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HEAT DISTRIBUTION ANALYSIS FOR A SEMI-CONDUCTING SAMPLE OF CONCRETELY GIVEN CONFIGURATION

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ABSTRACT

Without sufficient knowledge on temperature fields, one can not raise reliability, choose optimal technology of product making and operating conditions. Therefore the problems on heat distribution are of significant value in heat transfer processes to pass to higher operating parameters.
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1. INTRODUCTION

This work is denoted to the analytic theory of the heat distribution in the substance in the rest. In this work for a given semi-condicting sample the problem of the heat distribution is solved both analytically and numerically the distribution of the temperature is found.

2. PROBLEM STATEMENT

Find a solution of the equation

$$u_t = a^2 u_{xx} - qu + f \ , \quad x \in (\ell_j, \ell_{j+1}), t \in (T_{j-1}, T_j), j = \overline{1,3} \tag{1}$$

satisfying the initial conditions

$$\begin{aligned} &u(x,0) = u_{01}(x) \ , x \in [\ell_1, \ell_2], \\ &u(x,T_1) = u_{02}(x), \ x \in [\ell_2, \ell_3], \\ &u(x,T_2) = u_{03}(x), \ x \in [\ell_3, \ell_4], \end{aligned} \tag{2}$$

and boundary conditions

$$\begin{aligned} &u(\ell_1, t) = \varphi_1(t) \ , t \in [0, T_1], \\ &u(\ell_2, t) = \varphi_2(t) \ , t \in [0, T_2], \\ &u(\ell_3, t) = \varphi_3(t) \ , t \in [T_1, T_3], \\ &u(\ell_4, t) = \varphi_4(t) \ , t \in [T_2, T_3], \end{aligned} \tag{3}$$

on the domains

$$\begin{aligned} &\mathrm{D}_1 = \{\ell_1 \le \mathrm{x} \le \ell_2, \ 0 \le \mathrm{t} \le \mathrm{T}_1\}, \\ &\mathrm{D}_2 = \{\ell_2 \le \mathrm{x} \le \ell_3, \mathrm{T}_1 \le \mathrm{t} \le \mathrm{T}_2\}, \\ &\mathrm{D}_3 = \{\ell_3 \le \mathrm{x} \le \ell_4, \ \mathrm{T}_2 \le \mathrm{t} \le \mathrm{T}_3\} \end{aligned}$$

Here $a > 0,\ q \geq 0$ are constants, $u_{0i}(x)\ (i = 1,2,3), \varphi_k(t), k = \overline{1,4}$, $f(x,t)$ are the given functions, $u = u(x,t)$ is temperature, t is time. $f = f(x,t) = \sum_{k=0}^{n} a_k x^k + \sum_{k=0}^{m} b_k t^k$, $a^2 = k/\gamma\rho$ in thermal-conductivity coefficient, k is internal heat-conductivity coefficient, γ is heat capacity, ρ is density of the given material.

In the paper we give application of the net method to the solution of problem (1)-(3), prove the maximum principle for appropriate difference problem and convergence of the method.

First the solution of the problem is investigated in an analytical way and corresponding expressions are obtained for the solution on each domain, for example, on the domain D_1:

$$u_1 = \frac{2}{x_1 - x_0} \int_0^{t_0} \int_{x_0}^{x_1} \left\{ \sum_{n=1}^{\infty} e^{(-q_1 - a^2 \lambda_n)(t-\tau)} \cdot \sin \frac{\pi n (x - x_0)}{x_1 - x_0} \cdot \sin \frac{\pi n (\zeta - x_0)}{x_1 - x_0} \right\} f_1(\zeta, \tau) d\zeta d\tau \ ;$$

where $x_0 = \ell_1$, $x_1 = \ell_2$, $t_0 = t$; $\varphi_1(t) = \varphi_2(t) \equiv 0$; $t \in [0; T_1]$; $u_{01}(x) \equiv 0$; $x \in [\ell_1, \ell_3]$ and

$$\lambda_n = \left(\frac{\pi n}{x_1 - x_0} \right)^2 \quad (n = 1,2,...)$$

i.e. u_1 is the solution of the following equation under homogeneous conditions

$$u_t = a_1^2 u_{xx} - q_1 u + f_1(x,t)$$

Thus, on the rectangular domain

$$D_1 = \{ \ell_1 \leq x \leq \ell_2,\ 0 \leq t \leq T_1 \}$$

we choose a uniform net

$$V_{h\tau} = \left\{ (x_i = \ell_1 + ih; \quad t_j = j\tau), i = 0,1,...n, j = 0,1,...,m, \ nh = \ell_2; \ m\tau = T_1 \ , \tau = \sigma h^2 \right\}$$

