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A NOTE ON $\left|\overline{N},p_n;\delta\right|_k$ SUMMABILITY FACTORS

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Abstract: Two theorems of Lal [4] on $\left|\overline{N},p_n\right|$ summability method have been generalized for $\left|\overline{N},p_n;\delta\right|_k$ summability method, where $k \geq 1$ and $\delta \geq 0$.

$\left|\overline{N},p_n;\delta\right|_k$ TOPLANABİLME ÇARPANLARI ÜZERİNE BİR NOT

Özet: Lal [4] in $\left|\overline{N},p_n\right|$ toplanabilme metodu ile ilgili iki teoremi, $k \geq 1$ ve $\delta \geq 0$ olmak üzere $\left|\overline{N},p_n;\delta\right|_k$ toplanabilme metodu için genelleştirildi.

Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0,\ i \geq 1). \tag{1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of the $(\overline{N},p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [3]). The series $\sum a_n$ is said to be summable $\left|\overline{N},p_n\right|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty \tag{3}$$

and it is said to be summable $\left|\bar{N}, p_n; \delta\right|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [2])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |\Delta t_{n-1}|^k < \infty. \tag{4}$$

If we take $k = 1$ and $\delta = 0$, then $\left|\bar{N}, p_n; \delta\right|_k$ summability is the same as $\left|\bar{N}, p_n\right|$ summability.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. The Fourier series of $f(t)$ is

$$f(t) \cong \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) \,. \tag{5}$$

The Results of Lal

Lal [4] proved the folloving theorems for $\left|\bar{N}, p_n\right|$ summability methods.

Theorem A. If the sequence (s_n) is bounded and (λ_n) is a sequence such that

$$\sum_{n=1}^{m}\frac{p_n}{P_n}\left|\lambda_n\right| = O(1) \quad \text{as} \quad m \to \infty, \tag{6}$$

$$\sum_{n=1}^{m}\left|\Delta\lambda_n\right| = O(1) \quad \text{as} \quad m \to \infty, \tag{7}$$

then the series $\sum a_n \lambda_n$ is summable $\left|\bar{N}, p_n\right|$.

Theorem B. The summability $\left|\bar{N}, p_n\right|$ of the series $\sum A_n(t)\lambda_n$ at a point is a local property of the generating function if the conditions (6)-(7) of Theorem A are satisfied.

The Main Result

The aim of this paper is to generalize above theorems for $\left|\overline{N}, p_n\right|_k$ summability methods, where $k \geq 1$ and $\delta \geq 0$. Now, we shall prove the following theorems.

Theorem 1. Let k ≥ and $0 \leq \delta k < 1$. If the sequence (s_n) is bounded and the sequences (λ_n) and (p_n) satisfy the following conditions

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \left|\lambda_n\right|^k = O(1) \quad \text{as} \quad m \to \infty, \tag{8}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \left|\Delta\lambda_n\right| = O(1) \text{ as } \quad m \to \infty, \tag{9}$$

and

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right\}, \tag{10}$$

then the series $\sum a_n\lambda_n$ is summable $\left|\overline{N}, p_n; \delta\right|_k$.

Theorem 2. Let k ≥ and $0 \leq \delta k < 1$. The summability $\left|\overline{N}, p_n; \delta\right|_k$ of the series $\sum A_n(t)\lambda_n$ at a point is a local property of the generating function if the conditions (8) and (9) are satisfied.

Remark. It may be noted that, if we take $\delta = 0$ and $k = 1$ in Theorem 1 and Theorem 2, then we get Theorem A and Theorem B, respectively. In this case condition (10) reduces to

$$\sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = \sum_{n=v+1}^{m+1} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) = O\left(\frac{1}{P_v}\right) \quad as \quad m \to \infty,$$

which always holds.

Proof of Theorem 1. Let (T_n) denotes the $(\overline{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v .$$

Then

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v , \quad n \geq 1, \quad (P_{-1} = 0).$$

By Abel's transformation, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta(P_{v-1} \lambda_v) s_v + \frac{p_n s_n \lambda_n}{P_n}$$

$$= -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta \lambda_v + \frac{p_n s_n \lambda_n}{P_n}$$

$$= T_{n,1} + T_{n,2} + T_{n,3} , \quad say.$$

To complete the proof of Theorem 1, by Mikowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} | T_{n,r} |^k < \infty, \quad for \quad r = 1,2,3. \tag{11}$$

Now, applying Hölder's Inequality, we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} | T_{n,1} |^k \leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \{ \sum_{v=1}^{n-1} p_v | s_v || \lambda_v | \}^k$$

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} | T_{n,1} |^k = O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} | s_v |^k p_v | \lambda_v |^k \times \{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v \mid \lambda_v \mid^k$$

$$= O(1)\sum_{v=1}^{m} p_v \mid \lambda_v \mid^k \sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1} \mid \lambda_v \mid^k = O(1) \quad as \quad m \to \infty,$$

by virtue of hypotheses of Theorem 1. Again

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,2} \mid^k \le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}^k}\{\sum_{v=1}^{n-1} P_v \mid s_v \| \Delta\lambda_v \mid\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \| s_v \mid^k \times\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid$$

$$= O(1)\sum_{v=1}^{m} P_v \mid \Delta\lambda_v \mid \sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k} \mid \Delta\lambda_v \mid = O(1) \quad as \quad m \to \infty,$$

by virtue of hypotheses of Theorem 1.

Finally, we have that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,3} \mid^k = \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid \frac{p_n s_n \lambda_n}{P_n} \mid^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1} |\lambda_n|^k = O(1) \quad as \quad m \to \infty,$$

by virtue of hypotheses of Theorem 1. Therefore, we get that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,r}|^k = O(1) \quad as \quad m \to \infty, \quad for \quad r = 1,2,3.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Theorem 2 is a necessary consequence of Theorem 1.

If we take $\delta = 0$ and $p_n = 1$ for all values of n in these theorems, then we get two results related to $|C,1|_k$ summability methods.

References

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