1/2} \frac{d^2 u}{dt^2} \right)_{L_2} \quad .$$

We apply the Cauchy-Bunyakowskii inequality and have:

$$\left\| A\frac{du}{dt} \right\|_{L_2}^2 \le \frac{1}{2} \left(\left\| \rho^{1/2} A^2 u \right\|_{L_2}^2 + \left\| \rho^{-1/2} \frac{d^2 u}{dt^2} \right\|_{L_2}^2 \right) \quad . \tag{20}$$

But it follows from inequality (11) (lemma 1) that:

$$\left\| \rho^{-1/2} A^2 u \right\|_{L_2}^2 + \left\| \rho^{-1/2} \frac{d^2 u}{dt^2} \right\|_{L_2}^2 \le \left\| \rho^{-1/2} P_0 u \right\|_{L_2}^2 - 2\cos 2\varepsilon \left\| A\frac{du}{dt} \right\|_{L_2}^2 .$$

Considering this inequality in (20), we get:

$$\left\| A\frac{du}{dt} \right\|_{L_2}^2 \le \frac{1}{2} \cdot \left(\left\| \rho^{-1/2} P_0 u \right\|_{L_2}^2 - 2\cos 2\varepsilon \left\| A\frac{du}{dt} \right\|_{L_2}^2 \right) = \frac{1}{2} \left\| \rho^{-1/2} P_0 u \right\|_{L_2}^2 - \cos 2\varepsilon \left\| A\frac{du}{dt} \right\|_{L_2}^2 .$$

Hence it follows for $u \in \overset{o}{W}{}_2^2(R_+; H; 1)$

$$(1 + \cos 2\varepsilon) \left\| A\frac{du}{dt} \right\|_{L_2}^2 \le \frac{1}{2} \cdot \left\| \rho^{-1/2} P_0 u \right\|_{L_2}^2 \quad ,$$

i.e.

$$\left\| A\frac{du}{dt} \right\|_{L_2}^2 \le \frac{1}{4\cos^2\varepsilon} \cdot \left\| \rho^{-1/2} P_0 u \right\|_{L_2}^2 \le \frac{1}{4\cos^2\varepsilon} \cdot \max_t \rho^{-1} \| P_0 u \|_{L_2}^2 = \frac{1}{4\cos^2\varepsilon} \cdot \frac{1}{\min\left(\alpha^2;\beta^2\right)} \cdot \| P_0 u \|_{L_2}^2 \quad .$$

Consequently

$$\left\| A\frac{du}{dt} \right\|_{L_2} \le \frac{1}{2\cos\varepsilon} \cdot \frac{1}{\min(\alpha;\beta)} \| P_0 u \|_{L_2} \ .$$

The inequality (14) is proved.

Let's prove inequality (13). We first consider the case $0 \le \varepsilon \le \pi/4$, i.e. $\cos 2\varepsilon \ge 0$. Therefore in this case from (11) we get

$$\left\| \rho^{1/2} A^2 u \right\|_{L_2}^2 + \left\| \rho^{-1/2} \frac{d^2 u}{dt^2} \right\|_{L_2}^2 \le \left\| \rho^{-1/2} P_0 u \right\|_{L_2}^2 \ .$$

Hence it follows :

$$\left\| \rho^{1/2} A^2 u \right\|_{L_2}^2 \le \left\| \rho^{-1/2} P_0 u \right\|_{L_2}^2 \le \max_t \rho^{-1} \| P_0 u \|_{L_2}^2 \ .$$

On the other hand:

$$\left\| A^2 u \right\|_{L_2}^2 = \left\| \rho^{-1/2} \rho^{1/2} A^2 u \right\|_{L_2}^2 \le \max_t \rho^{-1}(t) \left\| \rho^{1/2} A^2 u \right\|_{L_2}^2 \le$$

$$\le \left(\max_t \rho^{-1}(t) \right)\left(\max_t \rho^{-1}(t) \right) \| P_0 u \|_{L_2}^2 = \max_t \rho^{-2}(t) \| P_0 u \|_{L_2}^2 = \frac{1}{\min(\alpha^4;\beta^4)} \| P_0 u \|_{L_2}^2 ,$$

i.e. for $0 \le \varepsilon \le \pi/4$ we have

$$\left\|A^2u\right\|_{L_2} \le \frac{1}{\min(\alpha^2;\beta^2)}\left\|P_0u\right\|_{L_2}. \tag{21}$$

But for $\pi/4 \le \varepsilon < \pi/2$ we have that $\cos 2\varepsilon \le 0$. From inequality (11) we obtain:

$$\max \rho^{-1}(t)\left\|P_0u\right\|_{L_2}^2 \ge \left\|\rho^{1/2}A^2u\right\|_{L_2}^2 + \left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|_{L_2}^2 + 2\cos 2\varepsilon \frac{1}{4\cos^2\varepsilon \min(\alpha^2;\beta^2)}\left\|P_0u\right\|_{L_2}^2.$$

