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FUZZY NUMBERS WITH NON-CENTRAL INFINITE-LEVEL INTERVAL NUMBERS

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Özet: G.Bojadziev ve M. Bojadziev farklı bir sekilde bir merkezli ve bir maksimumlu sonsuz seviyeli aralık sayılarına limit uygulayarak fuzzy number ve aritmetiğini tanımlamıştır. Biz de bu yöntemi merkezi olmayan ve çok maksimumlu sonsuz dereceli aralık sayılarına genelleştirdik.

Anahtar Kelimeler: Sonsuz seviye, Fuzzy sayısı,

MERKEZİ OLMAYAN SONSUZ SEVİYELİ ARALIK SAYILARI İLE FUZZY SAYILARI

Abstract: G.Bojadziev and M. Bojadziev defined differently a fuzzy number and its arithmetic from central infinite-level interval numbers with one maximum by using the limiting process. We generalised this process to the non-central and infinite-level interval numbers with more than one maximum.

Keywords: Fuzzy numbers, non-infinite

1. Introduction

The classical sets and mathematical logic divided everything sharply: yes and no, true and false, white and black, 0 and 1, etc. Fuzzy set and fuzzy logic deal with objects that are matter of degree, with all possible grades of truth between yes and no. The idea was founded in the mid-sixties by L. Zadeh. Fuzzy set and fuzzy logic have been applied virtually in all branches of science, engineering, and socio-economic sciences.

Fuzzy set and fuzzy logic theory are based on Fuzzy numbers which are a generalization of interval numbers. In the first section, we reviewed briefly the terminology, notation and basic properties of interval and its computations. Two excellent books on interval analysis are the ones by Moore [5], and Alefeld and Herzberger [1]. To get the central m-level interval number, G. Bojadziev and M. Bojadziev subdivided an interval into subintervals such that each subinterval has a level. From the central m-level interval number they illustrated central multi-level interval numbers with one maximum and its arithmetic by using limiting process in order to facilitate a direct definition of fuzzy numbers. This new approach was published for the first time in [2]. Nuhmios [6] and Dubois and Prade [3] contributed to the concept of fuzzy numbers. Kaufmann and Gupta [4] were the first to present theory and applications of fuzzy numbers in a comprehensive way. The aim of this paper is presented in the second section where we generalised the central multi-level interval number with one maximum to non-central multi-level interval number with more than one maximum to obtain the fuzzy numbers. Accordingly, in the final section, we described the arithmetic of fuzzy numbers.

Definition 1.1 (Interval Arithmetic) The set of all closed intervals in real number $\Re$ is denoted by $I(\Re)$ and its members, intervals, are denoted by capital letters A, B, C,... . An interval A in $I(\Re)$ is defined as follows:

$$A = [a_1, a_2] = \{x \mid a_1 \le x \le a_2, x \in \Re\} \quad (1.1)$$

where a_1 and a_2 are called *endpoints* of the interval. If for particular real numbers $a_1 = a_2 = a$ the interval number A is given by (1.1) reduces to the real number $a = [a, a]$ which is called *a point interval*. Hence an interval number is a generalisation of a real number.

For the interval $A = [a_1, a_2]$ we define width $w(A)$, magnitude $|A|$, image $-A$, and inverse A^{-1} as follows respectively,

$$w(A) = w[a_1, a_2] = a_2 - a_1 \quad (1.1.a)$$

$$|A| = |[a_1, a_2]| = \max(|a_1|, |a_2|) \quad (1.1.b)$$

$$-A = -[a_1, a_2] = [-a_2, -a_1] \quad (1.1.c)$$

$$A^{-1} = [a_1, a_2]^{-1} = \left[\frac{1}{a_1}, \frac{1}{a_2}\right] \text{ provided } 0 \notin [a_1, a_2] \quad (1.1.d)$$

Definition 1.2 (Arithmetic Operations) For the intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ the *addition* and *multiplication* operations can be established respectively as follows:

$$A + B = [a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2] \quad (1.2.a)$$

$$\begin{aligned} A \cdot B &= [a_1, a_2] \cdot [b_1, b_2] \\ &= [\min(a_1b_1, a_1b_2, a_2b_1, a_2b_2), \max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)] \end{aligned} \quad (1.2.b)$$

The subtraction operation is, in fact, addition of the image of B to A. It means A-B=A+(-B) and the division operation is the multiplication of the inverse of B with A, A:B=AB^{-1}.

