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PARALLEL (OFFSET) CURVES IN LORENTZIAN PLANE

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ABSTRACT

In this study, we have defined parallel (offset) curves in the Lorentzian plane. Also, we have given the relations between of the curvatures of these curves.Then, we have defined involute and evolute curves of a curve and we have given some theorems in lorentzian plane

Keywords: Plane curves; Involute and evolute curves; Parallel curves; Lorentzian plane.

LORENTZ DÜZLEMİNDE PARALEL DENGE EĞRİLERİ

ÖZET

Bu çalışmada, lorentz düzleminde paralel eğrileri tanımladık. Ayrıca bu eğrilerin eğrilikleri arasındaki ilişkileri verdik. Sonra, lorentz düzleminde bir eğrinin involute ve evolute eğrilerini tanımladık ve bazı teoremler verdik.

Anahtar Kelimeler: Düzlem eğrisi; İnvolute ve evolute eğriler; Paralel eğriler; Lorentz düzlemi.

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1.INTRODUCTION

Let $R^2 = \{(x_1, x_2) | x_1, x_2 \in R\}$ be a 2-dimensional vector space, $x = (x_1, x_2)$ and $x = (x_1, x_2)$ be two vectors in R^2.The Lorentzian scalar product of x and y is defined by $\langle x, y \rangle_L = x_1 y_1 - x_2 y_2$.

$E_1^2 = (R^2, \langle x_1, x_2 \rangle_L)$ is called Lorentzian plane.The vector x in E_1^2 is called a spacelike vector, null vector or a timelike vector if $\langle x, y \rangle_L > 0$, $\langle x, y \rangle_L = 0$ or $\langle x, y \rangle_L < 0$ respectively. For $x \in E_1^2$, the norm of the vector x defined by $\|x\|_L = \sqrt{|\langle x, x \rangle_L|}$ and x is called a unit vector if $\|x\|_L = 1$.Denote by $\{T(s), N(s)\}$ the moving Frenet frame along the curve $\alpha(s)$.Then T and N are the tangent and the principal normal vector of the curve $\alpha(s)$ respectively. Let be T spacelike and N timelike.This set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties

$$T' = \kappa N,\ N' = \kappa T \tag{1.1}$$

where κ curvature of the curve $\alpha(s)$ respectively. Let be T timelike and N spacelike. This set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties [4].

$$T' = \kappa N,\ N' = \kappa T \tag{1.2}$$

2. PARALLEL (OFFSET) CURVES

Definition 2.1. A planar parametric curve $\alpha(t)$ is given by $\alpha(t) = (\alpha_1(t), \alpha_2(t)), t \in I$ where α_1 and α_2 are differentiable functions of a parameter t.

Definition 2.2. Let $\alpha(t): I \to E_1^2$ be a plane curve. For a planar curve $\alpha(t)$ with the well defined normalized orthogonal vector $N(t)$. We define the offset curve at the distance d as

$$\beta(t) = \alpha(t) + dN_\alpha(t) \tag{2.1}$$

where $N_\alpha(t) = \left(\alpha_2^{'}(t), \alpha_1^{'}(t)\right)$.It is a displacement of the original curve in the direction of the normal vector. The unit normal vector is

$$N_\alpha(t) = \frac{\left(\alpha_2^{'}(t), \alpha_1^{'}(t)\right)}{\sqrt{\alpha_2^{'2}(t), \alpha_1^{'2}(t)}}.$$

Lemma 2.1. Let α be spacelike curve.The curve $\alpha(s)$ is part of a circle(in the lorentzian mean) if and only if $\kappa > 0$ is constant and $\tau = 0$.

Proof. Suppose α is part of a circle. α is a plane curve, so $\tau = 0$.Also by definition, for all s, $|\alpha(s) - p| = r$ .Squaring both sides gives

$$\langle \alpha(s) - p, \alpha(s) - p \rangle_L = r^2.$$

If we differentiate this expression, we get

$\langle 2T, \alpha(s) - p \rangle_L = 0$ or $\langle T, \alpha(s) - p \rangle_L = 0$.

If we differentiate again, then we obtain

$$\begin{aligned}
&\langle T', \alpha(s) - p \rangle_L + \langle T, T \rangle_L = 0, \\
&\langle \kappa N, \alpha(s) - p \rangle_L + 1 = 0, \\
&\langle \kappa N, \alpha(s) - p \rangle_L = -1.
\end{aligned} \tag{2.2}$$

This means, in particular, that $\kappa > 0$ and $\langle N, \alpha(s) - p \rangle_L \neq 0$. Now differentiating (2.2) produces

$$\left\langle \frac{d\kappa}{ds} N, \alpha(s) - p \right\rangle_L + \langle \kappa N', \alpha(s) - p \rangle_L + \langle \kappa N, T \rangle_L = 0$$
$$\left\langle \frac{d\kappa}{ds} N, \alpha(s) - p \right\rangle_L + \langle \kappa^2 T, \alpha(s) - p \rangle_L = 0.$$

