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NUMERICAL SOLUTIONS OF BURGERS-LIKE EQUATIONS : A LINEARIZED IMPLICIT FINITE DIFFERENCE METHOD

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Abstract: A linearized implicit finite-difference method is presented for numerical solutions of the one-dimensional Burgers-like equations. The method has been used successfully to obtain accurate numerical solutions even for small values of viscosity term ν. Results obtained by the present method using Gauss elimination technique for some values of ν have been compared with the exact one which are found to be in good agreement with each other.

Key Words: Burgers equation; linearized implicit finite-difference.

BURGERS TİPİ DENKLEMLERİN NÜMERİK ÇÖZÜMLERİ: LİNEERLEŞTİRİLMİŞ BİR KAPALI SONLU FARK YÖNTEMİ

Özet: Bir-boyutlu Burgers tipi denklemlerin nümerik çözümleri için lineerleştirilmiş bir kapalı sonlu fark yöntemi sunuldu. Yöntem, viskosite term ν'nin küçük değerleri için de doğru nümerik çözümleri elde etmek için başarılı bir şekilde kullanıldı.Viskositenin bazı değerleri için Gauss eleme tekniği kullanılarak sunulan yöntem ile elde edilen sonuçlar tam çözümle karşılaştırıldı ve sonuçların birbiri ile iyi uyuştuğu gözlendi.

Anahtar Kelimeler: Burgers denklemi, lineerleştirilmiş kapalı sonlu fark.

Introduction

A study of properties of Burgers equation is great importance since it is used as a mathematical model in turbulence problems and in the theory of shock waves. The one-dimensional Burgers equation, which was first introduced by Bateman [1] and later treated by Burger [2],

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=\nu\frac{\partial^2 U}{\partial x^2},\ 0<x<1,\ t>0, \tag{1}$$

is one of a few well known non-linear partial differential equation which can be solved analytically for a restricted set of initial conditions [3, 4].

Burgers equation has motivated considerable research into numerical methods by many authors [5-12] since the parameter $\nu>0$ in Eq. (1), which is the so-called viscosity term, plays an important role in determining the behaviour of the solution. They have used a variety of numerical techniques specially based on finite difference, finite element and boundary element methods to solve Eq. (1) particularly for small values of ν. Benton and Platzman [13] surveyed exact solutions of the one-dimensional Burgers-like equations. In many cases, these solutions involve infinite series which may converge very slowly for small values of viscosity $\nu > 0$ (see e.g., [8]). Recently, Kutluay et al. [14] proposed the exact-explicit finite difference method to the Burgers-like problems to obtain numerical solutions of adequate accuracy.

In this paper, we have applied a linearized implicit finite difference method to the Burgers equation with a set of initial and boundary conditions to obtain its numerical solutions. To make a comparison of numerical solutions with exact ones we have chosen two test problems given in the following section so that each of them has an exact (Fourier) solution.

Statements of Problems

We consider the Burgers equation (1) with the boundary conditions

$$U(0,t) = 0,\ t > 0$$
$$U(1,t) = 0,\ t > 0$$

and with the following initial conditions.

Problem (a) : For this problem, the initial condition is

$$U(x,0) = \sin(\pi x),\ 0 < x < 1.$$

The (exact) Fourier series solution of this problem given by Cole [3] is

$$U(x,t) = 2\pi\nu \frac{\sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\nu t) n \sin(n\pi x)}{a_0 + \sum_{n=1}^{\infty} a_n \exp(-n^2\pi^2\nu t)\cos(n\pi x)}, \tag{2}$$

where

$$a_0 = \int_0^1 \exp\{-(2\pi\nu)^{-1}[1-\cos(\pi x)]\}dx,$$
$$a_n = 2\int_0^1 \exp\{-(2\pi\nu)^{-1}[1-\cos(\pi x)]\}\cos(n\pi x)dx, \quad (n = 1,2,3,\ldots).$$

Problem (b) : The initial condition for this problem is

$$U(x,0) = 4x(1-x), 0 < x < 1.$$

The exact solution of this problem is given by Eq. (2) with the coefficients

$$a_0 = \int_0^1 \exp\{-x^2(3\nu)^{-1}(3-2x)\}dx,$$
$$a_n = 2\int_0^1 \exp\{-x^2(3\nu)^{-1}(3-2x)\}\cos(n\pi x)dx, \quad (n = 1,2,3,\ldots).$$

Method of Solution

The solution domain $0 \le x \le 1$, $t > 0$ is divided into intervals $h \equiv \Delta x$ in the direction of the spatial variable x and $k \equiv \Delta t$ in the direction of time t such that $x_i = ih$, $i = 0(1)N$ $(Nh = 1)$; $t_j = jk$, $j = 0(1)J$ and $U(x_i, t_j)$ is denoted by $U_{i,j}$.

