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A STUDY ON A LOD METHODS FOR TWO-DIMENSIONAL DIFFUSION EQUATION

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Abstract: Finite-difference techniques based on Noye-Hayman formula and 3-point backward time centered space (BTCS) method for one dimensional diffusion are used to solve the two-dimensional time dependent diffusion equation with boundary condition. In these cases locally one-dimensional (LOD) techniques are used to extend the one-dimensional techniques to solve the two-dimensional problem. The results of numerical testing shows that these schemes uses less central processor (CPU) time than the fully implicit schemes.
Key words: Finite difference, LOD, Noye-Hayman, BTCS.

İKİ BOYUTLU DİFFUZYON DENKLEMİ İÇİN LOD METODU ÜZERİNE BİR ÇALIŞMA

Özet: Bu çalışmada bir boyutlu diffuzyon denklemi için 3-Nokta Geri fark yöntemi ve Noye-Hayman yöntemini temel alan sonlu fark teknikleri , iki boyutlu zaman bağımlı diffuzyon denklemini çözmek için kullanıldı. Yerel bir boyut(LOD) yöntemi iki boyutlu diffuzyon denklemini çözmek için genişletildi. Nümerik sonuçlar ile bu yöntemin kapalı yöntemlere göre daha az zaman (CPU) harcadığı gösterildi.
Anahtar Kelimeler: Sonlu farklar,Yerel bir boyut yöntemi, Geri fark yöntemi, Noye-Hayman yöntemi.

1. Introduction

The constant-coefficient two-dimensional diffusion equation, namely

$$\frac{\partial u}{\partial t} = \alpha_x \frac{\partial^2 u}{\partial x^2} + \alpha_y \frac{\partial^2 u}{\partial y^2} \quad , \quad 0 \le x \le M \, , \; 0 \le y \le N \, , \quad 0 \le t \le T$$

1.1

where α_x and α_y are the coefficients of diffusion in the x and y directions respectively, has many applications to practical problems, including the flow of groundwater, and the diffusion of heat through solids. For many years the standart explicit two-level finite difference method for solving (1.1) was the classical explicit forward-time centred-space method described in Noye,B.J., Hayman,K.J.[1]

Recent improvements include the efficient alternating group explicit method of Dehghan M. [2]. The present article investigate the development of a fourth-order accurate two-level explicit finite difference method for solving (1.1) subject to Drichlet boundary condition. In particular locally one dimensional (LOD) methods and backward time centered space methods are investigated.

For convenience, a method which uses a computational molecule that involves m_1 grid points from time level (n+1) and m_2 grid points from time level n is denoted as an (m_1,m_2) methods. Also, the grid point $(i\Delta x, j\Delta y, n\Delta t)$ i=0,1,2,...I, j=0,1,2,...,J, n=0,1,2,...K where $\Delta x=M/I$, $\Delta y=N/J$, $\Delta t=T/K$, is referred to as the (i,j,n) grid point. At this point the partial differential equation (PDE) (1.1) is discretised to give the aproximating finite difference equation (FDE)

$$u_{i,j}^{\;n+1} = \sum_l \sum_m a_{l,m} u_{i+l,j+m}^{\;\;n} \qquad (1.2)$$

The coefficient $a_{l,m}$ are functions of the non dimensional diffusion numbers

$$r_x = \alpha_x \frac{\Delta t}{(\Delta x)^2} \; , \qquad r_y = \alpha_y \frac{\Delta t}{(\Delta y)^2}$$

Theoratical comparisons of the order of convergence of various finite-difference methods are based on the leading error terms in their modified equivalent partial differential equations (MEPDE) which have the general form

$$\frac{\partial u}{\partial t}-\alpha_x\frac{\partial^2 u}{\partial x^2}-\alpha_y\frac{\partial^2 u}{\partial t^2}+\sum_{p=3}^{\infty}\sum_{q=0}^{p}C_{p,q}\frac{\partial^p}{\partial x^{p-q}\partial y^q}=0 \qquad (1.3)$$

where the $C_{p,q}$ are coefficients of errors term. Given that (1.2) is consistent with the two-dimensional diffusion equation (1.1) which requires that

$$Lim_{\Delta x,\Delta y,\Delta t\to 0}C_{p,q}=0 \quad \text{for } p\geq 0, \qquad (1.4)$$

the error coefficient $C_{p,q}$ in the MEPDE can be written in the form;

$$C_{p,q}=\begin{cases}\dfrac{2\alpha_x(\Delta x)^{p-2}}{p!}\Gamma_{p,q}(r_x,r_y)..................q=0\\ \dfrac{2\alpha_y(\Delta y)^{p-2}}{p!}\Gamma_{p,q}(r_x,r_y)..................q=p\\ \dfrac{4\alpha_x(\Delta x)^{p-q-2}(\Delta y)^q}{(p-q)!q!}\Gamma_{p,q}(r_x,r_y).......otherwise\end{cases}$$

