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A POLYNOMIAL APPROACH FOR SOLVING HIGH-ORDER LINEAR COMPLEX DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS IN DISC

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ABSTRACT

The purpose of this study is to give a Taylor matrix method for approximately solving the high-order linear complex differential equations with variable coefficients under the mixed conditions in a circular domain. The method is based on first taking the truncated Taylor expansions of the expressions in equation and then substituting their matrix forms into the given equation. Hence the differential equation and conditions are transformed to the matrix equations. The solution of these equations yields the unknown Taylor coefficients of the solution function. To illustrate the pertinent features of the method, examples are presented and results are compared.
Keywords: Taylor approximation, Complex differential equations, Taylor matrix method.

DEĞİŞKEN KATSAYILI YÜKSEK MERTEBEDEN LİNEER COMPLEX DİFERANSİYEL DENKLEMLERİN DAİRESEL BİR BÖLGEDE POLİNOM ÇÖZÜMLERİ

ÖZET

Bu çalışmada dairesel bir bölgede karışık koşullar altında değişken katsayılı yüksek mertebeden lineer complex diferansiyel denklemlerin Taylor matris yöntemi ile nümerik çözümlerinin bulunması amaçlanmıştır. Belirtilen yöntem denklemdeki fonksiyonların kesilmiş Taylor polinomlarının matris formlarının denklemde yerine konması esasına dayanır. Böylece denklem ve koşullar matris denklemine dönüştürülür. Bu denklemlerin çözümleri Taylor polinomlarının katsayılarını oluştururlar.Yöntemin uygulaması çeşitli örneklerle açıklanmış ve sonuçlar tartışılmıştır.
Anahtar Kelimeler: Taylor yaklaşımları, Complex diferansiyel denklemler, Taylor matris yöntemi.

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1.INTRODUCTION

A Taylor-expansion approach for solving integral equations has been presented by Kanwal and Liu [1] and then this method has been extended by Sezer to Volterra integral equations [2] and second order linear differential equations [3], and by Nas,Yalçınbaş and Sezer to high order linear Fredholm integro-differential equations [4]. On the other hand, the method has been developed by Gülsu and Sezer [5,6,7] for solving high order linear difference equations and Fredholm integro-difference equations with mixed argument. In this paper, these methods are modified and developed to solve an m-th order linear complex differential equation with variable coefficients, which is extended to the linear complex differential equations [8-13],

$$\sum_{k=0}^{m} P_k(z) f^{(k)}(z) = g(z) \qquad (1)$$

under the mixed conditions

$$\sum_{k=0}^{m-1} \sum_{r=0}^{R} c_{jk}^{r} f^{(k)}(\xi_r) = \lambda_j \,, \qquad (j = 0,1,\ldots,m-1) \qquad (2)$$

where the coefficients $P_k(z)$ and the right-hand member $g(z)$ are single-valued analytic functions in the disc $D = \{z \in C : |z - z_0| \le \rho, \ \ 0 < \rho < \infty\}$; c_{jk}^{r} and λ_j are appropriate complex constants; and $z_0, \xi_r \in D$.

We assume that the aproximate solution of the equation (1) under the mixed condition (2) is expressed in the form

$$f(z) = \sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n \,, \ z_0 \in D \qquad (3)$$

which is a Taylor polynomial of degree N at $z = z_0$, where $f^{(n)}(z_0)$, $\quad n = 0,1,\ldots,N$, are the Taylor coefficients to be determined.

The rest of this paper is organized as follows. Fundamental relations for high-order linear complex differential equation with variable coefficients are presented in section 2. The new scheme is based on the Taylor matrix method. The method of finding approximate solution is described in section 3. To support our findings, we present numerical experiments in section 4. Section 5 concludes this article with a brief summary.

