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SINGULARITIES AND THE CONVERGENCE RATE OF A SOLUTION FOR STOKES FLOW IN A CAVITY WITH FREE SURFACES

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Abstract : Two-dimensional Stokes floequation in a rectangular cavity with two free surfaces and two moving lids is considered as a biharmonic boundary value problem which is solved analyticly for the streamfunction, ψ, expressed as a series of eigenfunctions. The series solution is shown to converge. In practice, the series must be turancated after n terms, so it is important to asses the number of terms required to achive a given accuracy in the streamfunction and velocity at any point in different cavities. The accuracy and convergence rate of the eigenfunction series are discussed. Furthermore, we study the influence of the singularities on the corners with respect to the accuracy of the solution. The singularities result in a reduce convergence of the eigenfunctions expansion in a small neighbourhood of the corners.

SERBEST YÜZEYLİ BİR OYUKTAKİ STOKES DENKLEMİNİN ÇÖZÜMÜNÜN SİNGÜLERİTELERİ VE YAKINSAKLIK ORANI

Özet: İki serbest yüzeyli ve iki hareketli kapak arasındaki dikdörtgen bir bölgede, iki boyutlu Stokes denklemi biharmonik sınır değer problemi olarak göz önüne alındı. Bu problem eigen fonksiyonların bir serisi ile analitik olarak çözüldü ve bu çözümün yakınsaklığı gösterildi. Uygulamada, sonsuz bir seri belli bir terimden sonra kesilmesi gerektiğinden. Bu kesmenin, boyutları farklı bölgelerin herhangi bir noktasında akış fonksiyonu ve türevinin yakınsaklık oranına etkisi tartışıldı. Ayrıca köşelerdeki singüler noktaların çözüme etkisi incelendi. Singülerlik, köşelerin küçük komşuluğundaki çözümün yakınsaklık oranını düşürdüğü belirlendi.

Introduction

Although Stokes equations have been investigated for nearly 150 years, analytic solutions have been obtained only for special geometries. For a semi--bounded domain, Smith [1] established an algorithm for solving the biharmonic equation. Smith also established conditions on the edge data sufficient to guarantee the convergence of his analytical solution which is expressed as an infinite series of complex eigenfunctions (Papkovich-Faddle functions). Smith's solution was then used by Joseph and Sturges [2] to analyse Stokes flow in a rectangular cavity heated from its side. This work, together with that of Joseph [3] revealed that Smith's conditions on the edge data were too estrictive for practical applications and showed that much less restrictive ones suffice to guarantee convergence of Smith's analytical solution.

Joseph and Sturges [4] examined numerically the convergence of an biorthogonal solution of Stokes flow in a single-lid-driven cavity with tree solid walls to the boundary conditions. It was found that by taking n=20 terms of the biorthogonal series solution the convergence of the streamfunctions and its normal derivatives to the boundary

onditions on the moving lid are given correct to 4 and one decimal place, espectively. Gürcan [5] investigated the truncation error of the infinity series for this problem. It was found that the truncation errors decreases when more terms in the series are taken, as expected. For the same problem Shankar [6] considered an analytic series expansion but used a least squares method to determine the coefficient in the series. He examined the convergence of the least squares solution on the moving lid, especially near to the corner at which there is a singularity. He concluded that both coefficient determination methods are indistinquishable from one another. Stokes Flow in a rectangular cavity with two free surfaces and two moving lids was proposed as a model for the flow in the bead of meniscus roll coater [7-9]. Generally they investigated flow structure by considering the behaviour of streamfunction close to stagnation points.

In this study, the series solution for this problem is considered. The convergence of the series solution is shown by determining the dominat contribution to each term in the series. We also determine the number of terms, n, of the series in order to ensure that the truncation error is acceptably small. The problem of interest exhibits corner singularities due to the inadmissibility of the no-slip hypothesis near the junction between a moving lid and a free surface [10]. The numerical study of flow singularities is very important since they typically increase computational errors [11-13]. A local singular analytical solution of [10] is used to correct the numerical methods in [12-13]. Here we investigate the influence of the singularities on the corners with respect to the accuracy of the series solution.