Since $\langle T, \alpha(s) - p \rangle_L = 0$ by above , we have

$$\left\langle \frac{d\kappa}{ds} N, \alpha(s) - p \right\rangle_L = 0.$$

Also, $\langle N, \alpha(s) - p \rangle_L \neq 0$ by above, so $\frac{d\kappa}{ds}$ .This means, of course, that $\kappa > 0$ is constant. Suppose now that $\kappa > 0$ is constant. To show $\alpha(s)$ is part of a circle we must show that each $\alpha(s)$ is a fixed distance from a fixed point. For the standard circle, from any point on the circle to the center we proceed in the normal direction a distance equal to the radius. That is, we go $rN = -\frac{1}{\kappa} N$ .We do the same here. Let $\gamma(s)$ denote the curve

$$\gamma(s) = \alpha(s) - \frac{1}{\kappa} N_\alpha(s).$$

Since we want γ to be a single point, the center of the desired circle, we must have $\gamma'(s) = 0$ .Computing, we obtain

$$\gamma'(s) = \alpha'(s) - \frac{1}{\kappa} N_\alpha^{'}(s)$$
$$= T - \frac{1}{\kappa}\kappa T$$
$$= 0$$

Hence $\gamma(s)$ is a constant p. Then we have

$$|\alpha(s) - p| = \left| -\frac{1}{\kappa} N_\alpha \right| = \frac{1}{\kappa},$$

so p is the centre of a circle $\alpha(s)$ of radius .

Theorem 2.2. Let $\alpha(t)$ be regular plane curve and spacelike curve. Then the parallel curve $\beta(t)$ is a regular at those t for which $1 + d\kappa_\alpha \neq 0$. Furthermore , its curvature is given by

$$\kappa_\beta = \frac{\kappa_\alpha}{|1 + d\kappa_\alpha|}.$$

The unit tangent vector of the offset curve is

$$T_\beta = \frac{1 + d\kappa_\alpha}{|1 + d\kappa_\alpha|} T_\alpha.$$

The unit normal vector of the offset curve is

$$N_\beta = \frac{-(1 + d\kappa_\alpha)}{|1 + d\kappa_\alpha|} N_\alpha.$$

Proof. Since the curve is planar, τ = 0. We can assume that α has unit speed.Then $\beta(s) = \alpha(s) + dN_\alpha(s)$ and using Eq (2.1),we have

$$\beta'(s) = \alpha'(s) + dN_\alpha^{'}(s)$$
$$= (1 + d\kappa_\alpha)T_\alpha$$

and

$$\beta''(s) = -d\kappa_\alpha^{'} T_\alpha + (1 + d\kappa_\alpha)\kappa_\alpha N_\alpha$$
$$= \kappa_\alpha (1 + d\kappa_\alpha)\kappa_\alpha N_\alpha$$

since $\kappa_\alpha = cons\tan t$. Hence

$$\kappa_\beta = \frac{\det(\beta', \beta'')}{\|\beta'\|_L^3} = \frac{\kappa_\alpha}{|1 + d\kappa_\alpha|}.$$

Then the parallel curve $\beta(t)$ is a regular at those t for which $(1 + d\kappa_\alpha) \neq 0$.Furthermore , its curvature is given by

$$\kappa_\beta = \frac{\kappa_\alpha}{|1 + d\kappa_\alpha|}.$$

The unit tangent vector of the offset curve is

$$T_\beta = \frac{\beta'}{\|\beta'\|_L} = \frac{1 + d\kappa_\alpha}{|1 + d\kappa_\alpha|} T_\alpha.$$

The unit normal vector of the offset curve is

$$N_\beta = T_\beta \wedge_L e_z = \frac{-(1 + d\kappa_\alpha)}{|1 + d\kappa_\alpha|} N_\alpha.$$

where $e_z = (0,0,1)$.

Theorem 2.3. Let $\alpha(t)$ be regular plane curve and timelike curve. Then the parallel curve $\beta(t)$ is a regular at those t for which $1 + d\kappa_\alpha \neq 0$.Furthermore, its curvature is given by

$$\kappa_\beta = \frac{-\kappa_\alpha}{|1 + d\kappa_\alpha|}.$$

The unit tangent vector of the offset curve is

$$T_\beta = \frac{1 + d\kappa_\alpha}{|1 + d\kappa_\alpha|} T_\alpha.$$

The unit normal vector of the offset curve is

$$N_\beta = \frac{-(1+d\kappa_\alpha)}{|1+d\kappa_\alpha|} N_\alpha .$$

3. INVOLUTES AND EVOLUTES OF THE PLANE CURVE

Definition 3.1. The center of the osculating circle is called the center of curvature, the radius of the osculating circle is called the radius of curvature of the given curve at the given point [1].

