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GEOMETRY OF THE PEDAL OF A SURFACE

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Abstract: Some characteristic properties of the pedal of a surface with respect to an origin point in Euclidean space E^3 have been investigated. The results about the Weingarten mapping and fundamental forms of the pedal have been given.

Key Words: Pedal, pedal transformation, surface pedal.

BİR YÜZEYİN PEDALİNİN GEOMETRİSİ

Özet: Euclidean uzay E^3 de bir yüzeyin, bir orijine göre pedalinin bazı karakteristik özellikleri incelendi. Pedalin Weingarten dönüşümü ve temel formları hakkında sonuçlar verildi.

Anahtar Kelimeler: Pedal, pedal dönüşümü, yüzey pedali.

Introduction

Let M be a smooth immersed regular surface in E^3, which also connected and oriented. We pick an origin O, which does not lie on any tangent plane of M. Hasanis and Koutroufiotis [1] were given the pedal of M with respect to the origin O by virtue of the smooth transformation $\pi: M \rightarrow M_\pi$ and obtained some characteristic properties of M_π giving the condition of being a regular surface of M_π. They have also decomposed the transformation π as $\pi = \sigma \circ \rho$, where and ρ are inversion and reciprocal transformations, respectively, and defined the optical system $\{M, O\}$ calling the surface M to be a reflector and choosing the origin O as a light source. Further more, they defined, at least locally, a smooth mapping τ of some part of the unit sphere with centre O into itself and called this τ as the characteristic mapping of $\{M, O\}$, and shown that if the characteristic mapping τ of M is diffeomorphism then τ^{-1} is the characteristic mapping of a reflector M'. Hence by virtue of τ it has been defined conjugate $\overline{M}$ of M and obtained the relationship between the curvatures of the reflectors M and $\overline{M}$.

Recently, Georgiou et al [2] have generalised some properties of the pedal of a surface in E^3 to a pedal of a hypersurface in E^{n+1}. They have also investigated the system $\{M^n, O\}$ taking an ovaloid M^n of E^{n+1} in place of the hypersurface *M*.

Basic Concepts

Let *M* be a smooth immersed regular connected and oriented surface in E^3. Let (u^1, u^2) be a local coordinate system on *M* and $x = x(u^1, u^2)$ be the parametric representation of *M*, where *x* is a position vector of *M*. The unit normal of *M* at a point *P* of *M* is defined as

$$N = \frac{x_1 \wedge x_2}{|x_1 \wedge x_2|},$$

where $x_i = \dfrac{\partial x}{\partial u^i}$, $i = 1,2$. Differentiable distance and support functions can be defined as $r = |x|$ and $f = -\langle x, N \rangle$ on *M*, respectively, where $\langle\ ,\ \rangle$ is the Euclidean inner product. Let $\overline{\nabla}$ be the standard connection of E^3, and ∇ be the induced connection on *M*. The equations of Gauss and Weingarten are

$$\overline{\nabla}_X Y = \nabla_X Y + \langle AX, Y \rangle, \quad \overline{\nabla}_X N = -AX,$$

respectively, where *X*, *Y* are vector fields tangent to *M* and *A* is the Weingarten mapping or the shape operator of *M*.

Now, let $I \subseteq IR$ be an open interval to be $0 \in I$. Let $\alpha\ I \rightarrow M$ be a differentiable curve satisfiying the condition $\alpha(0) = P$ and $\dfrac{d\alpha}{dt}\Big|_P = X_P$. The shape operator *A* of *M* along α is defined as

$$AX = -\frac{d(N \circ \alpha)}{dt}$$

(see for example, [3]). If *X* and *Y* are two vector fields on *M*, the first, second and third fundamental forms of *M* are $I(X,Y) = \langle X, Y \rangle$, $II(X,Y) = \langle AX, Y \rangle$ and $III(X,Y) = \langle AX, AY \rangle$, respectively. Let k_i's be the eigen values of *A*, then the Gauss and mean curvatures of *M* are $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$, respectively.

Definition. The pedal M_π with respect to the origin point *O* of a oriented surface *M* in E^3 is the surface defined by position vector $x^\pi = -f\,N$ with regard to *O*, and orientation of this surface is an orientation inherited from *M* through the pedal transformation $\pi : M \to M_\pi$. As being P is a point in M, the point $\pi(P)$ is the foot of the perpendicular through O to the tangent plane of M at P. Thus M_π is the locus of the points of intersection of tangent planes of M and the perpendiculars through O to these planes [1].

Suppose now that there exists a point O with the property that it lies on no tangent plane of M. Such a point will henceforth be called an admissible origin for M. If we choose such an O as origin, the corresponding support function clearly never vanishes. In this study we choose an orientation of M to be f>0.

Proposition 1. M_π is a regular surface if and only if the following two statements hold:

i) The Gauss curvature K of M is different from zero everywhere.

ii) The origin O is admissible for M [1].