2. FUNDAMENTAL MATRIX RELATIONS

Let us consider the linear complex differential equation with variable coefficients (1) and find the truncated Taylor series expansions of each term in this equation at $z = z_0$ and their matrix representations. We first consider the desired solution $f(z)$ of the problem, defined by the series (3). Then we can put series (3) in the matrix form

$$\left[f(z)\right] = \mathbf{Z}\mathbf{M_0}\mathbf{F} \tag{4}$$

where

$$\mathbf{Z} = \begin{bmatrix} 1 & (z-z_0) & (z-z_0)^2 & . & . & . & (z-z_0)^N \end{bmatrix}$$

$$\mathbf{M_0} = \begin{bmatrix} \frac{1}{0!} & 0 & 0 & . & . & . & 0 \\ 0 & \frac{1}{1!} & 0 & . & . & . & 0 \\ 0 & 0 & \frac{1}{2!} & . & . & . & 0 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 0 & 0 & 0 & . & . & . & \frac{1}{N!} \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} f^{(0)}(z_0) & f^{(1)}(z_0) & . & . & . & f^{(N)}(z_0) \end{bmatrix}^{\mathrm{T}}$$

Now we consider the expression $P_k(z)f^{(k)}(z)$ of equation (1). we can write it as the truncated series expansion of degree N at $z = z_0$ in the form

$$P_k(z)f^{(k)}(z)=\sum_{n=0}^{N}\frac{1}{n!}\left[P_k(z)f^{(k)}(z)\right]_{z=z_0}^{(n)}(z-z_0)^n . \tag{5}$$

By the Leibniz's rule we evaluate

$$\left[P_k(z)f^{(k)}(z)\right]_{z=z_0}^{(n)}=\sum_{i=0}^{n}\binom{n}{i}P_k^{(n-i)}(z_0)f^{(i+k)}(z_0)$$

and substitute it in expression (5). Thus the expression (5) becomes

$$P_k(z)f^{(k)}(z)=\sum_{n=0}^{N}\sum_{i=0}^{n}\frac{1}{(n-i)!i!}P_k^{n-i}(z_0)f^{(i+k)}(z_0)(z-z_0)^n \tag{6}$$

and its matrix form

$$\left[P_k(z)f^{(k)}(z)\right]=\mathbf{ZP_kF} \tag{7}$$

where

$$\mathbf{P_k}=\begin{bmatrix}
0 & \dots & 0 & \frac{P_k^{(0)}(z_0)}{0!\,0!} & 0 & 0 & \dots & 0 & 0\\
0 & \dots & 0 & \frac{P_k^{(1)}(z_0)}{1!\,0!} & \frac{P_k^{(0)}(z_0)}{0!\,1!} & 0 & \dots & 0 & 0\\
0 & \dots & 0 & \frac{P_k^{(2)}(z_0)}{2!\,0!} & \frac{P_k^{(1)}(z_0)}{1!\,1!} & \frac{P_k^{(0)}(z_0)}{0!\,2!} & \dots & 0 & 0\\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\
0 & \dots & 0 & \frac{P_k^{(N-k)}(z_0)}{(N-k)!\,0!} & \frac{P_k^{(N-k-1)}(z_0)}{(N-k-1)!\,1!} & \frac{P_k^{(N-k-2)}(z_0)}{(N-k-2)!\,2!} & \dots & \frac{P_k^{(1)}(z_0)}{1!\,(N-k-1)!} & \frac{P_k^{(0)}(z_0)}{0!\,(N-k)!}\\
0 & \dots & 0 & \frac{P_k^{(N-k+1)}(z_0)}{(N-k+1)!\,0!} & \frac{P_k^{(N-k)}(z_0)}{(N-k)!\,1!} & \frac{P_k^{(N-k-1)}(z_0)}{(N-k-1)!\,2!} & \dots & \frac{P_k^{(2)}(z_0)}{2!\,(N-k-1)!} & \frac{P_k^{(1)}(z_0)}{1!\,(N-k)!}\\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\
0 & \dots & 0 & \frac{P_k^{(N-1)}(z_0)}{(N-1)!\,0!} & \frac{P_k^{(N-2)}(z_0)}{(N-2)!\,1!} & \frac{P_k^{(N-3)}(z_0)}{(N-3)!\,2!} & \dots & \frac{P_k^{(k)}(z_0)}{k!\,(N-k-1)!} & \frac{P_k^{(k-1)}(z_0)}{(k-1)!\,(N-k)!}\\
0 & \dots & 0 & \frac{P_k^{(N)}(z_0)}{N!\,0!} & \frac{P_k^{(N-1)}(z_0)}{(N-1)!\,1!} & \frac{P_k^{(N-2)}(z_0)}{(N-2)!\,2!} & \dots & \frac{P_k^{(k+1)}(z_0)}{(k+1)!\,(N-k-1)!} & \frac{P_k^{(k)}(z_0)}{k!\,(N-k)!}
\end{bmatrix}$$

