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ON THE BRAİDS FOR 10_5 KNOT

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Abstract: A presentation of the 4-string braid is studied for 10_5 knot. Braids structure plays very important role in Knots Theory and Word problem..Braid structure connection can be found with knot and its word. In view of this structure, we obtained braids for 10_5 knot and we gave Artin and Garside presentations.

Key words: Braids, Positive word, Representations of Braids.

10_5 DÜĞÜMÜNÜN ÖRGÜSÜ ÜZERİNE

Özet: 10_5 düğümünün 4 bileşenli örgü temsili elde edildi. Örgü kavramı oldukça önemli bir rol oynar. Kelime ve Düğüm arasında ilişki örgü yoluyla bulunabilir.Bu makalede 10_5 düğümünün örgü temsili bulunarak Artin ve Garside gerenleri hesaplanmıştır.

Anahtar keimeler: Örgü, Pozitif kelime, Örgülerin Temsilleri

1. Introduction

The word problem in B_n was solved by Artin in [3]. Conjugacy problem in Bn was also posed in[3]. Moreover, its importance for the problem of recognizing knots and links algorithmicaly was noted. However it took 43 years before progres was made. A somewhat different question is the shortest word problem to find a representative of the word class which has shortest length in the Artin generators. Using Garside and Thurston and Birman's using new generator's will be able to solve the word problem.Starting with Braids structure. On the top and base of a cube, B, mark out n points, $A_1, A_2, \cdots, A_n$ ve $A_1^{'}, A'_2, \cdots, A'_n$ respectively. These points may be arbitrarily placed, however we shall express them in terms of specific coordinates.

Firstly, the coordinates for B in R^3 are,

$$B = \{(x, y, z) \mid 0 \le x, y, z \le 1\}$$

Let us choose $A_1, A_2, \cdots, A_n$, $A_1^{\prime\,'}, A'_2, \cdots, A'_n$ as follows,

$$A_1 = (\frac{1}{2}, \frac{1}{n+1}, 1) \cdots A_n = (\frac{1}{2}, \frac{n}{n+1}, 1)$$

$$A_1' = (\frac{1}{2}, \frac{1}{n+1}, 1) \cdots A_n' = (\frac{1}{2}, \frac{n}{n+1}, 0)$$

By the construction each A_i' is directly below the corresponding A_i Figure1. Now join the $A_1, A_2, \cdots, A_n$, $A_1', A'_2, \cdots, A'_n$ by means of n curves in B . As usual, they are joined in a such a way that these curves (including the end points) do not mutually intersect each other. We will call these polygonal arcs strings.

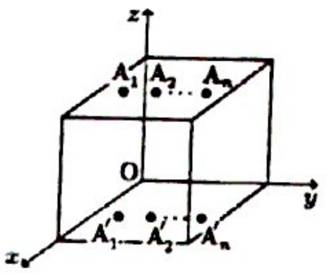


Figure 1

Suppose that we divide the cube into two parts by an arbitrary plane E that is parallel to base of the cube B . Then, if E intersects each strings at one and only one point, we say that these n string in B are an *n*-braid.

Suppose that B_n is the set of all *n*-braids (to be more precise all the equivalence class of these braids).

A fundemental result on the braid group B_n is that it has only the following two type of relations called the fundemental relations:

$$(1)\ \sigma_i\sigma_j = \sigma_j\sigma_i \qquad (|i-j| \geq 2)$$

$$(2)\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\ (i = 1, 2, \cdots, n-2).$$

Collecting together the various relations we have discussed so far we may write B_n in terms of its generators $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ and the these fundemantal relations,

$$B_n = \begin{pmatrix} \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \mid i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} (i = 1, 2, \cdots, n-2) \end{pmatrix}$$

where the right hand side is said to be a presentation of B_n (Artin generators).

In Garside genarators, i.e., positive braids, i.e., braids which are positive powers of the generators. Garside introduced the fundemantal braid Δ:

$$\Delta = (\delta_1\delta_2 \cdots \delta_{n-1})(\delta_1\delta_2 \cdots \delta_{n-2}) \cdots (\delta_1\delta_2)\delta_1. \tag{1}$$

He showed that every element $W \in B_n$ can be represented algorithmically by a word W of the form $\Delta^r P$, where r is an integer and P is a positive word, and r is a maximal for all such representations [2].