The Governing Equation and Boundary Conditions

Flow in a double-lid-driven cavity with 2 free surfaces and 2 moving lids, see figure 1, is governed by the biharmonic equation,

$$\nabla^4 \Psi = 0 . \tag{1}$$

The analysis is simplified by introducing the following dimensionless variables,

$u = U/V_2$, $v = V/V_2$, $x = X/L$, $y = Y/L$ and $\psi = \Psi/V_2 L$

$S = V_1/V_2$, $A = H_0/L$

where $2L$ is the bead width, V_1 the top lid velocity, V_2 the bottom lid velocity, $S = V_1/V_2$ the speed ratio, $2H_0$ the separation of the lids and $A = 2H_0/L$ the aspect ratio of the cavity.

For the lid-driven cavity with 2 free surfaces we assume that the liquid domain is closed and no liquid crosses the boundaries. This implies that

$$\psi(1, y) = \psi(-1, y) = 0 \text{ and } \psi(x, A) = \psi(x, -A) = 0 . \tag{2}$$

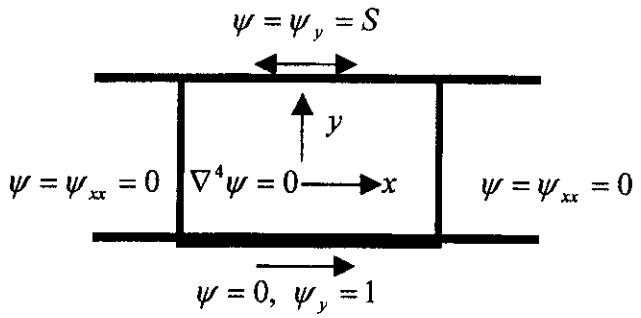

Figure 1: Dimensionless boundary value problem for rectangular cavity with two free surfaces.

Hence relations, $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$, enable the no-slip conditions to be written in terms of derivatives of the streamfunction.

$$\frac{\partial \psi}{\partial y}(x, A) = S,\ \frac{\partial \psi}{\partial y}(x,-A) = 1. \tag{3}$$

If $\hat{\underline{n}} = (\pm 1, 0)$, $\hat{\underline{t}} = (0, \pm 1)$ are unit vectors normal and tangential to the liquid-gas interface respectively, then the equation expressing the equilibrium of this interface [14-p69] is

$$\underline{\underline{\sigma}} \bullet \hat{\underline{n}} = \underline{\underline{\sigma}}_g \bullet \hat{\underline{n}} + \frac{T}{R_{curv}} \bullet \hat{\underline{n}}, \tag{4}$$

where $\underline{\underline{\sigma}}$ and $\underline{\underline{\sigma}}_g$ are the stress tensors for the liquid and gas respectively, T is the surface tension of the liquid, and R_{curv} the radius of curvature of the liquid-gas interface. For a Newtonian liquid the stress tensor is given by

$$\underline{\underline{\sigma}}_{ij} = -P\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{5}$$

where δ_{ij} is the Kronecker delta symbol. It can be shown, balancing shear stresses, that

$$\eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \eta_g\left(\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial x}\right) \tag{6}$$

where the subscript g refers to the gas. However $\eta_g / \eta << 1$, so equation (6) reduces to a 'zero-shear stress' condition, namely

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0 \text{ at } x=\pm 1. \tag{7}$$

Using relations, $u=\frac{\partial \psi}{\partial y}$ and $v=-\frac{\partial \psi}{\partial x}$, this may be written in terms of the streamfunction, giving

$$\frac{\partial^2 \psi}{\partial y^2}-\frac{\partial^2 \psi}{\partial x^2}=0 \text{ at } x=\pm 1. \tag{8}$$