It is interesting to see that if $a_1 \ne a_2$, then using (1.1.c) and (1.1.d)

$$A - A = [a_1, a_2] - [a_1, a_2] = [a_1 - a_2, a_2 - a_1] \ne 0 \text{, but } 0 \in A - A \text{ and}$$

$$A \cdot A^{-1} = [a_1, a_2] \cdot [a_1, a_2]^{-1} = [a_1, a_2] \cdot \left[\frac{1}{a_2}, \frac{1}{a_1}\right] \ne 1 \text{, but } 1 \in A \cdot A^{-1}.$$

If $a_1 = a_2 = a$, then in this particular case we obtain the basic results in $\Re$ since $[a, a] = a$. Above interesting points illustrate the fact that interval arithmetic is richer and more general than real numbers arithmetic, but also more complicated.

Theorem 1.3 For the intervals $A = [a_1, a_2]$, $B = [b_1, b_2]$ and $C = [c_1, c_2]$ the following rules hold.

(i) $A + B = B + A$ (iii) $A \cdot B = B \cdot A$

(ii) $(A + B) + C = A + (B + C)$ (iv) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

Proof: The proofs are trivial.

Note that the following rule, called *subdistributivity*, holds $A \cdot (B+C) \subseteq A \cdot B + A \cdot C$, but if $A = [a_1, a_2]$ is substituted for $a = [a, a]$, then the distributivity rule holds $a \cdot (B+C) = a \cdot B + a \cdot C$.

The point intervals $0 = [0,0]$ and $1 = [1,1]$ are the unique neutral elements for interval addition and multiplication. That is, $A = A + 0 = 0 + A$ and $A = A.1 = 1.A$.

Definition 1.4 (Distance) The distance between $A = [a_1, a_2]$ and $B = [b_1, b_2]$ is defined by $d(A,B) = d(B,A) = \max(|a_1 - b_1|, |a_2 - b_2|)$. If A and B are point intervals, $A = [a,a]$, $B = [b,b]$, then (1.4) is reduced to the distance between the real numbers. Really $d([a,a],[b,b]) = \max(|a-b|, |a-b|) = |a-b|$.

2. Fuzzy Numbers

Definition 2.1 (Levels) We call the area $L(\Re^2) = \{(x,y) : y \in [0,m], x, y \in \Re\}$ *levels area*. Levels are parallel lines of x-axis on area $L(\Re^2)$. The levels $y = \lambda_0, \lambda_1, \ldots, \lambda_m$ are geometrically presented on Figure (2.1)

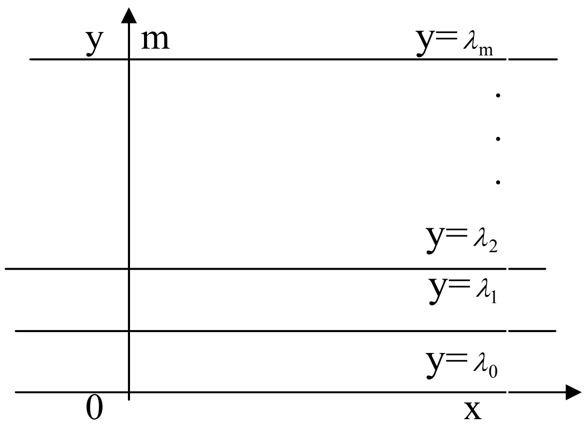

Figure 2.1

where it is clearly seen that $0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_m$.

It is not necessary that the distance between two levels are same. We call the λ_m maximum level.