(1.5)

It can be seen from (1.3) that the error term associated with the coefficients $C_{p,q}$ are of the order (p-2) in Δx and Δy. The order of accuracy of an FDE which approximately solves (1.1) is the smallest order of any error term present in the corresponding MEPDE. Hence if the leading error term in the MEPDE is $C_{P,q}$ for any q=0,1,2,...,P then the FDE is order (P-2) accurate.

In the following the time-stepping stability of the FDE (1.2) is established by means of the von Neumann method.
In order to verify theoretical predictions, numerical test were carried out on a two dimensional time-dependent diffusion equation:

$$\frac{\partial u}{\partial t}=\alpha_x\frac{\partial^2 u}{\partial x^2}+\alpha_y\frac{\partial^2 u}{\partial y^2} \qquad (1.6)$$

$$\begin{aligned}
&u(x,y,0)=f(x)=\exp(x+y), && 0\leq x,y\leq 1\\
&u(0,y,t)=g_0(y,t)=\exp(y+2t), && 0\leq t\leq T, 0\leq y\leq 1\\
&u(1,y,t)=g_1(y,t)=\exp(1+y+2t), && 0\leq t\leq T, 0\leq y\leq 1 \qquad (1.7)\\
&u(x,1,t)=h_1(x,t)=\exp(1+x+2t), && 0\leq t\leq T, 0\leq x\leq 1\\
&u(x,0,t)=h_0(x,t)=\exp(x+2t), && 0\leq t\leq T, 0\leq x\leq 1
\end{aligned}$$

2. Lod Methods

Partial Differential Equation (1.1) can be solved by splitting it into two one-dimensional equation

$$\frac{1}{2}\frac{\partial u}{\partial t}=\alpha_x\frac{\partial^2 u}{\partial x^2} \qquad (2.1a)$$

$$\frac{1}{2}\frac{\partial u}{\partial t}=\alpha_y\frac{\partial^2 u}{\partial y^2} \qquad (2.1b)$$

rather than discretising the complete two-dimensional diffusion equation to give an approximating finite-difference equation based on a two-dimensional computational molecule. Each of these equations is then solved over half of the time step used for the complete two-dimensional equation using techniques for the one dimensional problems. This is advantageous since accurate and stable techniques for one -dimensional diffusion are much easier to develop and use than single step methods for two-dimensional diffusion equation.

Commencing with the initial condition for each n=0,1,2,...,K the process of stepping from time t_n to t_{n+1} is carried out in two stages. In the first stage, in advancing from t_n=nk to the time $t_{n+\frac{1}{2}} = (t_n + \frac{k}{2})$, the partial differential equation

$$\frac{1}{2}\frac{\partial u}{\partial t} = \alpha_x \frac{\partial^2 u}{\partial x^2} \qquad (2.2)$$

is solved numerically at the spatial points (x_i, y_j) , i=1,2,...,I-1 for each j=0,1,...,J.
Commencing with previously computed values $u_{i,j}^{\ n}$ i,j=1,2,...,M-1 and boundary values:

$$u_{0,j}^{\ n} = g_0(y_j, t_n) \ , \quad j=0,1,2,...,J \qquad (2.3)$$
$$u_{M,j}^{\ n} = g_1(y_j, t_n) \ , \quad j=0,1,2,...,J$$

results in the set of approximate values $u_{i,j}^{\ n+\frac{1}{2}}$, i=1,2,...,I-1 , j=0,1,...,J being found at the intermediate time $t_{n+\frac{1}{2}}$. Then in advancing from the time $t_{n+\frac{1}{2}}$ to $t_{n+1} = (t_n + k)$ the equation:

$$\frac{1}{2}\frac{\partial u}{\partial t} = \alpha_y \frac{\partial^2 u}{\partial y^2} \qquad (2.4)$$