Definition 3.2. The locus of the centers of curvature of a curve is called the evolute of the curve. The evolute is defined for arcs along which the curvature is not zero. If $\alpha : I \rightarrow E_1^2$ is a curve with nowhere zero curvature function κ and unit normal vector field N_α,then the evolute can be parameterized by the mapping $\widetilde{\beta} : I \rightarrow E_1^2$,

$$\widetilde{\beta}(t) = \alpha(t) - \frac{1}{\kappa} N_\alpha(t).$$

Definition 3.3. Let α be a smooth parameterized curve, t a point of its domain. We say that $\alpha(t)$ is a singular point (or singular parameter) of the curve α if $\alpha'(t) = 0$ [1].

Theorem 3.1. Singular points of the parallel curves of a regular curve α sweep out the evolute of α.

Proof. Let α be spacelike curve.Since $\beta'(s) = (1+d\kappa_\alpha)T_\alpha$,singular parameters are characterized by $1+d\kappa_\alpha = 0$ .Then the corresponding singular points $\widetilde{\beta}(t) = \alpha(t) - \frac{1}{\kappa} N_\alpha(t)$ lie on the evolute of the curve.It is also easy to show that any evolute point is a singular point of a suitable parallel curve.

Theorem 3.2. The evolutes of a planar curve is helix.

Proof. Let the curve $\alpha(s)$ be spacelike and the evolute of the curve $\alpha(s)$ be planar curve. Then $T_\beta = N_\alpha$. So,

$$\alpha(s) = \beta(s) + \lambda T_\beta(s)$$

$$\beta(s) = \alpha(s) - \lambda N_\alpha(s)$$

$$T_\beta = \beta'(s) = (1 + d\kappa_\alpha)T_\alpha(s)\ .$$

Since $T_\beta \perp N_\alpha$, $1 - d\kappa_\alpha(s) = 0, \lambda = \dfrac{1}{\kappa_\alpha(s)}$. Hence,we have $\beta(s) = \alpha(s) - \dfrac{1}{\kappa_\alpha} N_\alpha$.The planar evolute of the curve $\alpha(s)$ is the locus of centre of curvature.If the nonplanar evolute of the curve $\alpha(s)$ is $\beta(s)$, then we have

$$\beta(s) = \alpha(s) - \lambda T_\beta(s)$$

$$T_\beta = T_\alpha - \lambda\kappa_\beta N_\beta + T_\beta$$

$$T_\beta(s) - \lambda\kappa_\beta N_\beta = 0$$

$$T_\alpha = \pm N_\beta .$$

If we derivate the equation $f = \langle T_\beta, T_\alpha \wedge_L N_\alpha \rangle_L$,we have

$$\begin{aligned} f' &= \langle \kappa_\beta N_\beta, T_\alpha \wedge_L N_\alpha \rangle_L + \langle T_\beta, \kappa_\alpha N_\alpha \wedge_L N_\alpha + T_\alpha \wedge_L (\kappa_\alpha N_\alpha) \rangle_L \\ &= \kappa_\beta \langle N_\beta, T_\alpha \wedge_L N_\alpha \rangle_L \\ &= \kappa_\beta \det(N_\beta, T_\alpha, N_\alpha) \\ &= \kappa_\beta .0 \\ &= 0. \end{aligned}$$

So, we find that f is constant.Hence,we have $\angle \langle T_\beta, T_\alpha \wedge_L N_\alpha \rangle_L = cons\tan t.$

Since the angle between the speed vector of the curve $\beta(s)$ and the normal of plane of the curve $\alpha(s)$ is constant, $\beta(s)$ is a helix.

Definition 3.4. Let $\alpha : I \to E_1^2$ be a unit speed curve with unit tangent vector field T. An involute of the curve α is a curve $\hat{\beta}$ of the form $\hat{\beta} = \alpha(s) - (l - s)T_\alpha(s)$,where l is a given real number. A curve has many involutes corresponding to the different choices of the length l of the thread.

Theorem 3.3. Let α be a unit speed curve with $\kappa > 0$,

$\hat{\beta} = \alpha(s) - (l - s)T_\alpha(s)$

an involute of it such that l is greater than the length of α. Then the evolute of $\hat{\beta}$ is α.