Then

$$\left\|\rho^{1/2}A^2u\right\|_{L_2}^2 \le \frac{1}{\min(\alpha^2;\beta^2)}\left(1-\frac{\cos 2\varepsilon}{2\cos^2\varepsilon}\right)\left\|P_0u\right\|_{L_2}^2 = \frac{1}{\min(\alpha^2;\beta^2)}\times$$
$$\times\frac{2\cos^2\varepsilon-\cos^2\varepsilon+\sin^2\varepsilon}{2\cos^2\varepsilon}\left\|P_0u\right\|_{L_2}^2 = \frac{1}{\min(\alpha^2;\beta^2)}\frac{1}{2\cos^2\varepsilon}\left\|P_0u\right\|_{L_2}^2.$$

On the other hand

$$\left\|A^2u\right\|_{L_2}^2 = \left\|\rho^{1/2}\rho^{-1/2}A^2u\right\|_{L_2}^2 \le \max_t \rho^{-1}(t)\left\|\rho^{1/2}A^2u\right\|_{L_2}^2 \le \frac{1}{\min(\alpha^2;\beta^2)}\frac{1}{\min(\alpha^2;\beta^2)}\frac{1}{2\cos^2\varepsilon}\left\|P_0u\right\|_{L_2}^2,$$

i.e. for $\pi/4 \le \varepsilon < \pi/2$ we have

$$\left\|A^2u\right\|_{L_2} \le \frac{1}{\sqrt{2}\cos\varepsilon}\frac{1}{\min(\alpha^2;\beta^2)}\left\|P_0u\right\|_{L_2}. \tag{22}$$

Inequalities (21) and (22) show the validity of (14).

Let’s prove inequality (15). Let $0 \le \varepsilon \le \pi/4$ $(\cos 2\varepsilon \ge 0)$. It follows from inequality (11) that

$$\left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|_{L_2}^2 \le \left\|\rho^{-1/2}P_0u\right\|_{L_2}^2 .$$

Then

$$\left\|\rho^{1/2}\rho^{-1/2}\frac{d^2u}{dt^2}\right\|_{L_2}^2 \le \max_t \rho(t)\left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|_{L_2}^2 \le \max_t \rho(t)\left\|\rho^{-1/2}P_0u\right\|_{L_2}^2 \le \frac{\max(\alpha^2;\beta^2)}{\min(\alpha^2;\beta^2)}\left\|P_0u\right\|_{L_2}^2 .$$

So, for $0 \le \varepsilon \le \pi/4$ $(\cos 2\varepsilon \ge 0)$ the inequality

$$\left\|\frac{d^2u}{dt^2}\right\|_{L_2} \le \frac{\max(\alpha;\beta)}{\min(\alpha;\beta)}\left\|P_0u\right\|_{L_2} \tag{23}$$

holds.

Now, let's assume that , $\pi/4 \le \varepsilon < \pi/2$.

Then $\cos 2\varepsilon \le 0$. In this case it follows from Lemma 1 that

$$\left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|_{L_2}^2 \le \left\|\rho^{-1/2}P_0u\right\|_{L_2}^2 - 2\cos 2\varepsilon\left\|A\frac{du}{dt}\right\|_{L_2}^2 . \tag{24}$$

Considering (14) in (24), one may get

$$\left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|^2_{L_2} \le \left\|\rho^{-1/2}P_0u\right\|^2_{L_2} - 2\cos 2\varepsilon\frac{1}{4\cos^2\varepsilon\min(\alpha^2;\beta^2)}\left\|P_0u\right\|^2_{L_2} \le \max_t \rho^{-1}\left\|P_0u\right\|^2_{L_2} -$$
$$-2\cos 2\varepsilon\frac{1}{4\cos^2\varepsilon\min(\alpha^2;\beta^2)}\left\|P_0u\right\|^2_{L_2} = \frac{1}{\min(\alpha^2;\beta^2)}\left\|P_0u\right\|^2_{L_2} - \frac{\cos 2\varepsilon}{2\cos^2\varepsilon}\times \tag{25}$$
$$\times\frac{1}{\min(\alpha^2;\beta^2)}\left\|P_0u\right\|^2_{L_2} = \frac{1}{\min(\alpha^2;\beta^2)}\left(1-\frac{\cos 2\varepsilon}{2\cos^2\varepsilon}\right)\left\|P_0u\right\|^2_{L_2} = \frac{1}{\min(\alpha^2;\beta^2)}\frac{1}{2\cos^2\varepsilon}\left\|P_0u\right\|^2_{L_2}.$$

With regard to inequality (25) we get, that for $\pi/4 \le \varepsilon < \pi/2$ the inequality

$$\left\|\frac{d^2u}{dt^2}\right\|^2_{L_2} = \left\|\rho^{1/2}\rho^{-1/2}\frac{d^2u}{dt^2}\right\|^2_{L_2} \le \max_t \rho(t)\left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|^2_{L_2} \le \frac{\max(\alpha^2;\beta^2)}{\min(\alpha^2;\beta^2)}\cdot\frac{1}{2\cos^2\varepsilon}\left\|P_0u\right\|^2_{L_2}$$

holds.