Definition 2.2 (m-level Interval) An interval is called *supporting interval* if it is subdivided into two or more subintervals. Suppose that the supporting interval $A = [a_0, a_n]$ is subdivided by the numbers $a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n$ into n subintervals $[a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n]$

Each subinterval is presumed to have a level selected from the set $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, such that if $n = m$, then each level has a subinterval. If $n > m$, then at least one of the subintervals will have the same level. In this case, it is possible for each of the subintervals to have the same level with one or more subintervals. If $n < m$, of course, it is nonsense, because subinterval chooses the level, not vice versa.

m-level interval is a union of all the subintervals on each level respectively. For instance,

$$A_{\lambda_2 \lambda_1 \lambda_4 \lambda_3 \cdots \lambda_{m-1} \lambda_m} = [a_0, a_1]_{\lambda_2} \cup [a_1, a_2]_{\lambda_2} \cup \cdots \cup [a_{n-2}, a_{n-1}]_{\lambda_m} \cup [a_{n-1}, a_n]_{\lambda_{m-1}} \quad (2.2)$$

is an m-level interval. Where $[a_i, a_{i+1}]_{\lambda_k}$ indicates subinterval $[a_i, a_{i+1}]$ on level λ_k. The subscript $\lambda_2 \lambda_1 \lambda_4 \lambda_3 \cdots \lambda_m \lambda_{m-1}$ in $A_{\lambda_2 \lambda_1 \lambda_4 \lambda_3 \cdots \lambda_m \lambda_{m-1}}$ is a combination of the level set $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ which gives the address of subintervals as in (2.2). Notation $\cup$ is for union, meaning here that n-subintervals are limited to form a new interval on m-levels. m-level interval is not unique. For every possible combination of the levels $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, the number of these combinations is m!=1.2.3.....(m-1).m, we get an m-level interval. So we present m! different m-level interval.

Definition 2.3 (m-level Interval Number) *m-level interval number*, given as $A_{\langle \lambda_1 \lambda_2 \cdots \lambda_m \rangle}$, is a set of all possible interval numbers. The subscript $\langle \lambda_1, \lambda_2, \ldots, \lambda_m \rangle$ indicates the set of all possible combination of the levels $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. Here the formula $A_{\langle \lambda_1 \lambda_2 \cdots \lambda_m \rangle}$ represents a set of m! different m-level interval.

Consider the general m-level interval number, as in Definition 2.2 with $n > m$, say $n = f(m)$ is an increase linear function. The supporting interval is subdivided into $n = f(m)$ subintervals $[a_i, a_{i+1}]$, $i = 0,1,2,\ldots,n-1$, whose width $w[a_i, a_{i+1}]$ is not necessarily the same. Any point a_i, $1 \le i \le n$, is considered to belong to both level of $[a_{i-1}, a_i]$ and level of $[a_i, a_{i+1}]$. It is graphically represented by a step function on Figure 2.2.