Lemma 1: The word $\Delta^{\#}$ in S_n has the following properties:

(i) For each word $V \in S_n$, $\Delta^{\#} V = \widehat{V} \Delta^{\#}$.

(ii) For each j $(1 \le j \le n-1)$, $D(\Delta^{\#})$ contains a word with initial letter s_j and word with final letter s_j [6].

From now on the semigroup S_n will be identified with its image $e(S_n)$ in B_n.

Lemma 2: For each j ($(1 \le j \le n-1)$, there is a positive word X_j such that

$\sigma_j^{-1} = \Delta^{-1} X_j$ .Also, $\Delta \sigma_j^{-1} = \sigma_{n-j}^{-1} \Delta$ (which, together with Lemma 1, implies that $\Delta^{-1} V = \widehat{V} \Delta^{-1}$ for every braid word V) [6].

Theorem 1 (Garside Solution to the Word Problem in B_n): If $\beta \in B_n$ then β is represented by unique word of the form $\Delta^m \overline{P}$, where the integers m and the positive word $\overline{P}$ are computed from any representative $\sigma_{\mu_1}^{\varepsilon_1} \cdots \sigma_{\mu_r}^{\varepsilon_r}$ of the word β in the following manner:

(i) List the positive word $X_1, \cdots, X_{n-1}$ whose existence is established by Lemma 1.

(ii) Replace every letter $\sigma^{-1}{}_{\mu_i}$ which occurs in the braid word $\sigma_{\mu_1}^{\varepsilon_1} \cdots \sigma_{\mu_r}^{\varepsilon_r}$ by $\Delta^{-1} X_{\mu_i}$.

(iii) Using the property $\Delta^{-1} V = \widehat{V} \Delta^{-1}$ (Lemma 2) collect all Δ^{-1}'s introduced in (ii) at the left, so that β is represented by a word of the form $\Delta^k P_0$, where P_0 is positive. Note that $k \le 0$.

(iv) Construct $D(P_0)$

(v) In $D(P_0)$, choose a word $\Delta^h P$ such that h maximal. Let m = h + k

(Note that $h \ge 0$)

(vi) Construct D(P). Let $\overline{P}$ be base of D(P).

Proof: See [6].

Now, let α be a braid and let us connect by a set of parallel arcs that lie outside the square, the points $A_1, A_2, \cdots, A_n$ on the top of a rectangular diagram of a braid α to the points $A_1', A'_2, \cdots, A'_n$ respectively, on the bottom of the same diagram. Then in a natural way we form regular diagram of knot or link from a braid. A knot that has been created in this way is said to be a knot, K, created from the braid α.

Theorem 2 (Alexander's Theorem): Given arbitrary (oriented) knot (or link), then it is equivalent (with orientation) to a knot (or link) that has been formed from a braid [1].

Proof: Let D be an oriented regular diagram of a knot K. Firstly cut the D at a point (not a crossing point) P_0 , and then pull the loose ends apart so that we now have a (1,1) tangle. Figure 2.

We shall show that we can change this tangle into a braid α. The knot, in a sense induced as described previously from the braid, is equivalent to K. If the tangle T has *m* local maxima, then it also has *m* local minima. In the case m = 0, T is a 1-braid and so no proof is required.

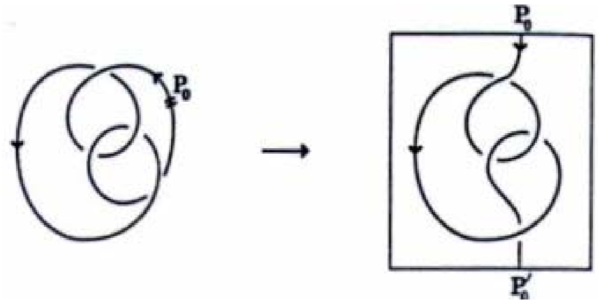
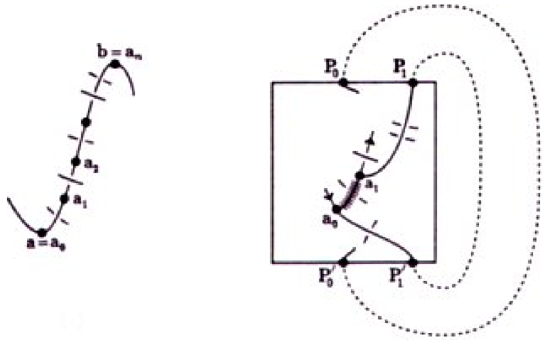