As a result of boundary condition (2)

$$\frac{\partial \psi}{\partial y}=\frac{\partial^2 \psi}{\partial y^2}=0 \text{ at } x=\pm 1, \tag{9}$$

and the zero-shear stress condition becomes

$$\frac{\partial^2 \psi}{\partial x^2}=0 \text{ at } x=\pm 1. \tag{10}$$

General Solution

The boundary value problem, figure 1, can be solved for ψ using 'natural' eigenfunctions of the biharmonic equation which are even functions of x since the flow is symmetric about $x=0$. In fact $\psi(x,y)$ has the form

$$\psi=\sum_{n=1}^{\infty}\left\{y(A_n e^{\lambda_n y}+B_n e^{-\lambda_n y})+C_n e^{\lambda_n y}+D_n e^{-\lambda_n y}\right\}\cos\lambda_n x, \tag{11}$$

where λ_n are eigenvalues and A_n, B_n, C_n, D_n are constant coefficients to be determined from the boundary conditions. The conditions, $\psi=\frac{\partial^2 \psi}{\partial x^2}=0$ at $x=\pm 1$ are satisfied if $\cos\lambda_n=0$ for all λ_n, giving the eigenvalues

$$\lambda_n=(n-1/2)\pi \text{ for } n=1,2,... \tag{12}$$

Since the streamfunction $\psi(x,y)$ satisfies the boundary condition (2) on $y=A$ then $\psi(x,y)$ can be written in the form

$$\psi = \sum_{n=1}^{\infty} \psi_n \cos \lambda_n x \,, \tag{13}$$

where

$$\psi = (y - A)(A_n e^{\lambda_n y} + B_n e^{-\lambda_n y}) - A \frac{A_n e^{-2\lambda_n y} + B_n}{\sinh 2\lambda_n A} e^{\lambda_n y}(1 - e^{2\lambda_n (A-y)}) \,. \tag{14}$$

Since $\frac{\partial \psi}{\partial y}(x, A) = S$ and also $\frac{\partial \psi}{\partial y}(x, -A) = 1$ then using the orthogonality condition A_n and B_n are obtained

$$A_n = \frac{(-1)^{n+1}\left[(e^{\lambda_n A} - \frac{2A\lambda_n}{\sinh 2A\lambda_n} e^{-\lambda_n A})S - (e^{-\lambda_n A} - \frac{2A\lambda_n}{\sinh 2A\lambda_n} e^{\lambda_n A} \right]}{\frac{\lambda_n}{\sinh 2A\lambda_n}(\sinh^2 2A\lambda_n - 4A^2\lambda_n^2)} \tag{15}$$

$$B_n = \frac{(-1)^{n+1}\left[(e^{\lambda_n A} - \frac{2A\lambda_n}{\sinh 2A\lambda_n} e^{-\lambda_n A}) - S(e^{-\lambda_n A} - \frac{2A\lambda_n}{\sinh 2A\lambda_n} e^{\lambda_n A} \right]}{\frac{\lambda_n}{\sinh 2A\lambda_n}(\sinh^2 2A\lambda_n - 4A^2\lambda_n^2)} \,. \tag{16}$$

Series Truncation and Convergence

Since, in practice, it is only possible to include a finite number of terms in the series (13), it is necessary to first establish that the series converges and then to determine the number of terms which need to be taken in order to ensure that the series has converged satisfactorily.