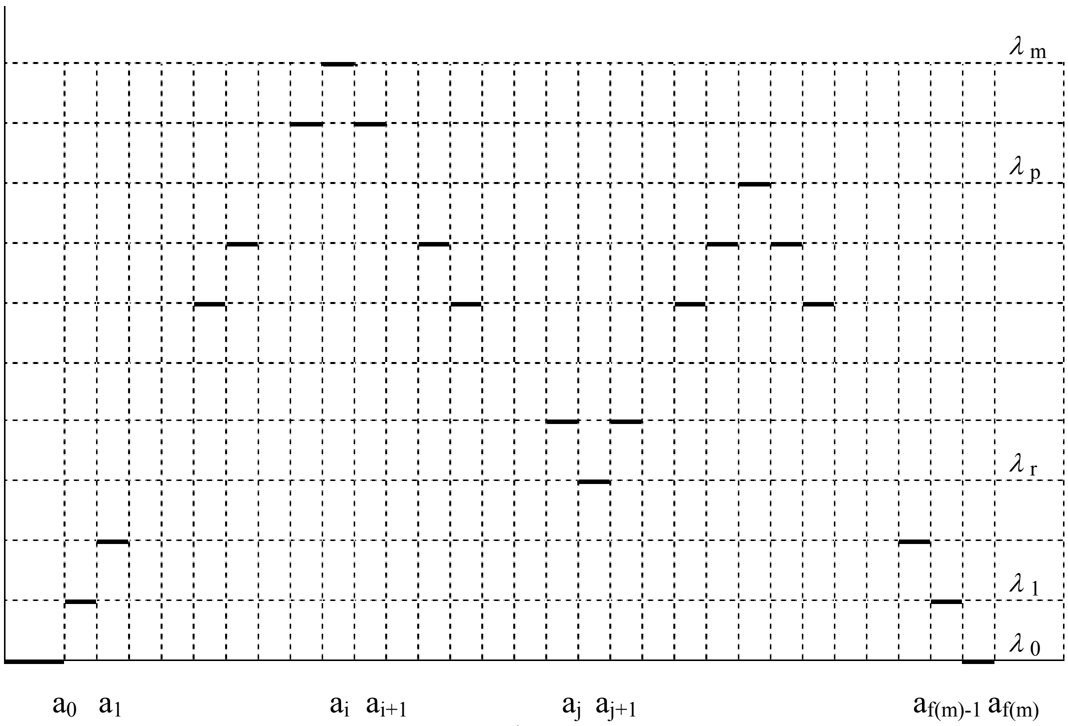

Figure 2.2

where $a_0<a_1<\ldots<a_i<a_{i+1}<\ldots<a_j<a_{j+1}<\ldots<a_k<a_{k+1}<\ldots<a_{f(m)}=a_n$, we have one overall maximum subinterval which is $[a_i, a_{i+1}]$, its level λ_m is the highest. One local maximum subinterval is $[a_k, a_{k+1}]$, its level λ_p is smaller than λ_m. We also have two overall minimum subintervals which are $[a_0, a_1]$ and $[a_{f(m)-1}, a_{f(m)}]$, their level λ_0 is the lowest one. Local minimum subinterval is $[a_j, a_{j+1}]$, its level λ_r. Figure 2.2 has no central. So we call the interval number on Figure 2..2 *non-central m-level interval number* abbreviated by NCmLIN. Each level λ_t has some number of intervals as in following list:

s=1 for t=m, s=2 for $0 \le t < r$ and p<t<m, s=3 for t=r=p, s=4 for r<t<p.

Definition 2.4 (λ_t-Cut) Consider NCmLIN in figure 2.2. If all intervals a level higher than λ_t, $0 \le t \le m$, projected on level λ_t, then we get the interval C_{λ_t} called λ_t*-cut*. That is, λ_t-cut contains subintervals having levels higher than λ_t. The λ_t-cut, C_{λ_t}, represents an interval on level of presumption, λ_t cutting off higher levels $\lambda_{t+1}, \ldots, \lambda_m$, whose subintervals are downgraded to the level λ_t.

Definition 2.5 (Infinite-level Interval Number) Consider again NCmLIN as in figure 2.2. Let us set $\lambda_0 = 0$ and $\lambda_{f(n)} = \lambda_m = 1$. Suppose now that the number of levels between $\lambda_0 = 0$ and $\lambda_{f(n)} = \lambda_m = 1$ increases without bound, i.e. $m \to \infty$. Since $n = f(m)$, $m \to \infty$ implies $n \to \infty$. The width gets smaller and approaches 0; each subinterval $[a_s, a_{s+1}]$ approaches a point interval. As a result of this limiting process, the step function on Figure 2.2 representing the NCmLIN will approach a continuous monotonous function

$$\lambda = F_A(x) : [a_0, a_n] \to [0,1]$$

we call this function an *infinitive-level interval number* abbreviated by ILIN. It is graphically represented in Figure 2 .3.