is solved numericaly at the spatial points (x_i, y_j) , commencing with initial values $u_{i,j}^{\ n+\frac{1}{2}}$, i=1,2,...,I-1 , j=1,2,...,J-1 and using as boundary values $u_{i,0}^{\ n+\frac{1}{2}}$ and $u_{i,M}^{\ n+\frac{1}{2}}$ i=1,2,...,I-1. Not that the boundary conditions (1.7) are not used at the intermediate time $t_{n+\frac{1}{2}}$. This is because in the time interval t_n to $t_{n+\frac{1}{2}}$, the process of diffusion in the x-direction has been applied with a diffusion coefficient which is twice that in the original equation (1.1) as can be seen by rearranging in the form

$$\frac{\partial u}{\partial t} = 2\alpha_x \frac{\partial^2 u}{\partial x^2} \qquad (2.5)$$

Not that the values of $u_{i,j}^{\ n+\frac{1}{2}}$ i=1,2,...,I , j=1,2,...,J are not approximate solutions to the original problem.
Let's running the LOD process using Noye-Hayman formula for which the correct two-stage procedure is:
At the half-time level it is necessary to compute values at grid points next to the boundary and on the boundary using the one-dimensional diffusion procedure with the appropriate diffusion coefficient. The values for the two-dimensional diffusion problem should not be used at the grid points on the boundary at intermediate times.
With x–sweeps of this formula used to solve (2.1) in the first half time step when proceeding from the time t_n to t_{n+1} the formula used with i=2,3,...,I-2 is

$$u_{i,j}^{\ n+\frac{1}{2}} = \frac{r_x}{12}(6r_x - 1)(u_{i-2,j}^{\ n} + u_{i+2,j}^{\ n}) + \frac{2r_x}{3}(2 - 3r_x)(u_{i-1,j}^{\ n} + u_{i+1,j}^{\ n}) + \frac{1}{2}(2 - 5r_x + 6r_x^{\ 2})u_{i,j}^{\ n} \qquad (2.6)$$

for each j=0,1,..,J.

The problem of finding values of $u_{1,j}^{\ n+\frac{1}{2}}$ and $u_{I-1,j}^{\ n+\frac{1}{2}}$ which can not be found using

(2.6) can be find by using a re-arrangement of the unconditionally stable inverted version of (2.6) namely

$$r_x(6r_x+1)u_{i-2,j}^{n+\frac{1}{2}} - 8r_x(2+3r_x)u_{i-1,j}^{n+\frac{1}{2}} + 6(2+5r_x+6r_x^2)u_{i,j}^{n+\frac{1}{2}}$$

$$-8r_x(2+3r_x)u_{i+1,j}^{n+\frac{1}{2}} + r_x(6r_x+1)u_{i+2,j}^{n+\frac{1}{2}} = 12u_{i,j}^{n} \qquad (2.7)$$

which is obtained by setting -Δt for Δt in (2.6). Putting i=3 and re-arranging gives

$$u_{1,j}^{n+\frac{1}{2}} = \frac{8(2+3r_x)}{(6r_x+1)}\left(u_{2,j}^{n+\frac{1}{2}} + u_{4,j}^{n+\frac{1}{2}}\right) - \frac{6(2+5r_x+6r_x^2)}{r_x(6r_x+1)}u_{3,j}^{n+\frac{1}{2}} - u_{5,j}^{n+\frac{1}{2}} + \frac{12}{r_x(6r_x+1)}u_{3,j}^{n}$$

(2.8)

This gives values of $u_{1,j}^{n+\frac{1}{2}}$ for all j=0,1,...,J because all the values on its right hand side are knownn. The values of $u_{I-1,j}^{n+\frac{1}{2}}$ may be calculated using a similar formula obtained by setting i=I-3 in (2.7).

$$u_{I-1,j}^{n+\frac{1}{2}} = -u_{I-5,j}^{n+\frac{1}{2}} + \frac{8(2+3r_x)}{(6r_x+1)}u_{I-4,j}^{n+\frac{1}{2}} - \frac{6(2+5r_x+6r_x^2)}{r_x(6r_x+1)}u_{I-3,j}^{n+\frac{1}{2}}$$

$$+\frac{8(2+3r_x)}{(6r_x+1)}u_{I-2,j}^{n+\frac{1}{2}} + \frac{12}{r_x(6r_x+1)}u_{I-3,j}^{n} \qquad (2.9)$$

