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AN APPLICATION OF DEGREE THEORY II

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Abstract: We calculate the degree of some functions.

DERECE TEORİSİNİN UYGULAMASI II

Özet: Bazı fonksiyonların dereceleri hesaplanmıştır.

Introduction

Let $\Omega \subset R^n$ open and bounded in R^n and $f:\Omega \to R^m$. Recall that, f is said to smooth if there exists an open set $U \supset \Omega$ in R^n and a function $F:U \to R^m$ such that if $F=(F_1,F_2,...;F_m)$, then F_k has partial derivates of all orders for $i \le k \le m$ and $F|_\Omega = f$, where $F|_\Omega$ is the restriction of F to Ω [2]. Let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω, respectively [2]. $B_r(x_0)=\{x \in R^n : \|x-x_0\| < r$ denotes open ball of center x_0 and radius $r>0$, where $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$.

Definition 1. Let $f:\Omega \subseteq R^n \to R^m$ be smooth.Then $q \in R^m$ is called a regular value of f if $x \in f^{-1}(q)$ imlies that the matrix

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

has rank m ($(m \leq n)$.(Otherwise, q is called a critical value of f) [2].

Note. If $q \in R^n$ is a regular value of f with $q \notin f(\partial\Omega)$, then $f^{-1}(q)$ is a finite set [1].

Definition 2. Let A be a finite set and card A denote the Cardinality of A .
$f^{-1}(q)^+ = \{x \in f^{-1}(q) : \det DF(x) > 0$, and $f^{-1}(q)^- = \{x \in f^{-1}(q) : \det DF(x) < 0\}$. Then $d(f,\Omega,q) = \operatorname{Card} f^{-1}(q)^+ - \operatorname{Card} f^{-1}(q)^-$ is called the (Brouwer) degree of f with respect to Ω and q [1].

Theorem 1. (Special case of Homotopy Invariance Theorem). $H : \overline{U} \times [0,1] \subset R^{n+1} \to R^n$ be smooth and let $U \subseteq R^n$ be open and bounded.Suppose that $f(x) = H(x,0), \forall x \in \overline{U}$ and $g(x) = H(x,1), \forall x \in \overline{U}$ .Suppose that $q \in R^n$ is a regular value for $H|_{U \times [0,1]}, f|_U$, and $g|_U$ and also that $q \notin H(\partial U x[0,1])$. Then $d(f,U,q) = d(g,U,q)$ [1].

It is well known (see [1]) that there is only one function

$$d : \{(f,\Omega,y) : \Omega \subset R^n \text{ open and bounded, } f : \overline{\Omega} \to R^n \text{ continuous, } y \in R^n \setminus f(\partial\Omega)\} \to Z$$

satifying

(1) $d(id,\Omega,y) = 1$ for $y \in \Omega$.

(2) $d(f,\Omega,y) = d(f,\Omega_1,y) + d(f,\Omega_2,y)$, whenever Ω_1, Ω_2 are disjoint open subsets of Ω such that $y \notin f(\overline{\Omega}) \setminus (\Omega_1 \cup \Omega_2))$.

(3) $d(h(t,\cdot),\Omega,y(t))$ is independent of $t \in J = [0,1]$ whenever $h : J \times \overline{\Omega} \to R^n$ is continuous, $y : J \to R^n$ is continuous and $y(t) \notin h(t,\partial\Omega)$ for all $t \in J$.

Definition 3. Let $\Omega \subset R^n$ be open and bounded, $f \in \overline{C^1(\Omega)}$ and $y \in R^n \setminus f(\partial\Omega \cup S_f)$. Then we define $d(f,\Omega,0) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} \det Df(x)$ [1].

Definition 4. If $f : \overline{\Omega} \to R^n$ is smooth and $q \notin f(\partial\Omega)$, then $d(f,\Omega,q) = d(f,\Omega,q_1)$ where q_1 is any rugular value of $f_{|\Omega}$ such that $|q_1 - q| < \min|f(x) - q|$, $x \in \partial\Omega$ [1].

Note that, if A is a linear map with $\det A \neq 0$, then $d(A,\Omega,0) = \operatorname{sgn} \det A$, the sign of $\det A$.