Let the function $g(z)$ be approximated by a truncated Taylor series at $z = z_0$

$$g(z) = \sum_{n=0}^{N} \frac{1}{n!} g^{(n)}(z_0)(z - z_0)^n$$

Then we can put this series in the matrix form

$$[g(z)] = \mathbf{Z M_0 G}, \tag{8}$$

where

$$\mathbf{G} = \begin{bmatrix} g^{(0)}(z_0) & g^{(1)}(z_0) & . & . & . & g^{(N)}(z_0) \end{bmatrix}^T$$

Substituting the matrix forms (7) and (8) corresponding to the functions $P_k(z) f^{(k)}(z)$ and $g(z)$, into equation (1), and then simplifying the resulting equation, we have the matrix equation

$$\left(\sum_{k=0}^{m-1} \mathbf{P}_k\right)\mathbf{F} = \mathbf{M_0 G} \tag{9}$$

The matrix equation (9) is a fundamental relation for the m-th order linear differential equations with variable coefficients (1).

Next, let us form the matrix representations for the conditions (2) as follows. The derivatives of expression (3) can be written in the form

$$f^{(k)}(z) = \sum_{n=k}^{N} \frac{1}{(n-k)!} f^{(n)}(z_0)(z - z_0)^{n-k}, \quad k = 0, 1, \ldots, m-1$$

and the corresponding matrix equations

$$\left[f^{(k)}(z)\right] = \mathbf{ZM_kF}\,,\quad k=0,1,\ldots,m-1 \tag{10}$$

where

$$\mathbf{M}_k = \begin{bmatrix} 0 & 0 & & \frac{1}{0!} & 0 & & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{(N-k)!} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

The matrix equations (10) for $z=\xi_r$ become

$$\left[f^{(k)}(\xi_r)\right] = \mathbf{MZ}(\boldsymbol{\xi}_\mathbf{r})\mathbf{M_kF}\,,\quad k=0,1,\ldots,m-1;\ r=0,1,\ldots,R\,, \tag{11}$$

where

$$\mathbf{Z}(\boldsymbol{\xi}_\mathbf{r}) = \left[1 \quad (\xi_r-z_0) \quad (\xi_r-z_0)^2 \quad . \quad . \quad . \quad (\xi_r-z_0)^N\right]$$

Substituting quantities (11) into expression (2) and then simplifying, we obtain the matrix forms corresponding to the mixed conditions (2) as

$$\mathbf{U_jF} = \left[\lambda_j\right],\quad j=0,1,\ldots,m-1 \tag{12}$$

where

$$\mathbf{U_j} = \sum_{k=0}^{m-1}\sum_{r=0}^{R} c_{jk}^{r}\mathbf{Z}(\boldsymbol{\xi}_\mathbf{r})\mathbf{M}_k \equiv \left[u_{j0} \quad u_{j1} \quad \cdots \quad u_{jN}\right]$$

and the complex constants $u_{jn}, n=0,1,\ldots,N; j=0,1,\ldots,m-1$ are related to the coefficients c_{jk}^{r} and ξ_r.