In the finite difference method, the dependent variable and its derivatives are approximated by the finite difference approximation. This approximation will lead to either a single explicit equation or a system of difference equations. Applying the classical implicit finite-difference method to non-linear problems normally give non-linear system of equations which cannot be solved directly.

In practice, usually a very specialised form of non-linear equation is considered rather than the more general form of non-linear equation since the analysis of stability becomes more complicated. For example, Richtmyer and Morton [15] considered the non linear problem of the form

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U^n}{\partial x^2}$$

with $n = 5$ and Douglas [16] considered the quasi-linear parabolic equation

$$\frac{\partial^2 U}{\partial x^2} = f(x,t,U)\frac{\partial U}{\partial t} + g(x,t,U),\ f \ge 0.$$

In a way, this specialised approach probably relaxes the complications or difficulties which may arise in the analysis of convergence and stability. For non-linear problems, stability depends not only on the form of the finite difference system but also generally upon the solution being obtained. In practice, in the case of conditionally stable difference approximations it is necessary to alter the stability parameter $r = k/h^2$ in order to restore the stability.

The symbol is the central difference operator defined by $\delta_x U_{i,j} = U_{i+1,j} - U_{i-1,j}$. Using the forward difference approximation for $\partial U/\partial t$, the weighted central difference approximation for $\partial U^2/\partial x$ and the central difference approximation for $\partial^2 U/\partial x^2$ at the point $(i, j+1)$, i.e.,

$$\frac{\partial U}{\partial t} \cong \frac{U_{i,j+1} - U_{i,j}}{k},$$

$$\frac{\partial U^2}{\partial x} \cong \frac{1}{2h}\{\theta(U_{i+1,j+1}^2 - U_{i-1,j+1}^2) + (1-\theta)(U_{i+1,j}^2 - U_{i-1,j}^2)\},$$

and

$$\frac{\partial^2 U}{\partial x^2} \cong \frac{1}{h^2}(U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}),$$

respectively. The Burgers equation (1) can be written as

$$\frac{\partial U}{\partial t}+\frac{1}{2}\frac{\partial U^2}{\partial x}=\nu\frac{\partial^2 U}{\partial x^2}. \tag{3}$$

Using the above difference approximations, Eq. (3) yields the system of algebraic equations

$$\begin{aligned}&\frac{U_{i,j+1}-U_{i,j}}{k}+\frac{1}{4h}\left\{\theta\left(U^2_{i+1,j+1}-U^2_{i-1,j+1}\right)+(1-\theta)\left(U^2_{i+1,j}-U^2_{i-1,j}\right)\right\}=\\&\frac{\nu}{h^2}\left(U_{i+1,j+1}-2U_{i,j+1}+U_{i-1,j+1}\right)\end{aligned} \tag{4}$$

for $i=1(1)N-$ and $j=0(1)J$ with a truncation error of $O(k)+O(h^2)$. For $=0$, the scheme (4) gives a linear system of equations in $U_{i,j+1}$. For $0<\theta\le 1$, the scheme is a non-linear system of equations in $U_{i,j+1}$ and it needs to use an iteration technique to evaluate the solution. Caldwell and Smith [17] solved iteratively various finite-difference schemes of Burgers-like equations by using the Gauss-Seidel method. However, in some cases a linearization technique is also possible.