where $F_1^{-1}(\lambda_m) = F_2^{-1}(\lambda_m)$, $F_2^{-1}(\lambda_r) = F_3^{-1}(\lambda_r)$ and $F_3^{-1}(\lambda_p) = F_4^{-1}(\lambda_p)$. $\lambda = F(x)$ defined on the supporting interval $A = [a_0, a_n] \subset \Re$; $F(x) \in [0,1]$ is normalised. Then we can express the function on Figure 2.3 as:

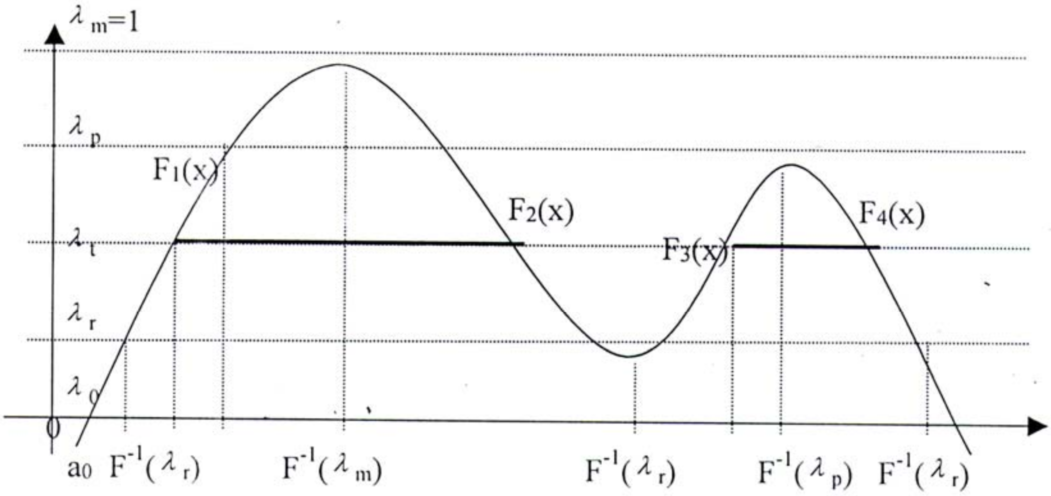

Figure 2.3

$$\lambda = F(x) = \begin{cases} F_1(x) & \text{for} \quad a_0 \le x \le F_1^{-1}(\lambda_m), \\ F_2(x) & \text{for} \quad F_1^{-1}(\lambda_m) \le x \le F_2^{-1}(\lambda_r), \\ F_3(x) & \text{for} \quad F_2^{-1}(\lambda_r) \le x \le F_3^{-1}(\lambda_p), \\ F_4(x) & \text{for} \quad F_3^{-1}(\lambda_p) \le x \le a_n \end{cases}$$

If $\lambda = F(x)$ is a smooth function, then $F_i(x)$, $i = 1,2,3,4$, can be combined in one equation. If however some $F_i(x)$, $i = 1,2,3,4$ or all are piecewise continuous then more equations are needed to express analytically $F(x)$.

As a result of this limiting process the λ_t-cut C_{λ_t} will approach an interval number, written A_{λ_t}, $\lambda_t \in [0,1]$. That is, any λ_t- level interval number A_{λ_t} represents a λ_t-cut of the ILIN. As seen in Figure 2.3 A_{λ_t} is defined by

$$A_{\lambda_t} = \left[F_1^{-1}(\lambda_t),\ F_2^{-1}(\lambda_t)\right]_{\lambda_t} + \left[F_3^{-1}(\lambda_t),\ F_4^{-1}(\lambda_t)\right]_{\lambda_t}$$

By using (1.2.a), A_{λ_t} will be

$$A_{\lambda_t} = \left[F_1^{-1}(\lambda_t) + F_3^{-1}(\lambda_t), F_2^{-1}(\lambda_t) + F_4^{-1}(\lambda_t)\right]_{\lambda_t} \tag{2.5}$$