3. METHOD OF SOLUTION

The fundamental matrix equation (9) for high-order linear complex differential equation (1) corresponds to a system of *(N+1)* algebraic equations for the (N+1) unknown complex coefficients $f^{(0)}(z_0), f^{(1)}(z_0), \ldots, f^{(N)}(z_0)$. Briefly we can write Eq.(9) in the form

$$\mathbf{WF=M_0G} \text{ or } [\mathbf{W;M_0\,G}] \tag{13}$$

so that

$$\mathbf{W}=\left[w_{pq}\right]=\sum_{k=0}^{m}\mathbf{P}_k\,;\;\; p,q=0,1,\ldots,N\,.$$

We can write the matrix equations (12) for the conditions (2) in the augmented matrix forms

$$\left[\mathbf{U}_j;\lambda_j\right] \text{ or } \left[u_{j0} \quad u_{j1} \quad \cdots \quad u_{jN};\lambda_j\right],\; j=0,1,\ldots,m-1\,. \tag{14}$$

To obtain the approximate solution of Eq(1) under the conditions (2) in the terms of Taylor polynomials, by replacing the m row matrices (14) by last m rows of augmented matrix (13), we have new augmented matrix

$$\left[\mathbf{W}^*;\mathbf{G}^*\right]=\begin{bmatrix} w_{00} & \cdots & w_{0N} & ; & \dfrac{g^{(0)}(z_0)}{0!} \\ w_{10} & \cdots & w_{1N} & ; & \dfrac{g^{(1)}(z_0)}{1!} \\ \cdots & \cdots & \cdots & ; & \cdots \\ w_{N-m,\,0} & \cdots & w_{N-m,N} & ; & \dfrac{g^{(N-m)}(z_0)}{(N-m)!} \\ u_{00} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & \cdots & u_{1N} & ; & \lambda_1 \\ \cdots & \cdots & \cdots & ; & \cdots \\ u_{m-1,\,0} & \cdots & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix} \tag{15}$$

If $rank\;\mathbf{W}^*=rank\left[\mathbf{W}^*;\mathbf{G}^*\right]=N+1$, then we have

$$\mathbf{F} = (\mathbf{W}^*)^{-1}\mathbf{G}^* \tag{16}$$

so that

$$\mathbf{W}^* = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} \\ w_{10} & w_{11} & \cdots & w_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} \\ u_{00} & u_{01} & \cdots & u_{0N} \\ u_{10} & u_{11} & \cdots & u_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} \end{bmatrix} \qquad \mathbf{G}^* = \begin{bmatrix} \dfrac{g^{(0)}(z_0)}{0!} \\ \dfrac{g^{(1)}(z_0)}{1!} \\ \vdots \\ \dfrac{g^{(N-m)}(z_0)}{(N-m)!} \\ \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{m-1} \end{bmatrix}$$

Thus the coefficients $f^{(n)}(z_0)$, $n = 0,1,\ldots,N$ are uniquely determined by Eq.(16).Therefore the solution of Eq.(1) with conditions (2) is obtained by the truncated Taylor series (3).

We can easily check the accuracy of this solution as follows [4] :

Since the Taylor polynomial (3) is an approximate solution of Eq.(1), when the solution $f(z)$ and its derivatives are substituted in Eq.(1), the resulting equation must be satisfied approximately; that is, for $z = z_i \in D$

$$E(z_i) = \left| \sum_{k=0}^{m} P_k(z_i) f^{(k)}(z_i) - g(z_i) \right| \cong 0$$

or

$E(z_i) \le 10^{-k_i}$, (k_i is any positive integer).

If max 10^{-k} (k is any positive integer) is prescribed, then the truncation limit N is increased until the values $E(z_i)$ at each of the points z_i becomes smaller than the prescribed 10^{-k} .

4. EXAMPLES

Example 1.