By using the operator the replacement (4) can be written as

$$\begin{aligned}&\frac{U_{i,j+1}-U_{i,j}}{k}+\frac{1}{4h}\left\{\theta\delta_x\left(U^2_{i,j+1}\right)+(1-\theta)\delta_x\left(U^2_{i,j}\right)\right)\\&=\frac{\nu}{h^2}\left(U_{i+1,j+1}-2U_{i,j+1}+U_{i-1,j+1}\right)\end{aligned}$$

By Taylor expansion of $U^2_{i,j+1}$ about the point (i,j) we obtain

$$\begin{aligned}U^2_{i,j+1}&=U^2_{i,j}+k\frac{\partial U^2_{i,j}}{\partial t}+\cdots\\&=U^2_{i,j}+k\frac{\partial U^2_{i,j}}{\partial U_{i,j}}\frac{\partial U_{i,j}}{\partial t}+\cdots\end{aligned}$$

Hence to terms of order k, $U^2_{i,j+1}\cong U^2_{i,j}+2U_{i,j}(U_{i,j+1}-U_{i,j})$ and taking

$$W_i=U_{i,j+1}-U_{i,j} \tag{5}$$

Eq. (4), with some manipulations, leads to

$$\begin{aligned}&(\theta rhU_{i-1,j}+2\nu r)W_{i-1}-2(1+2r\nu)W_i+(2r\nu-\theta rhU_{i+1,j})W_{i+1}\\&=\frac{rh}{2}(U^2_{i+1,j}-U^2_{i-1,j})-2\nu r(U_{i-1,j}-2U_{i,j}+U_{i+1,j}),\ i=1(1)N-1\end{aligned} \tag{6}$$

a system of linear equations for W_i, where $0\le\theta\le 1$ and $r=k/h^2$. This approximation is second order in both space and time as regards truncation error. Obviously, the solution at the $(j+1)th$ time level is obtained from (5) as $U_{i,j+1}=U_{i,j}+W_i$.

Numerical Results and Conclusions

All calculations were performed in double precision arithmetic on a Pentium II processor using Microsoft FORTRAN Compiler. A system of algebraic equations corresponding to the scheme (6) has been solved directly by the Gauss elimination method.

In order to show how good the numerical solutions of the above problems (a) and (b) with the exact ones we shall use the weighted 1-norm $\|e\|_1$ defined by

$$\|e\|_1 = \frac{1}{N}\sum_{i=1}^{N-1}\left|\frac{U(x_i,t_j)-U_{i,j}}{U(x_i,t_j)}\right|, \; e=[e_1\cdots e_{N-1}]^T. \tag{7}$$

Table 1 illustrates results obtained by the replacement (6) of the problem (a) at various values of the weighted factor for $\nu =$. It is clearly seen that numerical solutions are in good agreement with the exact one. So it is a simple matter the choice of satisfying the inequality $0 \le \theta \le 1$.

Table 1. Comparison of results for $\nu = 1$, $h = 0.025$ and $k = 0.00001$ at various values of θ

x	t_f	Numerical				Exact
		$= 0$	$\theta = 0.1$	$= 0.$	$= 1.0$	
0.25	0.01	0.62903	0.62903	0.62903	0.62904	0.62904
	0.05	0.41319	0.41319	0.41315	0.41315	0.41307
	0.10	0.25374	0.25374	0.25374	0.25374	0.25364
	0.15	0.15672	0.15672	0.15670	0.15670	0.15660
	0.20	0.09654	0.09654	0.09653	0.09653	0.09644
	0.25	0.05929	0.05929	0.05929	0.05929	0.05922
0.50	0.01	0.90568	0.90568	0.90568	0.90568	0.90571
	0.05	0.60923	0.60923	0.60917	0.60917	0.60907
	0.10	0.37173	0.37173	0.37173	0.37173	0.37158
	0.15	0.22700	0.22700	0.22698	0.22698	0.22682
	0.20	0.13862	0.13862	0.13860	0.13860	0.13847
	0.25	0.08465	0.08465	0.08464	0.08464	0.08454
0.75	0.01	0.65237	0.65237	0.65237	0.65237	0.65244
	0.05	0.45025	0.45025	0.45021	0.45021	0.45018
	0.10	0.27269	0.27269	0.27269	0.27269	0.27258
	0.15	0.16450	0.16450	0.16448	0.16448	0.16437
	0.20	0.09954	0.09954	0.09953	0.09953	0.09944
	0.25	0.06043	0.06043	0.06042	0.06042	0.06035

Table 2 displays the expected convergence as the grid size h is refined. Again, good agreement with the analytic values is evident, as is convergence. In fact, applying Richardson's extrapolation (see, e.g., [18, Section 2]) to the value of the weighted 1-norm error measure (given by (7)) shown in Table 2 yields convergence rates of, approximately, 1.7121. This agrees with the theoretical expectation of $O(h^2)$.