Results

Lemma 1. Let A be a real $n \times n$ matrix and $e^A = \sum_{m \geq} \frac{A^m}{m!}$. Then $\det e^A > 0$.

Proof. Let $M = \{B : B \text{ is } n \times n \text{ matrix}\}$, $\Omega = (0,1)$, and define $H : [0,1] \to M$ by $H(t) = e^{tA}$. Note that $H(0) = e^0 = id$, where id is the $n \times n$ matrix and $H(1) = e^A$. By property (3) of d,wehave $d(H(1),\Omega,0) = d(H(0),\Omega,0) = d(id,\Omega,0) = 1$. Hence, $1 = d(H(1),\Omega,0) = d(e^A,\Omega,0)$ $= \operatorname{sgn} \det e^A$ and consequently $\det e^A > 0$.

Lemma 2. Let A be a real $n \times n$ matrix with $\det A > 0$. Then there exists a continuous map H from $[0,1]$ into the space of all $n \times n$ matrices such that $H(0) = id$, $H(1) = A$ and $\det H(t) > 0$ for all $t \in [0,1]$.

Proof. Let $M = \{B : B \text{ is } n \times n \text{ matrix}\}$ and $\Omega = (0,1)$. Define $H : [0,1] \to M$, by $H(t) = tA + (1-t)id$. Note that $H(0) = id$, $H(1) = A$ and H is continuous. $H(t)$ is a linear map for all $t \in [0,1]$, $d(H(t),\Omega,0) = \text{sign} \det H(t)$. But $H(1) = A$, and by property (3) of d, $\operatorname{sgn} \det H(t) = d(H(t),\Omega,0) = d(H(1),\Omega,0) = \operatorname{sgn} \det A = 1$ by assumption that $\det A > 0$. Therefore $\det H(t) > 0$.

Lemma 3. Let $\Omega \subset R$ be open interval with $0 \in \Omega$ and $f(x) = \alpha x^k$ with $\alpha \neq 0$. Then

$$d(f,\Omega,0) = \begin{cases} 0, & \text{if } k \text{ is even,} \\ \operatorname{sgn} \alpha, & \text{if } k \text{ is odd.} \end{cases}$$

Proof. $f'(x) = k\alpha x^{k-1} = 0$, $x = 0$ is a critical point. Choose $\varepsilon > 0$ with

$$|\varepsilon - 0| \le \text{dist}(0, \{f(a), f(b)\} = f(\partial\Omega)) = \min\{|f(a)|, |f(b)|\} = r.$$

Then ε is a regular value of f and by definitions 3 and 4, we have

$$d(f,\Omega,0) = d(f,\Omega,\varepsilon) = \sum_{x\in f^{-1}(\varepsilon)} \text{sgn} \det Df(x).$$

Notice that $Df(\varepsilon) = f'(\varepsilon) = k\alpha\varepsilon^{k-1} \ne 0$. Therefore

$$d(f,\Omega,0) = \sum_{x\in f^{-1}(\varepsilon)} \text{sgn} \det Df(x) = \sum_{x\in f^{-1}(\varepsilon)} \text{sgn}(k\alpha\varepsilon^{k-1}) = \begin{cases} 1-1=0, & \text{if } k \text{ is even,} \\ \text{sgn}\alpha, & \text{if } k \text{ odd.} \end{cases}$$

Example 1. Let $f : R^2 \to R^2$

$$(x,y) \to f(x,y) = (x^2 - y^2,\ 2xy) \text{ and } \Omega = B_r(0).$$

Show that $d(f,\Omega,(0,0)) = 2$.

Proof. Notice that $Df(x,y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$. Hence,

$$\det Df(x,y) = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) = 0$$

which shows that $(0,0)$ is a critical point. Let $p = (0,0)$ and $q = (\varepsilon,0)$, where $0 < \varepsilon < r$.