From now on, when we speak of the pedal M_π, we shall tacitly assume that it is regular.

Proposition 2. The unit normal N^π of M_π at the point $\pi(P)$ is given by

$$N^\pi = \frac{\operatorname{sgn} K}{r}\left(x + 2f\, N\right), \tag{1}$$

where x, N etc. are computed at P [1].

The Shape Operator of the Pedal

Let M be a smooth regular, oriented and connected surface and O be an admissible origin for M. If α is a differentiable curve on M, then $\pi \circ \alpha$ is a differentiable curve on M_π.

Proposition 3. Let M be a smooth regular oriented and connected surface and O be an admissible origin for M. As being M_π is the pedal of M with respect to O, the following two statements hold:

i) $d\pi(X) = -(Xf)N + f\,AX, \quad d\pi(X) \in \chi(M_\pi),$

ii) $A^\pi(d\pi(X)) = \dfrac{\operatorname{sgn} K}{r} X - \dfrac{\langle x, X\rangle}{r^2} N^\pi - \dfrac{2\operatorname{sgn} K}{r} d\pi(X),$

where X is the vector field tangent to M, A and A^π are shape operators of M and M_π, respectively.

Proof. (*i*) Let α be differentiable curve on M such that $\alpha(0) = P$ and $\frac{d\alpha}{dt}|_P = X_P$. Thus, we obtain

$$d\pi(X_{\alpha(t)}) = \frac{d}{dt}\pi(\alpha(t)) \tag{2}$$

$$= -\frac{d\ f(\alpha(t))}{dt} N_{\alpha(t)} - f(\alpha(t))\frac{d\ N_{\alpha(t)}}{dt},$$

since $\frac{d\, f(\alpha(t))}{dt} = X_{\alpha(t)} f$ and $\frac{d\, N_{\alpha(t)}}{dt} = -AX_{\alpha(t)}$,

we have

$$d\pi(X) = -(Xf)N + f\, AX.$$

(*ii*) It is known that

$$A^{\pi}(d\pi(X)) = -\frac{dN^{\pi}_{\pi(\alpha(t))}}{dt}. \tag{3}$$

Therefore, from (1) the unit normal vector of M_{π} at the point $\pi(\alpha(t))$ is

$$N^{\pi} = \frac{\operatorname{sgn} K}{|\alpha(t)|}\left(\alpha(t) + 2f(\alpha(t))N_{\alpha(t)}\right).$$

In this case, the expression (3) is obtained as

$$A^{\pi}(d\pi(X)) = \operatorname{sgn} K\left[\frac{1}{|\alpha(t)|}\frac{d\alpha(t)}{dt} - \frac{\frac{d|\alpha(t)|}{dt}}{|\alpha(t)|^2}\alpha(t) + \frac{2}{|\alpha(t)|}\frac{df(\alpha(t))}{dt}N_{\alpha(t)}\right.$$
$$\left. + \frac{2}{|\alpha(t)|}f(\alpha(t))\frac{dN_{\alpha(t)}}{dt} - \frac{2}{|\alpha(t)|^2}\frac{d|\alpha(t)|}{dt}f(\alpha(t))\,N_{\alpha(t)}\right]. \tag{4}$$

Here

$$\frac{d|\alpha(t)|}{dt} = \frac{\langle \alpha(t), X\rangle}{|\alpha(t)|}. \tag{5}$$

Since every points of M is on a parameter curve as being x is the position vector of M, we can take x in place of $\alpha(t)$ and r in place of $|\alpha(t)|$. Substitution of these with (5) in (4) leads to

$$A^{\pi}(d\pi(X)) = \operatorname{sgn} K\left[\frac{1}{r}X - \frac{\langle x, X\rangle}{r^3}x + \frac{2}{r}(Xf)\,N - \frac{2}{r}f\ AX - \frac{2\langle x, X\rangle}{r^3}f\ N\right]. \tag{6}$$

Using $\frac{\langle x, X\rangle}{r^3}x+\frac{2\langle x, X\rangle}{r^3}f\,N=\operatorname{sgn}K\frac{\langle x, X\rangle}{r^2}N^\pi$ and $\frac{2}{r}(Xf)\,N-\frac{2}{r}f\,AX=-\frac{2}{r}d\pi(X)$ we have

$$A^\pi(d\pi(X))=\frac{\operatorname{sgn}K}{r}X-\frac{\langle x, X\rangle}{r^2}N^\pi-\frac{2\operatorname{sgn}K}{r}d\pi(X).$$

Corollary 1. If the position vector of a point P of M is in the direction of the unit normal at this point, the following two statements hold:

i) $d\pi(X)=f\,AX$,

ii) $A^\pi(d\pi(X))=\operatorname{sgn}K\frac{1-2fk}{fkr}d\pi(X)$,

where X is the principal vector and k is the principal curvature corresponding to X.