Hence , it follows that for the inequality

$$\left\|\frac{d^2u}{dt^2}\right\|_{L_2} \le \frac{\max(\alpha;\beta)}{\min(\alpha;\beta)}\cdot\frac{1}{\sqrt{2}\cos\varepsilon}\left\|P_0u\right\|_{L_2} \tag{26}$$

is valid.

Inequalities (23) and (26) affirm the validity of inequality (15). The Lemma is proved.

Now we shall prove the main theorem.

Theorem 2. Let the operator A satisfies the condition i), but operators $B_j = A_jA^{-j}$ $(j = 0,1,2)$ satisfy condition ii), moreover

$$K(\varepsilon;\alpha;\beta) = \sum_{j=0}^{2} c_j(\varepsilon;\alpha;\beta)\left\|B_{2-j}\right\| < 1,$$

where numbers $c_j(\varepsilon)$ are defined from lemma 2. Then boundary problem (1),(2) is regularly solvable.

Proof. As in the proof of Theorem 1 we consider the boundary - value problem (1),(2) in the form of equation (3) and $P_0 u$ and $P_1 u$ as equations (4) and (5) . .

By the proved Theorem 1 the operator $P_0 : \overset{o}{W}{}_2^2(R_+;H;1) \to L_2(R_+;H)$ is an isomorphism. Then the inverse P_0^{-1} exists and bounded. Therefore, for any $u \in \overset{o}{W}{}_2^2(R_+;H;1)$ there responds a unique $\vartheta \in L_2(R_+;H)$:

$$\vartheta = P_0^{-1} u .$$

By means of this substitution we reduce equation (3) to the equation in $L_2(R_+;H)$:

$$\vartheta + P_1 P_0^{-1} \vartheta = f \quad , \tag{27}$$

where $\vartheta \in L_2(R_+;H)$, $f \in L_2(R_+;H)$.

Let's show that

$$\left\| P_1 P_0^{-1} \right\|_{L_2(R_+;H) \to L_2(R_+;H)} < 1.$$

Since for any $\vartheta \in L_2(R_+;H)$:

$$\left\|P_1 P_0^{-1}\vartheta\right\|_{L_2} = \left\|P_1 u\right\|_{L_2} = \left\|A_0 \frac{d^2u}{dt^2} + A_1 \frac{du}{dt} + A_2 u\right\|_{L_2} \leq \left\|A_0 \frac{d^2u}{dt^2}\right\|_{L_2} + \left\|A_1 \frac{du}{dt}\right\|_{L_2} +$$

$$+\left\|A_2 u\right\|_{L_2} \leq \left\|A_0\right\|_{H\to H}\left\|\frac{d^2u}{dt^2}\right\|_{L_2} + \left\|A_1 A^{-1}\right\|_{H\to H}\left\|A\frac{du}{dt}\right\|_{L_2} + \left\|A_2 A^{-2}\right\|_{H\to H}\left\|A^2 u\right\|_{L_2} =$$

$$= \left\|B_0\right\|\left\|\frac{d^2u}{dt^2}\right\|_{L_2} + \left\|B_1\right\|\left\|A\frac{du}{dt}\right\|_{L_2} + \left\|B_2\right\|\left\|A^2 u\right\|_{L_2} .$$

From lemma 2 we get

$$\left\|P_1 P_0^{-1}\vartheta\right\|_{L_2} \leq \left\|B_0\right\| c_2(\varepsilon,\alpha;\beta)\left\|P_0 u\right\|_{L_2} + \left\|B_1\right\| c_1(\varepsilon,\alpha;\beta)\left\|P_0 u\right\|_{L_2} + \left\|B_2\right\| c_0(\varepsilon,\alpha;\beta)\left\|P_0 u\right\|_{L_2} =$$

$$+\left(\left\|B_0\right\| c_2(\varepsilon,\alpha;\beta) + \left\|B_1\right\| c_1(\varepsilon,\alpha;\beta) + \left\|B_2\right\|_{L_2} c_0(\varepsilon,\alpha;\beta)\right)\left\|P_0 u\right\|_{L_2} = K(\varepsilon,\alpha;\beta)\left\|\vartheta\right\|_{L_2},$$

i.e.

$$\left\|P_1 P_0^{-1}\right\|_{L_2(R_+;H)\to L_2(R_+;H)} < 1.$$

Then from equation (27) we can find ϑ

$$\vartheta = \left(E + P_1 P_0^{-1}\right)^{-1} f ,$$

but

$$u = P_0^{-1}\left(E + P_1 P_0^{-1}\right)^{-1} f .$$

Hence it follows

$$\left\|u\right\|_{W_2^2(R_+;H)} \leq const\left\|f\right\|_{L_2(R_+;H)}.$$

The Theorem 2 is proved.

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