Table 3 displays finite difference solutions of the problem (a) for $\nu = 0.$ and $\nu = 0.01$ respectively, with $k = 0.0001$ at different values of t . It is observed that the numerical predictions are again in good agreement with the analytic solution.

In order to show how good the numerical predictions exhibit the correct physical behaviour of the problem, we only give the graphs in Figures 1, 2 and 3. Both solutions of the problem are drawn on the same diagram, but curves cannot be distinguishable since they are very close to each other.

Table 2. Comparison of results at $t_f = 0.1$ for $\nu = 1$, $k = 0.00001$ at various mesh sizes

	Numerical				Exact
x	$h=0.$	$h=0.0$	$h=0.02$	$h=0.012$	
0.1	0.10915	0.10959	0.10957	0.10955	0.10954
0.2	0.21020	0.21005	0.20987	0.20981	0.20979
0.3	0.29305	0.29233	0.29202	0.29192	0.29190
0.4	0.34964	0.34848	0.34807	0.34795	0.34792
0.5	0.37361	0.37219	0.37173	0.37161	0.37158
0.6	0.36109	0.35963	0.35919	0.35907	0.35905
0.7	0.31170	0.31041	0.31003	0.30993	0.30991
0.8	0.22913	0.22818	0.22791	0.22783	0.22782
0.9	0.12138	0.12088	0.12073	0.12069	0.12069
$\|e\|_1$	0.004281	0.001366	0.000370	0.000066	

Table 3. Comparison of results for $\nu = 0.1$ and $\nu = 0.01$ with $h = 0.0125$, $k = 0.0001$ and $\theta = 0.5$ at different times

		$\nu = 0.1$		$\nu = 0.01$	
x	t_f	Numerical	Exact	Numerical	Exact
0.25	0.4	0.30890	0.30889	0.34189	0.34191
	0.6	0.24075	0.24074	0.26890	0.26896
	0.8	0.19569	0.19568	0.22139	0.22148
	1.0	0.16258	0.16256	0.18810	0.18819
	3.0	0.02722	0.02720	0.07508	0.07511
0.50	0.4	0.56969	0.56963	0.66078	0.66071
	0.6	0.44726	0.44721	0.52946	0.52942
	0.8	0.35928	0.35924	0.43916	0.43914
	1.0	0.29146	0.29292	0.37442	0.37442
	3.0	0.04023	0.04021	0.15015	0.15018
0.75	0.4	0.62539	0.62544	0.91051	0.91026
	0.6	0.48723	0.48721	0.76738	0.76724
	0.8	0.37396	0.37392	0.64747	0.64740
	1.0	0.28752	0.28747	0.55610	0.55605
	3.0	0.02979	0.02977	0.22481	0.22481

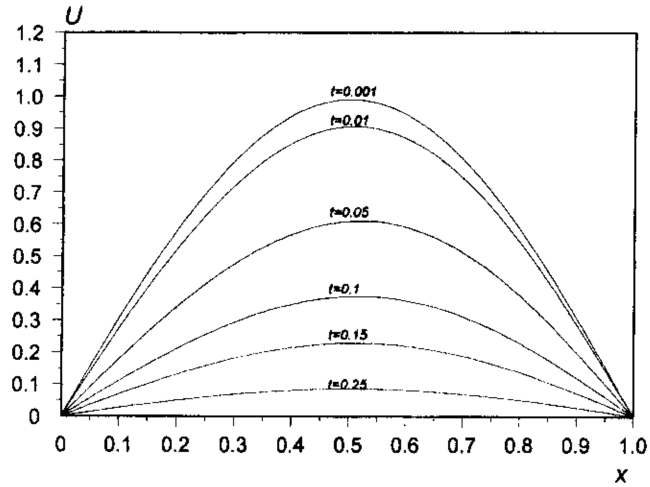

Figure 1. Solutions at different times for $\nu = 1.0,\ h = 0.0125,\ k = 0.001$.