The series for ψ

It is possible to investigate the convergence of the series for ψ by determining the dominant contribution to each term in the series for large n. First consider the coefficient A_n which may be rewritten as

$$A_n = \frac{(-1)^{(n+1)}\left[(\frac{1}{2} - \frac{e^{-4\lambda_n A}}{2} - 2\lambda_n A e^{-4\lambda_n A})S - (\frac{e^{-2\lambda_n A}}{2} - \frac{e^{-6\lambda_n A}}{2} - 2\lambda_n A e^{-2\lambda_n A}) \right]}{\frac{\lambda_n e^{\lambda_n A}}{4} - \frac{\lambda_n e^{-3\lambda_n A}}{2} + \frac{\lambda_n e^{-7\lambda_n A}}{4} - 4\lambda_n{}^3 A e^{-3\lambda_n A}}$$

Extracting the dominant term, we see that for large n,

$$A_n = (-1)^{(n+1)} \frac{2S}{\lambda_n e^{\lambda_n A}} + l.o.t \tag{17}$$

where 'l.o.t.' indicates lower order terms. Similarly for the coefficient B_n,

$$B_n = (-1)^{(n+1)} \frac{2}{\lambda_n e^{\lambda_n A}} + l.o.t \tag{18}$$

Substituting expressions (17) and (18) into ψ gives for large n

$$\psi_n = (-1)^{(n+1)} \left[\left(\frac{S(y-A)}{\lambda_n e^{\lambda_n (A-y)}} + \frac{(y+A)}{\lambda_n e^{\lambda_n (A+y)}} \right] + l.o.t \text{ ,for } -A < y < A \right. \tag{19}$$

Expression (19) gives the dominant terms in ψ both of which are

$$\psi_n = O((-1)^{(n+1)} \frac{1}{\lambda_n} e^{-k\lambda_n}), \; k > 0 \tag{20}$$

where either $k = A - y$ or $k = A + y$ for $y \in (-A, A)$. The series $\sum \psi_n$ for $S \in [-1,1]$ is clearly absolutely convergent and hence the series for ψ converges for $y \in [-A, A]$.

To see how convergence varies for different values of y and A, tables 1 and 2 indicate the accuracy on truncation after n terms of the streamfunction calculated from equation (13) for $A = 0.1$ and $A = 1$ respectively and $S = 2/3$. Table 1, for example, shows that for $A = 0.1$ and $y = A/2$ truncating series (13) after $n = 100$ terms gives an accuracy of 9-10 decimal places whereas for $n = 200$ terms the accuracy is 17 decimal places in ψ. This clearly shows that the accuracy of ψ improves very rapidly as the number of terms in the series (13) is increased. Table 2 shows that for $A = 1$ and $y = A/2$ truncating the series after only $n = 20$ terms gives ψ correct to 10 decimal places. Clearly for large A the convergence is much faster than for lower A values, but it also depends on y, as shown tables 1 and 2.

Table 1: A comparison of the accuracy of the streamfunction calculated by the series (13) after n terms (the accuracy is indicated by the underlined numbers) for $A = 0.1$ at $S = 2/3$. n denotes the number of terms used in the series.

$S = 2/3$ and $A = 0.1$				
y	n	actual values of ψ		
		$x = -0.7$	$x = -0.3$	$x = 0.0$
$y = A$	50	0.0	0.0	0.0
	100	0.0	0.0	0.0
	200	0.0	0.0	0.0
$y = A/$	50	-0.021887577212570	-0.021875137363653	-0.021874876919845
	100	-0.021885248376509	-0.021874999483326	-0.021874999976018
	200	-0.021885248427780	-0.021874999510103	-0.021874999999919
	300	-0.021885248427780	-0.021874999510103	-0.021874999999919
$y = 0$	50	-0.008352175975982	-0.008333332130091	-0.008333333302818
	100	-0.008352175913157	-0.008333332094946	-0.008333333334355
	200	-0.008352175913157	-0.008333332094946	-0.008333333334355
	300	-0.008352175913157	-0.008333332094946	-0.008333333334355
$y = -A/$	50	0.009365349725042	0.009375092392383	0.009374917946727
	100	0.009365174276404	0.009375000472198	0.009374999984145
	200	0.009365174310584	0.009375000490049	0.009375000000079
	300	0.009365174310584	0.009375000490049	0.009375000000079

Table 2: A comparison of the accuracy of the streamfunction given by the series (13) after n terms for $A = 1.00$ at $S = 2/3$.