Definition 2.6 (Fuzzy Number) We define a *fuzzy number* A by the function $\lambda = F(x)$ called *membership function* of the fuzzy number A and write

$$\mathbf{A}: =F(x) \tag{2.6}$$

where the symbol := means "defined as". By definition (2.6) the fuzzy number **A** is identical to its membership function $F(x)$. The ILIN obtained by a limiting process applied on a multi-level interval number is a fuzzy number. The fuzzy number **A** like an ILIN can be used to describe the level of presumption $\lambda \in [0,1]$ corresponding to each $x \in [a_0, a_n]$. Let us assign this level *grade* or *degree* of membership to x. A fuzzy number **A** can be thought of as containing real numbers within an interval A having varying levels of presumption or degrees of membership from 0 to 1. As the level of presumption for an element $x \in A$ increases from 0 to 1, so does our confidence that x belongs more and more to the interval $A = [a_0, a_n]$.

For $\lambda = 0$ the fuzzy number **A** reduces to the supporting interval $A = A_0 = [a_0, a_n]$ at level of presumption 0.

For $\lambda = \lambda_t$, $0 < \lambda_t < 1$, $A_{\lambda_t} = \left[F_1^{-1}(\lambda_t),\ F_2^{-1}(\lambda_t)\right]_{\lambda_t} + \left[F_3^{-1}(\lambda_t),\ F_4^{-1}(\lambda_t)\right]_{\lambda_t}$ is an interval of presumption λ_t.

For $\lambda = 1$ the fuzzy number **A** reduces to the point interval $A_{\lambda_m} = A_1 = \left[F_1^{-1}(1),\ F_2^{-1}(1)\right]_{\lambda_m}$, $F_1^{-1}(1) = F_2^{-1}(1)$.

The concept of fuzzy number is built here on the concept of interval numbers and their generalisation.

3. Arithmetics with Fuzzy Numbers

Fuzzy numbers are generalisation of interval and multi-level interval numbers. Naturally, fuzzy numbers have a greater expressive power than the interval numbers due to the capability of gradation levels of presumption $\lambda \in [0,1]$. Before defining fuzzy number arithmetic, we first have to define a generalisation of interval and multi-level interval arithmetic. Arithmetic operations with interval have been introduced in Section 1. For m-level or multi-level interval we follow the same procedure, but apply it level by level. Therefore it is natural to introduce fuzzy arithmetic as a generalisation of interval and multi-level interval arithmetic, performing operations with intervals at the same level λ of presumption. Consider two fuzzy numbers **A**, **B** with membership function $F(x)$ defined by $F_1, F_2, \ldots, F_i$ with convenient intervals and $G(x)$ defined by $G_1, G_2, \ldots, G_j$ with convenient intervals respectively. The λ-level intervals A_λ and B_λ will be respectively as follows:

$$A_\lambda = \left[F_1^{-1}(\lambda),\ F_2^{-1}(\lambda)\right]_\lambda + \cdots + \left[F_{i-1}^{-1}(\lambda),\ F_i^{-1}(\lambda)\right]_\lambda,\ \lambda \in [0,1]$$

$$B_\lambda = \left[G_1^{-1}(\lambda),\ G_2^{-1}(\lambda)\right]_\lambda + \cdots + \left[G_{j-1}^{-1}(\lambda),\ G_j^{-1}(\lambda)\right]_\lambda,\ \lambda \in [0,1]$$

By using (2.5) we than write addition of λ-level interval as follows:

$$A_\lambda + B_\lambda = \left[(F_1^{-1}(\lambda) + \cdots + F_{i-1}^{-1}(\lambda) + G_1^{-1}(\lambda) + \cdots + G_{j-1}^{-1}(\lambda)),\right.$$
$$\left.(F_2^{-1}(\lambda) + \cdots + F_i^{-1}(\lambda) + G_2^{-1}(\lambda) + \cdots + G_j^{-1}(\lambda))\right]$$

The *subtraction* , *multiplication* and *division* of fuzzy numbers can be defined similarly by using subtraction, multiplication and division of interval respectively.

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