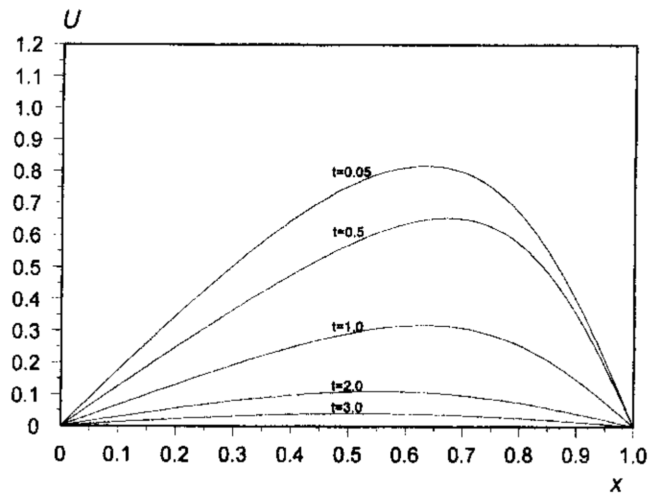

Figure 2. Solutions at different times for $\nu = 0.1,\ h = 0.0125,\ k = 0.001$.

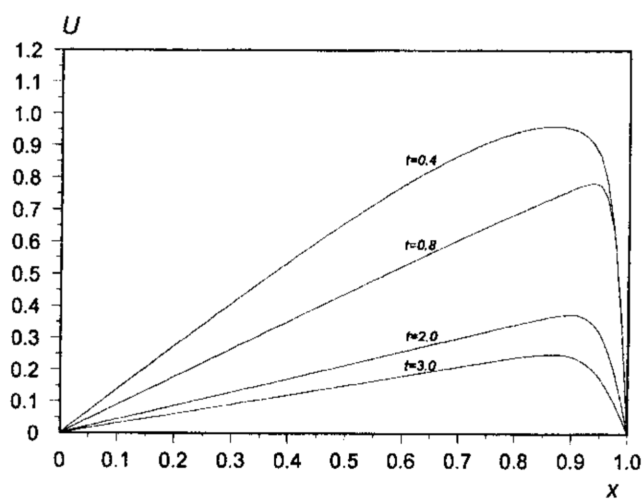

Figure 3. Solutions at different times for $\nu = 0.01,\ h = 0.0125,\ k = 0.00$.

Table 4 shows the numerical solutions for viscosity coefficients from $\nu = 0.005$ to $\nu = 0.0001$ at times from $t_f = 0.01$ to $t_f = 0.25$ which exhibit the correct physical behaviour of the problem. For these viscosity values it is obvious that the exact solution fails [8] because of the slow convergence of Fourier series (2).

The numerical solutions of Burgers equation for problem (b) obtained by the present method have been compared with the analytic solution in Tables 5-7 for various values of viscosity coefficient ν. It can be seen that numerical solutions are in good agreement with the analytic one. Table 5 shows that the accuracy of the numerical solutions which improves rapidly as the mesh size is reduced. Again, good agreement with the exact values is evident, as is convergence. The values of $\|e\|_1$ shown in Table 5 indicate a rate of convergence of about $.6957$ which is reasonably in agreement with the theoretical expectation of $O(h^2)$.

Table 4. Numerical results for various values of ν with $h = 0.025, k = 0.00001$ and $\theta = 0.5$ at different times

		Numerical			
x	t_f	$\nu = 0.00$	$\nu = 0.00$	$\nu = 0.000$	$\nu = 0.000$
0.25	0.01	0.69111	0.69137	0.69141	0.69143
	0.05	0.63220	0.63319	0.63331	0.63341
	0.10	0.56820	0.56962	0.56979	0.56994
	0.15	0.51409	0.51565	0.51584	0.51600
	0.25	0.42986	0.43133	0.43152	0.43166
0.50	0.01	0.99900	0.99940	0.99945	0.99949
	0.05	0.98569	0.98755	0.98778	0.98797
	0.10	0.95145	0.95461	0.95500	0.95532
	0.15	0.90488	0.90865	0.90912	0.90950
	0.25	0.80158	0.80531	0.80578	0.80615
0.75	0.01	0.72275	0.72305	0.72309	0.72312
	0.05	0.78663	0.78853	0.78877	0.78896
	0.10	0.86589	0.87068	0.87128	0.87176
	0.15	0.93250	0.94035	0.94134	0.94212
	0.25	0.98773	0.99776	0.99900	0.99999

Table 7 shows the numerical solutions of problem (b) for viscosity coefficients from $\nu = 0.005$ to $\nu = 0.0001$ at times from $t_f = 0.01$ to $t_f = 0.25$ which exhibit the correct physical behaviour of the problem. It is noticed that as the viscosity value ν decreases, there is no significant change in the values of $U_{i,j}$ at mesh points (x_i, t_j). It is also seen that the exact solution fails for these viscosity values since Fourier series (2) converges very slowly.