Proof. (*i*) Since the support function of M is $f=-\langle x, N\rangle$ and by hypothesis, it is clearly seen that $Xf=0$. Therefore, from Proposition 3 (i)

$$d\pi(X)=f\,AX.$$

(*ii*) If X is a principal vector and k is the principal curvature corresponding to X, then

$$AX=kX \tag{7}$$

and from (*i*)

$$X=\frac{1}{fk}d\pi(X). \tag{8}$$

Using the hypothesis of the Corollary 1 and Proposition 3 (*ii*), we obtain

$$A^\pi(d\pi(X))=\frac{\operatorname{sgn}K}{r}X-\frac{2\operatorname{sgn}K}{r}d\pi(X). \tag{9}$$

Substitution of (8) in (9), (9) becomes

$$A^\pi(d\pi(X))=\operatorname{sgn}K\frac{1-2fk}{fkr}d\pi(X).$$

Thus, the proof is completed.

We conclude from Corollary 1 (ii) that the principal curvature k^π corresponding to $d\pi(X)$ at the point $\pi(P)$ of M_π is

$$k^\pi=\operatorname{sgn}K\frac{1-2fk}{fkr}.$$

Corollary 2. Let us denote the third fundamental form of M by III and I^{π} the first fundamental form of M_{π}, there is a relationship between these two forms as follows:

$$\mathrm{I}^{\pi}\left(d\pi(X), d\pi(Y)\right) = (Xf)(Yf) + f^{2}\,\mathrm{III}(X,Y),$$

where X and Y are vector field tangent to M.

Proof. It can be written

$$\mathrm{I}^{\pi}\left(d\pi(X), d\pi(Y)\right) = \langle d\pi(X), d\pi(Y)\rangle .$$

Considering Proposition 3 (*i*) with this the proof is clear.

Corollary 3. The second fundamental form of M can be written as a linear combination of the first and second fundamental forms of M_{π}.

Proof. The second fundamental form of M_{π} is

$$\mathrm{II}^{\pi}\left(d\pi(X), d\pi(Y)\right) = \langle A^{\pi}\left(d\pi(X)\right), d\pi(Y)\rangle ,$$

from Proposition 3

$$\begin{aligned}\mathrm{II}^{\pi}\left(d\pi(X), d\pi(Y)\right) &= \frac{\operatorname{sgn} K}{r}\langle X, d\pi(Y)\rangle - \frac{2\operatorname{sgn} K}{r}\langle d\pi(X), d\pi(Y)\rangle \\ &= \frac{\operatorname{sgn} K}{r} f\,\mathrm{II}(X,Y) - \frac{2\operatorname{sgn} K}{r}\mathrm{I}^{\pi}\left(d\pi(X), d\pi(Y)\right),\end{aligned}$$

with a small manipulation it is written as the following simplified form

$$\mathrm{II} = \frac{2}{f}\mathrm{I}^{\pi} + \operatorname{sgn} K \frac{r}{f}\mathrm{II}^{\pi} ,$$

which completes the proof.

Theorem 1. The pedal transformation $\pi\colon M \to M_{\pi}$ preserves the asymptotic vector X on M if and only if $d\pi(X)$ is conjugate to each vector of $T_{M_{\pi}}\left(\pi(P)\right)$.

Proof. Let π preserve the asymptotic vector X on M, i.e., $\langle AX, X\rangle = 0$ implies $\langle A^{\pi}\left(d\pi(X)\right), d\pi(X)\rangle = 0$. In this case from Proposition 3 (*ii*)

$$\frac{1}{r}\langle d\pi(X), X\rangle - \operatorname{sgn} K \frac{2}{r}\langle d\pi(X), d\pi(X)\rangle = 0. \tag{10}$$

The first term on the left hand side of this equation is equal to zero, since $\langle d\pi(X), X\rangle = 0$. So, from equation (10) we obtain $\langle d\pi(X), d\pi(X)\rangle = 0$ which implies $d\pi(X)=0$. Thus, $A^{\pi}(d\pi(X)) = 0$. As a result of this, we obtain $\langle A^{\pi}(d\pi(X)), d\pi(Y)\rangle = 0$ for each element $d\pi(Y)$ of $T_{M_\pi}(\pi(P))$. This means that $d\pi(X)$ is conjugate to each vector of $T_{M_\pi}(\pi(P))$.

Conversely, let $d\pi(X)$ be conjugate to each vector of $T_{M_\pi}(\pi(P))$, i.e., for each element $d\pi(Y)$ of $T_{M_\pi}(\pi(P))$

$$\langle A^{\pi}(d\pi(X)), d\pi(Y)\rangle = 0. \tag{11}$$

Equation (11) is also satisfied by taking $d\pi(X)$ in place of $d\pi(Y)$. Thus, equation (11) becomes $\langle A^{\pi}(d\pi(X)), d\pi(X)\rangle = 0$ which is meant to be the asymptotic direction of $d\pi(X)$.

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