Figure 2



a) (b)

Figure 3

So suppose that $m > 0$, then there exists an arc *ab* in T, which we may say “is rising upwards”, connecting a local minimum *a* to a local maximum *b*. Figure 3.

Further we may assume that *ab* intersects with the other parts of the tangle at *n* places. Let us now mark *n+1* points on *ab*, i.e., $a = a_0, a_1, \cdots, a_n = b$, such that the arc $a_i a_{i+1}$ intersects only one part of the tangle, see Figure 3(b). Next replace the arc $a_0 a_1$ by the much larger arc $a_0 P_1' P_1 a_1$. The large arc $P_1 P_1'$ lies outside the tangle *T*, and the arcs $a_0 P_1'$ and $a_1 P_1$ are selected in such a way that if $a_0 a_1$ passes over (or under) the other segment, then they also pass over (or under) all the other segments. The result of the above manipulations is a (2,2) tangle. Figure 3(b).

It follows immediately that the oriented knot obtained by joining (outside the square) the four endpoints of this (2,2) tangle by curves is equivalent to original knot. By using the same methods as above, with regard to the arcs $a_1 a_2, a_2 a_3, \cdots, a_{n-1} a_n$ we will eventually form a $(n+1, n+1)$ tangle *T* that does not have a local minimum and maximum at *a* and *b* respectively, and further has at most $m-1$ local maxima anda minima. Continuing this procedure, we shall finally create a tangle that has no local maxima and minima. This tangle is our required braid .

Now, we will find braids for 10_5 knots. Regular diagrams of 10_5 given in Figure 2.

Figure 4

Then Theorem 2 is applied for the knot. Then we get 4-braids for this knot. 4 braid is given in Figure 5.

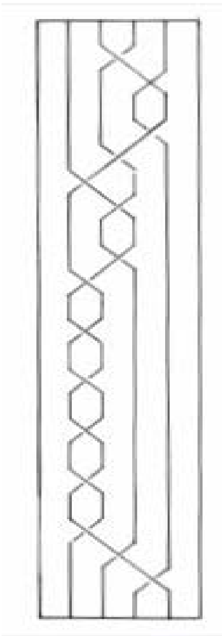


Figure 5

From Figure 5 ,we write $\beta = \alpha_2^{-1}\alpha_3^2\alpha_2^{-1}\alpha_1^{-1}\alpha_2^2\alpha_1^{-6}\alpha_2^{-1}\alpha_3^{-1}$ (Artin Presentation).At the same time, we will write the Garside generators for this braid. We apply Theorem 1 to β braid which we get it from Figure 3. Then we write:

$$D(P) = \{A\sigma_3\, AC\sigma_2^2\,(BC)^6\, A(\sigma_2\sigma_3\sigma_1\,\sigma_2\sigma_1\,)\}$$

where,

$$A = \sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\,, B = \sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\,,\ \ C = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\,.$$

These equations are the representation of 10_5 knots (Garside Rep.).

References

1. Murasugi.,K., Knots Theory and Its Applications, Birkhauser Boston. 1996.
2. Birman,J., Ko,K.,H., Lee,S.,J, A New Aproach To The Word And Conjugacy Problems In The Braid Group, Advances In Math 139,322-353, 1998.
3. Artin, E.,Theorie der Zopfe Hamburg Abh., 47-72., 1925.
4. Garside, F.A., The braid group and other groups. Quart. J. Math. Oxford 20, No 78, 235-254,1969.
5. Paterson,M.S. and Rasborov,A.A., The set minimal braids is co_NP-complete, J. Algorithms. 12 , 393-408, 1991
6. Birman, J., Braids,Links and Mapping Class Groups, Princeton Uni..Press, p-79. 1974.