Let us illustrate our method by the second order complex differential equation with variable coefficients

$$f'' - 2z^3 f' + 10z^2 f = (1 + 10z^2 - 2z^3)e^z$$

with the conditions

$$f(0) = 1,\ f'(0) = 2$$

and approximate the solution f(z) by the polynomial

$$f(z) = \sum_{n=0}^{6} \frac{f^{(n)}(0)}{n!} z^n ,$$

where N=6, $P_0(z) = 10z^2$, $P_1(z) = -2z^3$, $P_2(z) = 1$, $g(z) = (1 + 10z^2 - 2z^3)e^z$. We first reduce this equation, from Eq.(9), to the matrix form

$$(\sum_{k=0}^{2} \mathbf{P}_k)\mathbf{F} = \mathbf{M_0 G}$$

The quantities in the equation are computed as

$$\mathbf{P}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5/12 & 0 & 0 \end{bmatrix} \quad \mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/3 & 0 & 0 \end{bmatrix},$$

$$\mathbf{P}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 1 \\ 1 \\ 21/2 \\ 49/6 \\ 73/24 \\ 81/120 \\ 61/720 \end{bmatrix},$$

$$\mathbf{W} = \mathbf{P}_0 + \mathbf{P}_1 + \mathbf{P}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 1/6 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1/24 \\ 0 & 0 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 & 0 \end{bmatrix}$$

and the augmented matrix is

$$[\mathbf{W};\mathbf{G}] = \left[\begin{array}{ccccccc:c} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 0 & 1/2 & 0 & 0 & 21/2 \\ 0 & 8 & 0 & 0 & 0 & 1/6 & 0 & 49/6 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1/24 & 73/24 \\ 0 & 0 & 0 & 2/3 & 0 & 0 & 0 & 81/120 \\ 0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 61/720 \end{array}\right]$$

The augmented matrix based on the conditions $f(0) = 1,\ f'(0) = 2$ is obtained as

$$\left[\mathbf{W}^*;\mathbf{G}^*\right] = \left[\begin{array}{ccccccc:c} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 0 & 1/2 & 0 & 0 & 21/2 \\ 0 & 8 & 0 & 0 & 0 & 1/6 & 0 & 49/6 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1/24 & 73/24 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

Solving this system, the Taylor coefficients are obtained as

$$f^{(0)}(0)=1,\ f^{(1)}(0)=2,\ f^{(2)}(0)=1,\ f^{(3)}(0)=1,\ f^{(4)}(0)=1,\ f^{(5)}(0)=-47,\ f^{(6)}(0)=25$$

and thereby the polynomial solution becomes

$f(z)=1+2z+1/2z^2+1/6z^3+1/24z^4-47/120z^5+5/144z^6$

Taking N=6 the obtained solution are compared with the exact solution $f(z)=z-\frac{2}{5}z^5+\exp(z)$ in Table 1 .

Table 1. The absolute error for Example 1.($I=\sqrt{-1}$)

z	Exact Solution	Present Met(z_0=0)	N_e=6
-0.5-0.5I	-0.0177083333 - 0.8407986000 I	-0.0177083330 - 0.8449652800 I	0.41666667000e-2
-0.4-0.4I	0.2010239999 - 0.6774215111 I	0.2010239990 - 0.6785137778 I	0.10922667000e-2
-0.3-0.3I	0.4038430000 - 0.5228151000 I	0.4038430000 - 0.5230095000 I	0.19440000000e-3
-0.2-0.2I	0.6018986667 - 0.3631687111 I	0.6018986670 - 0.3631857778 I	0.17066700000e-4
-0.1-0.1I	0.8003009999 - 0.1903490111 I	0.8003009990 - 0.1903492778 I	0.26670000000e-6
0.0 -0.0I	0.0000000000 + 0.000000000 I	0.0000000000 + 0.000000000 I	0.00000000000
0.1+0.1I	1.1996656670 + 0.2103489889 I	1.1996656670 + 0.2103487222 I	0.26670000000e-6
0.2+0.2I	1.3975679990 + 0.4431672889 I	1.3975679990 + 0.4431502222 I	0.17066700000e-4
0.3+0.3I	1.5934570000 + 0.7027989000 I	1.5934570000 + 0.7026045000 I	0.19400000000e-6
0.4+0.4I	1.7904426670 + 0.9973304889 I	1.7904426670 + 0.9962382222 I	0.10922667000e-2
0.5+0.5I	1.9968749990 + 1.3404513890 I	1.9968749990 + 1.3362847220 I	0.41666670000e-2