Proof. Let be T spacelike and N timelike.We have

$$\hat{\beta}' = (l - s)\kappa_\alpha(s)N_\alpha(s)$$
$$\hat{\beta}'' = ((l - s)\kappa_\alpha(s))' N_\alpha(s) + (l - s)\kappa_\alpha^2(s)T_\alpha(s).$$

Computing the curvature $\hat{\kappa}_\beta$ of $\hat{\beta}$,

$$\hat{\kappa}_\beta = \frac{\det(\hat{\beta}', \hat{\beta}'')}{\|\hat{\beta}'\|_L^3} = \frac{(l - s)^2 \kappa_\alpha^3(s)}{(l - s)^3 \kappa_\alpha^3(s)} = \frac{1}{(l - s)}.$$

Thus, the evolute of $\hat{\beta}$ is

$$\hat{\beta} - \left(\frac{1}{\hat{\kappa}_\beta}\right) T_{\hat{\beta}} = \alpha(s) - (l - s)T_\alpha(s) + (l - s)T_\alpha(s) = \alpha(s).$$

Theorem 3.4. Suppose that the regular curves γ_1 and γ_2 have regular evolutes. Then, γ_1 and γ_2 are parallel if and only if their evolutes are the same.

Proof. Let γ_1 be spacelike curve and let γ_1 and γ_2 parallel curve.In this case $\gamma_2 = \gamma_1 + d\gamma_1$.

If the evolute of γ_1 is β_1 and the evolute γ_2 is β_2 ,then we can write that

$$\beta_1 = \gamma_1 - \frac{1}{\kappa_{\gamma_1}} N_{\gamma_1} \tag{3.1}$$

$$\beta_2 = \gamma_2 - \frac{1}{\kappa_{\gamma_2}} N_{\gamma_2}. \tag{3.2}$$

We will show that $\beta_1 = \beta_2$ .From the equations (3.1) and (3.2) ,we have

$$\begin{aligned}\beta_2 - \beta_1 &= (\gamma_2 - \gamma_1) + \frac{1}{\kappa_{\gamma_1}} N_{\gamma_1} - \frac{1}{\kappa_{\gamma_2}} N_{\gamma_2} \\ &= dN_{\gamma_1} + \frac{1}{\kappa_{\gamma_1}} N_{\gamma_1} - \frac{1}{\kappa_{\gamma_2}} N_{\gamma_2}.\end{aligned}$$

Since the curves γ_1 and γ_2 are parallel, we can take to equal the normals of these curves i.e $N_\gamma = N_{\gamma_1} = N_{\gamma_2}$ .Therefore, we have

$$\beta_2 - \beta_1 = \left(d + \frac{1}{\kappa_{\gamma_1}} - \frac{1}{\kappa_{\gamma_2}} \right) N_\gamma . \tag{3.3}$$

On the other hand,let the curvature functions of the curves γ_1 and γ_2 be κ_{γ_1} and κ_{γ_2} , respectively.Since

$$\kappa_{\gamma_2} = \frac{\kappa_{\gamma_1}}{1+\kappa_{\gamma_1}} . \tag{3.4}$$

If we substitute the equation (3.2) in the equation (3.1),we have

$$
\begin{aligned}
\beta_2 - \beta_1 &= \left(d + \frac{1}{\kappa_{\gamma_1}} - \frac{1}{\kappa_{\gamma_2}} \right) N_\gamma \\
&= \left(d + \frac{1 - 1 - d\kappa_{\gamma_1}}{\kappa_{\gamma_1}} \right) N_\gamma = 0 \\
\beta_2 &= \beta_1 .
\end{aligned}
$$

The contrary proof of this theorem is obvious.Because it is trivial.

Theorem 3.5. If two involutes of a regular curve are regular, then they are parallel.

Proof. Let $\alpha(t)$ be spacelike curve.Let the frenet vectors of the curve β_1 be $\{T_{\beta_1}, N_{\beta_1}\}$ and let the frenet vectors of the curve β_2 be $\{T_{\beta_2}, N_{\beta_2}\}$ and let the frenet vectors of the curve $\alpha(t)$ be $\{T_\alpha, N_\alpha\}$.Let two involutes of the curve $\alpha(t)$ be β_1 and β_2 ,respectively.Then, we have

$$
\begin{aligned}
\beta_1(t) &= \alpha(t) - (l_1 - t)T_\alpha, l_1 \in R \\
\beta_2(t) &= \alpha(t) - (l_2 - t)T_\alpha, l_2 \in R
\end{aligned}
\tag{3.5}
$$

From the equation (3.5),we have

$$
\begin{aligned}
\beta_2(t) - \beta_1(t) &= \alpha(t) + (l_2 - t)T_\alpha - \alpha(t) - (l_1 - t)T_\alpha \\
\beta_2(t) - \beta_1(t) &= (l_2 - l_1)T_\alpha .
\end{aligned}
$$

Since $T_{\beta_1} = N_\alpha$ and $N_{\beta_1} = T_\alpha$, we have

$$\beta_2(t) = \beta_1(t) + (l_2 - l_1)N_{\beta_1}$$
$$\beta_2(t) = \beta_1(t) + dN_{\beta_1}, d \in R.$$

Hence, the curves β_1 and β_2 are parallel.

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