When computing values of $u_{i,j}^{n+1}$ from the values of $u_{i,j}^{n+\frac{1}{2}}$ in the y-sweep used in the second stage , the formula used with j=2,3,...,J-2 is:

$$u_{i,j}^{n+1} = \frac{r_y}{12}(6r_y-1)\left(u_{i,j-2}^{n+\frac{1}{2}} + u_{i,j+2}^{n+\frac{1}{2}}\right) + \frac{2r_y}{3}(2-3r_y)\left(u_{i,j-1}^{n+\frac{1}{2}} + u_{i,j+1}^{n+\frac{1}{2}}\right)$$

$$+\frac{1}{2}(2-5r_y+6r_y^2)u_{i,j}^{n+\frac{1}{2}} \qquad (2.10)$$

for each i=1,2,...,I-1. As in the first sweep , values at gridpoints adjacent to the boundaries y=0,1 are not found and supplementary equations must be used. The inverted formula corresponding to (2.9) is again used which , with j=2 gives on re-arrangement

$$u_{i,1}^{n+1} = \frac{(6r_y+1)}{8(2+3r_y)}\left(u_{i,0}^{n+1} + u_{i,4}^{n+1}\right) + \frac{3(2+5r_y+6r_y^2)}{4r_y(2+3r_y)}u_{i,2}^{n+1} - u_{i,3}^{n+1} - \frac{3}{2r_y(2+3r_y)}u_{i,2}^{n}$$

(2.11)

for each j=1,2,...,J-1. All the values on the right hand side of (2.10) are known and include the known boundary values $u_{i,0}^{n+1}$ which apply at the end of the complete procedure which involves diffusion in both x and y directions. The values of $u_{i,J-1}^{n+1}$ are found using a similar formula.

$$u_{i,J-1}^{n+1} = \frac{(6r_y+1)}{8(2+3r_y)}u_{i,J-4}^{n+1} - u_{i,J-3}^{n+1} + \frac{6(2+5r_y+6r_y^2)}{8r_y(2+3r_y)}u_{i,J-2}^{n+1} + \frac{(6r_y+1)}{8(2+3r_y)}u_{i,J}^{n+1} - \frac{12}{8r_y(2+3r_y)}u_{i,J-2}^{n}$$

(2.12)

Because known boundary values at the end of a fractional step can be used only for the second half-time step , whereas at the intermediate level they must be computed using an approximating finite-difference formula, the order of x- and y- sweeps is reversed for alternate time steps.

This procedure is stable, in

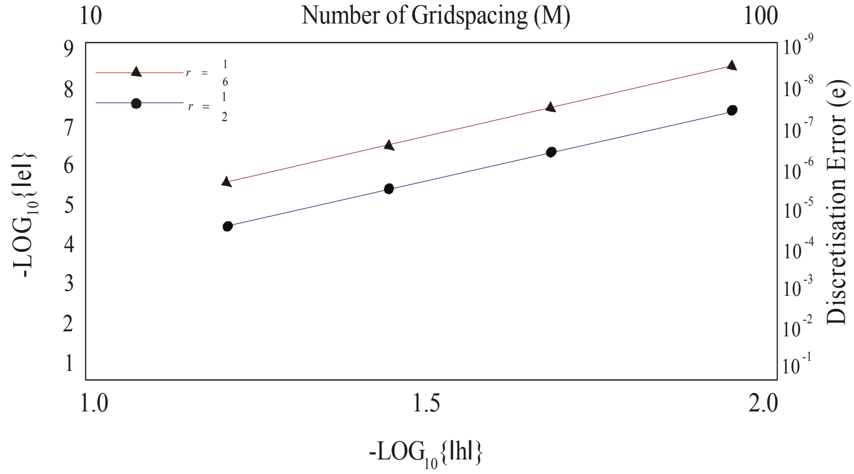

Figure1. Relation between error u and gridspacing for Noye-Hayman Method

The numerical results obtai
Figure1. The slopes of the

3. Point BTCS Method

The use of the 3-point BTC
solving the two one-dimens
In the y –sweep of the LO

$$-r_y u_{i,j-1}^{\,n+\frac{1}{2}} + (1+2r_y)u_{i,}$$

which is applied with j=1,2

values of $u_{i,j}$ computed at t

procedure is unconditionally von Neumann stable and solvable.
In the x-sweep the following formula is used , with i=1,2,...,I-1 for each j=1,2,...,J-1:

$$-r_x u_{i-1,j}^{\,n+1} + (1+2r_x)u_{i,j}^{\,n+1} - r_x u_{i+1,j}^{\,n+1} = u_{i,j}^{\,n+\frac{1}{2}} \qquad (3.2)$$