$\varepsilon = |f(x,y) - q| \le \min\{\ |f(x,y) - p|\ \} \le \min|f(x,y)| = r, \quad (x,y) \in \partial\Omega.$ $\qquad f(x,y) = q = (\varepsilon,0)$. It follows that $x^2 - y^2 = \varepsilon$ and $2xy = 0$. Hence $y = 0$ and $x = \pm\sqrt{\varepsilon}$. So $q_1 = (\sqrt{\varepsilon},0)$ and $q_2 = (-\sqrt{\varepsilon},0)$ are regular points of f. Since

$$\det Df(q_1) = \begin{vmatrix} 2\sqrt{\varepsilon} & 0 \\ 0 & 2\sqrt{\varepsilon} \end{vmatrix} = 4\varepsilon > 0 \text{ and } \det Df(q_2) = \begin{vmatrix} -2\sqrt{\varepsilon} & 0 \\ 0 & -2\sqrt{\varepsilon} \end{vmatrix} = 4\varepsilon > 0$$

we get $d(f,\Omega,p) = \sum_{(x,y)\in f^{-1}(q)} \text{sgn} \det Df(x) = 1 + 1 = 2$. Therefore, $d(f,\Omega,p) = 2$.

Example 2. Let $f:R^2 \to R^2$, $f(x,y)=\left(e^x \cos y,\ e^x \sin y\right)$, $\Omega=(-a,a)\times(-b,b)$, where $a,b>0$ and $p=(1,0)$. Show that $d(f,\Omega,(1,0))=2m+1$, where $m\in N$.

Proof. Note that $Df(x,y)=\begin{bmatrix} e^x\cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y\end{bmatrix}$. We have

$$\det Df(x,y)=\begin{vmatrix} e^x\cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y\end{vmatrix}=e^{2x}(\cos^2 y+\sin^2 y)=e^{2x}\neq 0.$$

Hence $\det Df(x)>0$. Let $f(x,y)=p=(1,0)$. It follows from $e^x\cos y=1$ and $e^x \sin y=0$ that $x=0$ and $y=\pm 2n\pi$, $n\in N$. Let $2\pi m<b<2\pi(m+1)$. Then

$$d(f,\Omega,p)=\sum_{(x,y)\in f^{-1}(1,0)} \operatorname{sgn}\det Df(x,y)=2m+1,$$

since there are 2m+1 points in the interval $(-b,b)$.

Theorem 2. Let $\Omega=(a,b)\subset R$ and $f:[a,b]\to R$ be a continuous map such that $f(a)f(b)\neq 0$. Then $d(f,\Omega,0)=\frac{1}{2}\{\operatorname{sgn} f(b)-\operatorname{sgn} f(a)$.

Proof. Let $f(a)f(b)\neq 0$ and g be a linear function between $(a,f(a))$ and $(b,f(b))$, i.e.,

$$g(x)=\frac{f(b)-f(a)}{b-a}(x-a)+f(a).$$

Define a homotopy $H:[0,1]\times[a,b]\to R$, by $H(t,x)=t\,f(x)+(1-t)g(x)$. Hence $H(0,x)=g(x)$, $H(1,x)=f(x)$. It follows easily that $H(t,a)\neq 0\neq H(t,b)$ for all $t\in[0,1]$. We have $Dg(x)=g'(x)=\frac{f(b)-f(a)}{b-a}$. Note that 0 is a regular value of g, f and H. By Theorem 1.1, we get $d(f,\Omega,0)=d(g,\Omega,0)$. By definiton 4, $d(g,\Omega,0)=\sum_{x\in g^{-1}(0)} \operatorname{sgn}\det Dg(x)$.

Since $g'(x)=\frac{f(b)-f(a)}{b-a}$ and $b-a>0$, we need to consider the sign of $f(b)-f(a)$.

$$\text{If } f(b)f(a) > 0, \text{ then } \operatorname{sgn} f(b) - \operatorname{sgn} f(a) = 0,$$
$$\text{If } f(b)f(a) < 0, \text{ then } \operatorname{sgn} f(b) - \operatorname{sgn} f(a) = \pm 2.$$

Hence ,

$$d(f,\Omega,0) = d(g,\Omega,0) = \begin{cases} 0, & \text{if } f(a)f(b) > 0, \\ 1, & \text{if } f(a)f(b) < 0 \text{ and } f(b) > 0, \\ -1, & \text{if } f(a)f(b) < 0 \text{ and } f(b) < 0, \end{cases}$$

which shows that $d(f,\Omega,0) = \frac{1}{2}\{\operatorname{sign} f(b) - \operatorname{sign} f(a)\}$.

References

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[2] J.T. Schwartz, Nonlinear functional analysis, Gordon and Breach, New York, 1969.