Example 2 : [9, Example 2]

Let us consider the equation

$$f''' - f' + (-8e^{6z} - 12e^{5z} - 30e^{4z} - 19e^{3z} - 9e^{2z})f = 0$$

The exact solution of this equation is $f(z)=\exp(e^{z}+e^{2z})$ and the Taylor expansion of f(z) about z=0 gives

$$f(z) = 7.389056099 + 22.16716830z + 51.72339269z^2 + 99.75225734z^3 + 169.6404129z^4 + 262.6809443z^5 + 377.6936550z^6 + \cdots$$

The solution obtained using the matrix method for N=6 is

$$f_{approx}(z) = 7.389056099 + 22.16716830z + 51.72339269z^2 + 99.75225734z^3 + 169.6404129z^4 + 262.6809443z^5 + 377.6554273z^6.$$

which is the first six terms of the Taylor expansion of the exact solution at z=0. The obtained polynomial solution is compared with the exact solution in Table 2**.**

Table 2. The absolute error for Example 2.($I=\sqrt{-1}$)

z	Exact Solution	Present Met(z_0=0)	N_e=6
-0.4 - 0.4 I	4.6787109160 - 6.699272158I	4.6787109160 - 6.700524803I	0.1252645000e-2
-0.3 - 0.3 I	3.1824369060 - 2.375789374I	3.1824369060 - 2.376012318I	0.2229440000e-3
-0.2 - 0.2 I	3.8021915230 - 1.748726332I	3.8021915230 - 1.748745904I	0.1957200000e-4
-0.1 - 0.1 I	5.3144948570 - 1.374267496I	5.3144948570 - 1.374267802I	0.3060000000e-6
-0.05-0.05I	6.3017230900 - 0.874398371I	6.3017230900 - 0.874398376I	0.4500000000e-8
0.0 - 0.0 I	7.3890560990 + 0.000000000I	7.3890560990 + 0.000000000I	0.0000000000
0.05+0.05I	8.4679070880 + 1.391537884I	8.4679070880 + 1.391537880I	0.4000000000e-8
0.1 + 0.1 I	9.3279050110 + 3.437160717I	9.3279050110 + 3.437160411I	0.3060000000e-6
0.2 + 0.2 I	8.8045233900 + 9.637750007I	8.8045233900 + 9.637730434I	0.1957300000e-4
0.3 + 0.3 I	0.6029765340 + 16.59123784I	0.6029765340 + 16.59101490I	0.2229400000e-3
0.4 + 0.4 I	-24.64295528 + 15.05221740I	-24.64295528 + 15.05096475I	0.1252650000e-2

Example 3: Let us consider second order linear complex differential equation

$$f'' - 2z^3 f' + 10z^2 f = 0$$

with the condition

$f(0) = 0$, $f'(0) = 1$

For N=6, the matrix form of the problem is defined by

$$(\sum_{k=0}^{2} \mathbf{P}_k)\mathbf{F} = \mathbf{M_0G}$$

After the augmented matrices of the equation and conditions are computed, we obtain the Taylor coefficients matrix

F= [0 1 0 0 0 -48 0].^{T}

Therefore, by the present method, we are able to find the exact solution

$f(z) = z - \frac{2}{5}z^5$.