and to problem (1)-(3) we assign the difference problem:

$$(\vartheta_{ij+1} - \vartheta_{ij}) / \tau = a^2 (\vartheta_{i+1,j} - 2\vartheta_{ij} + \vartheta_{i-1j}) / h^2 - q\vartheta_{ij} + f_{ij} \qquad (i = 1,2,...,n-1) \qquad (4)$$

$$\vartheta_{i0} = F_i \ , \qquad (i = 0,1,....,n) \ , \qquad (5)$$

$$\begin{cases} \vartheta_{\ell_1 j} = \varphi_1(t_j) = \varphi_1, \\ \vartheta_{\ell_2 j} = \varphi_2(t_j), = \varphi_2 \ , (j = 0,1,...,m) \end{cases} \qquad (6)$$

Rewrite this problem in the following form:

$$(-1/\tau + 2a^2/h^2 + q)\vartheta_{ij} + 1/\tau \cdot \vartheta_{ij+1} - a^2/h^2 (\vartheta_{i-1,j} + \vartheta_{i+1,j}) = f_{ij} \qquad (i = 1,2,...,n-1) \qquad (7)$$

$$\vartheta_{i0} = F_i \ , \qquad (i = 0,1,....,n) \ , \qquad (8)$$

$$\begin{cases} \vartheta_{\ell_1 j} = \varphi_{1j}, \\ \vartheta_{\ell_2 j} = \varphi_{2j}, \quad (j = 0,1,...,m) \end{cases} \qquad (9)$$

It is known that the maximum principle holds for the solution of the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

By fulfilling the conditions
$a > 0$ and $q \geq 0$ the analogue of the maximum principle is proved for problem (7)-(9).

Theorem 1 (Maximum principle) Let a grid function $V_{h\tau}$ determined on ϑ_{ij}, satisfy problem (7)-(9) and

$$f_{ij} \leq 0, \quad i = 1,2,...,n-1 \ , \varphi_{1j} \leq 0, \ \varphi_{2j} \leq 0$$
$$(f_{ij} \geq 0, \quad i = 1,2,...,n-1, \ \varphi_{1j} \geq 0, \ \varphi_{2j} \geq 0), \qquad j = 0,1,....,m \ .$$

Then the solution ϑ_{ij}, differ from a constant may not take the greatest positive (the least negative) value in internal nodes of the net $V_{h\tau}$.

Proof. Let's prove the first part of the theorem, i.e. prove that if $f_{ij} \le 0, \varphi_{1j} \le 0, \varphi_{2j} \le 0$, the solution ϑ_{ij} may not take the greatest positive value in the nodes of the net $V_{h\tau}$.

Let at some internal node of the net ϑ_{ij} the function accept the greatest positive value $V_{h\tau}$. Then, since $\vartheta_{ij} \ne const$, there will be found a point (x_{i0}, t_{j0}) at which

$$\vartheta_{i0j0} = \max_{\substack{0 \le i \le n \\ o \le j \le m}} \vartheta_{ij} = \overline{M} > 0,$$

and even if at one of the neighboring points (x_{i0-1}, t_{j0}), (x_{i0+1}, t_{j0}) and (x_{i0}, t_{j0-1}) the value of the function ϑ_{ij} will be strongly less than $\overline{M}$.

Let $1 \le i_0 \le n-1$. Then at the node (x_{i0}, t_{j0}) we have:

$$\begin{aligned} f_{i_0 j_0} &= (-1/\tau + 2a^2/h^2 + q)\vartheta_{i_0 j_0} + 1/\tau \cdot \vartheta_{i_0 j_0 - 1} - a^2/h^2 (\vartheta_{i_0+1 j_0} + \vartheta_{i_0-1 j_0}) > \\ &> (-1/\tau + 2a^2/h^2 + q)\overline{M} - 1/\tau \cdot \overline{M} - \left(2a^2/h^2\right)\overline{M} = q\overline{M} \ge 0 , \end{aligned}$$

i.e. $f_{i_0 j_0} > 0$, that contradicts the requirement $f_{i_0 j_0} < 0$. We can show in a similar way that ϑ_{ij} may not take the greatest positive value, i.e. the second part of the theorem is proved.