The resulting system is diagonally dominant , which guarantees that it is solvable. In Table1 the results are shown for $u_{i,j}^{\,n}$ with Δx=Δy=h=0.05 and r=1/2 at T=1.0 using 3-point BTCS and the Noye-Hayman formula with given boundary values everywhere.
When the absolute value of the error;

$$e_{i,j}^{\,n} = u(ih, jk, nk) - u_{i,j}^{\,n} \qquad (3.3)$$

at the point (0.5,0.5) at time T=1.0 was graphed against h on a logarithmic scale for various of r it was found that the slope of lines was always close to 2 for 3-point BTCS formula. These results illustrate the theoretical orders of accuracy evident from the modified equivalent equation.

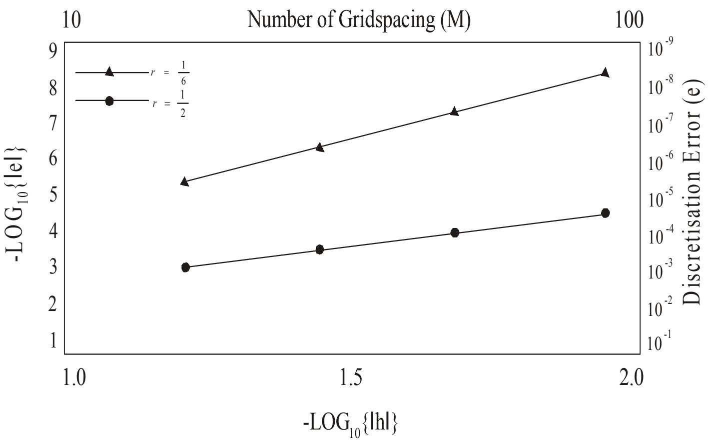

Figure 2. Relation between error u and gridspacing for 3-point BTCS method

The numerical results obtained with the fourth-order one dimensional equation are shown in Figure1. The slopes of the lines of best fit to the results are very close to 4 for each r. These results indicate that this fourth-order technique is much more accurate then the second-order LOD method based on the BTCS formula. For example with r=1/2 and Δx=0.5 the error produced by the second –order method was about 10^{-4} while that produced by the fourth-order technique was about 10^{-7}.

Table1. Results for u with T=1.0 , h=0.05, r=1/2

x	y	NH-Method	BTCS Method	NH- Error	BTCS -Error	Analitical Solutions
0.1	0.1	9.025012169	9.0248130	0.00000133	0.00020	9.025013499
0.2	0.2	11.02317240	11.022476	0.00000398	0.00070	11.02317638
0.3	0.3	13.46373102	13.461738	0.00000701	0.00100	13.46373804
0.4	0.4	16.44463704	16.442647	0.00000973	0.00200	16.44464777
0.5	0.5	20.08553577	20.093243	0.00000115	0.00200	20.08553692
0.6	0.6	24.53252900	24.530530	0.00000120	0.00200	24.53253020
0.7	0.7	29.96409897	29.962100	0.00000108	0.00200	29.96410005
0.8	0.8	36.59822730	36.597234	0.00000714	0.00100	36.59823444
0.9	0.9	44.70118088	44.700532	0.00000361	0.00050	44.70118449

Overall, it can be seen that LOD techniques provide an effective solution to the two-dimensional problem. However , it must be kept in mind that proper treatment is required with the LOD procedure to obtain the correct values to be used on the boundary at intermediate time levels.

4. Conclusion

In this paper time-split finite difference method have been used to solve the two-dimensional constant coefficient diffusion equation with given boundary values. Using the Noye-Hayman method for one-dimensional diffusion equation in a LOD procedure with special treatment on the boundaries at the intermediate time level gave fourth-order accuracy. Without the special boundary treatment at the intermediate time levels high-order methods used at interior grid points in an LOD procedure only produce low-order results. A comparison with the implicit scheme for the test problem clearly demonstrates that this technique are computationally superior. The numerical test obtained by using these methods give acceptable results. Also BTCS procedure produced second-order results. It used more CPU time than the fourth-order LOD procedure to get results of the same accuracy.

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