Example 4: [12, p.304]

Our last example is the second order linear complex differential equation

$$(1-z^2)f''(z) - 2zf'(z) + 6f = (z^2+5)\sin z - 2z^2 \cos z$$

with the condition

$$f(0) = -\frac{1}{2}, f'(0) = 1.$$

This equation has the axact solution $f(z) = -\frac{1}{2} + \frac{3}{2}z^2 + \sin z$ whose Taylor expansion is

$$f(z) = -\frac{1}{2} + z + \frac{3}{2}z^2 - \frac{1}{6}z^3 + \frac{1}{120}z^5 + ...$$

After the augmented matrices of the system and conditions are computed, for N=5, we obtain the Taylor coefficients matrix

F= [-1/2 1 3 -1 0 15]^T

Therefore, we find the exact solution

$$f_{approx}(z) = -\frac{1}{2} + z + \frac{3}{2}z^2 - \frac{1}{6}z^3 + \frac{15}{120}z^5 .$$

The obtained polynomial solution is compared with the exact solution

$f(z) = -\frac{1}{2} + \frac{3}{2}z^2 + \sin z$ in Table 3 .

Table 3. The absolute error for Example 4.($I=\sqrt{-1}$)

z	Exact Solution	Present Met(z_0=0)	N_e=5
-0.5 - 0.5I	-1.026041667 + 0.307291666i	-1.0406250000 + 0.2927083334i	0.206239475e-1
-0.4 - 0.4I	-0.916213333 + 0.106453333i	-0.9209920000 + 0.1016746666i	0.675805525e-2
-0.3 - 0.3I	-0.807785000 - 0.019785000i	-0.8089190000 - 0.0209190000i	0.160371818e-2
-0. 2- 0.2I	-0.702506667 - 0.077173333i	-0.7026560000 -0.0773226667i	0.211189199e-3
-0. 1- 0.1I	-0.600328333 - 0.069661667i	-0.6003330000 - 0.0696663340i	0.659968921e-5
-0.05-.05I	-0.550041510 - 0.042458177i	-0.5500416560 - 0.0424583229i	0.206284267e-6
0.0 + 0.0I	-0.500000000 + 0.000000000i	-0.5000000000 + 0.0000000000i	0.000000000
0.05+.05I	-0.449958489 + 0.057458177i	-0.4499583437 + 0.0574583220i	0.206284267e-6
0.1 + 0.1I	-0.399671667 + 0.129661666i	-0.3996670000 + 0.1296663334i	0.659971043e-5
0.2 + 0.2I	-0.297493333 + 0.317173333i	-0.2973440000 + 0.3173226666i	0.211189178e-3
0.3 + 0.3I	-0.192215000 + 0.559785000i	-0.1910810000 + 0.5609190000i	0.160371818e-2
0.4 + 0.4I	-0.083786666 + 0.853546667i	-0.0790079999 + 0.8583253340i	0.675805524e-2
0.5 + 0.5I	0.026041668 + 1.192708330i	0.04062500001 +1.2072916660i	0.206239476e-1

5. CONCLUSIONS

High-order linear complex differential equations are usually difficult to solve analyticaly. Then it is required to obtain approximate solutions. For this reason, the present method has been proposed for approximate solution and also analytical solution.

The method presented in this study is a method for computing the coefficients in the Taylor expansion of the solution of a linear complex differential equations, and is valid when the functions $P_k(z)$ and g(z) are defined in the disc $D = \{z \in C : |z - z_0| \leq \rho\}$

The Taylor method is an effective method for the cases that the known functions have the Taylor series expansion at $z=z_0$. In this case, the Taylor polynomial solution f(z) and the values $f(z_j)$, $z_j \in D$ can be easily evaluated at low-computation effort. To get the best approximating solution of the equation, the truncation limit N must be chosen large enough. For computational efficiency, some estimate for N, the degree of the approximating polynomial (the truncation limit of Taylor series) to f(z), should be available. Because the choice of N determines the precision of the solution f(z). If N is chosen too large, unnecessary labour may be done; but if N is taken a small value, the solution will not be sufficiently accurate. Therefore N must be chosen sufficiently large to get a reasonable approximation. For example, an interesting feature, this method finds the analytical solutions if the equation has an exact solution that is a polynomial of degree N or less than N.

The method can also be extended to the system of linear complex equations with variable coefficients, but some modifications are required.

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