Corollary 1. If

$f_{ij} \le 0,\ i = 1,2,...,n-1, \varphi_{1j} \le 0,\ \varphi_{2j} \le 0,\ (f_{ij} \ge 0, i = 1,2,...,n-1,\ \varphi_{1j} \ge 0,\ \varphi_{2j} \ge 0),\ j = 1,2,...,m$ and $\vartheta_{i0} \le 0, (\vartheta_{i0} \ge 0),\ i = 0,1,...,n$, the solution of problem (4)-(6) ϑ_{ij} is non-positive (non-negative):

$\vartheta_{ij} \leq 0, (\vartheta_{ij} \geq 0), i = 0,1,...,n, \ j = 0,1,...,m$.

Corollary 2. For $f_{ij} = 0, \ i = 1,2,...,n-1, \ \varphi_{1j} = 0, \ \varphi_{2j} = 0 \quad j = 1,....,m$ and $\vartheta_{i0} = 0, \ (i = \overline{0,n})$ problem (4)-(6) has only a trivial solution: $\vartheta_{ij} = 0, \ i = 0,1,...,n, \ j = 0,1,...,m$ and consequently, problem (4)-(6) is uniquely solvable for any $f_{ij}, \varphi_{1j}, \varphi_{2j}, F_i$.

Before we investigate the solution of problem (1)-(3), we prove the following theorems:

Theorem 2. (Comparison theorem) Let ϑ_{ij} be a solution of problem (4)-(6), and $\overline{\vartheta_{ij}}$ be a solution of the problem which is obtained when replacing the functions $f_{ij}, \varphi_{1j}, \varphi_{2j}, F_i$ by $\overline{f_{ij}}, \overline{\varphi_{1j}}, \overline{\varphi_{2j}}$ and $\overline{F_i}$, respectively, in problem (4)-(6). Then, if

$$|f_{ij}| \leq \overline{f_{ij}}, \ i = 1,2,...,n-1,$$

$$|\varphi_{kj}| \leq \overline{\varphi_{kj}}, \ (k = 1,2), \ j = 1,2,...,m$$

and $$|F_i| \leq \overline{F_i}, \ i = 0,1,...,n$$

then the inequality:

$$|\vartheta_{ij}| \leq \overline{\vartheta_{ij}}, \ i = 0,1,...,n, j = 0,1,...,m$$

holds.

This theorem is easily proved and by virtue of this theorem we get the validity of the following statement:

Theorem 3. Let

$$K = \max\left\{ \max_{\overline{D}_1} |f(x,t)|, \max_{\substack{0 \le t \le T \\ k=1,2}} |\varphi_k(t)|, \max_{\ell_1 \le x \le \ell_2} |F(x)| \right\} ;$$

then for the solution of problem (4)-(6) it holds the inequality:

$|\vartheta_{ij}| \le K \cdot e^{T_1}$, $i = 0,1,...,n,\ j = 0,1,...,m$.

Convergence. Let u_{ij} be the value of the exact solution of problem (1)-(3) at the nodes, ϑ_{ij} be a solution of problem (4)-(6). Let's determine the function

$$z_{ij} = \vartheta_{ij} - u_{ij} .$$

If we substitute $\vartheta_{ij} = u_{ij} + z_{ij}$ in (4)-(6), we get:

$$(-1/\tau + 2a^2/h^2 + q_1)z_{ij} + 1/\tau \cdot z_{ij+1} - a^2/h^2\,(z_{i-1,j} + z_{i+1j}) = R_{ij} \qquad (i = 1,2,...,n-1)$$

$$z_{io} = 0, \quad i = 0,1,...,n$$

$$z_{\ell_1 j} = R_{\ell_1 j}$$

$$z_{\ell_2 j} = R_{\ell_2 j} \qquad j = 1,2,...,m$$

where

$$|R_{ij}| \le (L/2)(\tau + \sigma h),\ i = 0,1,...,n,\ j = 0,1,...,m.$$

$$L = \max_{\overline{D}_1}\left\{ \left|\frac{\partial^2 u}{\partial x^2}\right|, \left|\frac{\partial^3 u}{\partial x^3}\right|, \left|\frac{\partial^2 u}{\partial t^2}\right| \right\}, \sigma = \max(2a/3) > 0 . \qquad (10)$$

By theorem 2 the following statement is true.

Theorem 4. If the solution of problem (1)-(3) is $u(x,t) \in C^2(\overline{D_1})$, the solution of problem (4)-(6) converges to the solution of problem (1)-(3) with velocity $\mathrm{O}(\mathrm{h}+\tau)$. And it holds the estimation:

$$\left|\vartheta_{ij} - u_{ij}\right| \le (L/2) \cdot e^{T_1}(\tau + \sigma h),$$

where L and σ are determined, by equalities (10).

Numerical solution of the problem is obtained by applying the net method.

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