Table 5. Comparison of results at $t_f = 0.1$ for $\nu = 1, k = 0.00001$ and various mesh sizes

	Numerical table5				Exact
x	$h = 0.$	$h = 0.05$	$h = 0.025$	$h = 0.0125$	
0.1	0.11245	0.11294	0.11293	0.11290	0.11289
0.2	0.21662	0.21651	0.21634	0.21627	0.21625
0.3	0.30208	0.30140	0.30109	0.30099	0.30097
0.4	0.36056	0.35942	0.35901	0.35889	0.35886
0.5	0.38543	0.38404	0.38358	0.38345	0.38342
0.6	0.37270	0.37125	0.37081	0.37068	0.37066
0.7	0.32185	0.32057	0.32019	0.32009	0.32007
0.8	0.23668	0.23574	0.23546	0.23539	0.23537
0.9	0.12541	0.12491	0.12476	0.12473	0.12472
$\|e\|_1$	0.004151	0.001339	0.000366	0.000066	

Table 6. Comparison of results for $\nu = 0.1$ and $\nu = 0.01$ with $h = 0.0125$, $k = 0.0001$ and $\theta = 0.5$ at different times

x	t_f	$\nu = 0.1$		$\nu = 0.01$	
		Numerical	Exact	Numerical	Exact
0.25	0.4	0.31753	0.31752	0.36226	0.36226
	0.6	0.24615	0.24614	0.28197	0.28204
	0.8	0.19957	0.19956	0.23036	0.23045
	1.0	0.16561	0.16560	0.19559	0.19469
	3.0	0.02777	0.02776	0.07610	0.07613
0.50	0.4	0.58459	0.58454	0.68375	0.68368
	0.6	0.45803	0.45798	0.54838	0.54832
	0.8	0.36745	0.36740	0.45375	0.45371
	1.0	0.29839	0.29834	0.38568	0.38568
	3.0	0.04109	0.04107	0.15214	0.15218
0.75	0.4	0.64557	0.64562	0.92067	0.92050
	0.6	0.50269	0.50268	0.78311	0.78299
	0.8	0.38538	0.38534	0.66280	0.66272
	1.0	0.29591	0.29586	0.56937	0.56932
	3.0	0.03049	0.03044	0.22774	0.22774

Table 7. Numerical results for various values of ν with $h = 0.025$, $k = 0.00001$ and $\theta = 0.5$ at different times

x	t_f	Numerical			
		$\nu = 0.00$	$\nu = 0.00$	$\nu = 0.000$	$\nu = 0.000$
0.25	0.01	0.73456	0.73487	0.73491	0.73494
	0.05	0.67609	0.67738	0.67754	0.67767
	0.10	0.61003	0.61208	0.61234	0.61254
	0.15	0.55250	0.55496	0.55526	0.55551
	0.25	0.46081	0.46349	0.46382	0.46409
0.50	0.01	0.99919	0.99951	0.99955	0.99958
	0.05	0.98829	0.98983	0.99002	0.99017
	0.10	0.95952	0.96228	0.96263	0.96290
	0.15	0.91887	0.92241	0.92285	0.92320
	0.25	0.82346	0.82748	0.82798	0.82838
0.75	0.01	0.76479	0.76512	0.76516	0.76520
	0.05	0.82332	0.82522	0.82546	0.82565
	0.10	0.89190	0.89612	0.89664	0.89707
	0.15	0.94646	0.95282	0.95361	0.95424
	0.25	0.99002	0.99815	0.99915	0.99996

The performance of the method has been examined by comparing all the numerical results with the exact ones. It is concluded that the method is capable of solving the Burgers-like equations by a direct method since it produces very accurate results, even for small values of viscosity ($\nu < 0.01$).

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