$S = 2/3$ and $A = 1.0$				
y	n	actual values of ψ		
		$x = -0.7$	$x = -0.3$	$x = 0.0$
$y = A$	20	0.0	0.0	0.0
	50	0.0	0.0	0.0
	100	0.0	0.0	0.0
$y = A/$	20	-0.12198463011580	-0.19330687552501	-0.20444704539851
	50	-0.12198463012672	-0.19330687554985	-0.20444704539602
	100	-0.12198463012672	-0.19330687554985	-0.20444704539602
	200	-0.12198463012672	-0.19330687554985	-0.20444704539602
$y = 0$	20	-0.031460348869680	-0.059018667895093	-0.065249136291313
	50	-0.031460348869680	-0.059018667895093	-0.065249136291313
	100	-0.031460348869680	-0.059018667895093	-0.065249136291313
	200	-0.031460348869680	-0.059018667895093	-0.065249136291313
$y = -A/$	20	0.063269110456510	0.094031267337532	0.097425348491845
	50	0.063269110454900	0.094031267339420	0.097425348492717
	100	0.063269110454900	0.094031267339420	0.097425348492717
	200	0.063269110454900	0.094031267339420	0.097425348492717

The series for $\dfrac{\partial \psi}{\partial y}$

Similarly, substituting expressions (17) and (18) into $\dfrac{\partial \psi}{\partial y}$ gives for large n

$$\frac{\partial \psi_n}{\partial y} = (-1)^{(n+1)} \left[\frac{S(\frac{1}{\lambda_n} + (y - A))}{e^{\lambda_n (A-y)}} + \frac{\frac{1}{\lambda_n} - (y + A)}{e^{\lambda_n (A+y)}} \right] + l.o.t \tag{21}$$

For $y \in (-A, A)$

$$\frac{\partial \psi_n}{\partial y} = O((-1)^{(n+1)} e^{k\lambda_n}) \quad \text{where} \quad k > 0 \tag{22}$$

and so the series $\sum \frac{\partial \psi_n}{\partial y}$ is absolutely convergent and hence the series for $\frac{\partial \psi}{\partial y}$ converges for $y \in (-A, A)$.

To see how convergence varies for different values of y and A, table 3 indicates the accuracy of the truncated for $\frac{\partial \psi}{\partial y}$ for various aspect ratios, speed ratios and positions within the cavity. For $A = 0.1$, $S = -1/2$ at $y = -A/2$ truncating $\frac{\partial \psi}{\partial y}$ after 100 terms gives 8 decimal place accuracy and for $n = 200$ sixteen decimal places, see table 3.

This table shows also that for $A = 1, S = -1$ and at the same location $y = -A/2$ with only 20 terms, $\frac{\partial \psi}{\partial y}$ shows an accuracy of 14 decimal places. Results clearly show that the convergence of the series for $\frac{\partial \psi}{\partial y}$ depends on y and A. In general for large A the convergence is much faster than for small A but that the convergence decreases as the lids are approached.

On The Lids

The coefficients A_n, B_n are chosen so that the lid velocity boundary conditions are approximated by Fourier series. Hence on the top lid

$$\frac{\partial \psi}{\partial y} = \sum ((-1)^{(n+1)} \frac{S \cos \lambda_n x}{\lambda_n}) \tag{23}$$

and on the bottom lid

$$\frac{\partial \psi}{\partial y} = \sum ((-1)^{(n+1)} \frac{\cos \lambda_n x}{\lambda_n}). \tag{24}$$

Since these series satisfy the `Dirichlet' conditions for Fourier series convergence they converge to the boundary conditions on the lids everywhere except close to the corners.

Corner Singularities

It can be seen that this problem contains singularities at all corners formed at the junction between a moving lid and a free surfaces. We computed the singular behaviour near the upper-left corner with the procedure described by Moffatt [10] to compore the present results. In this procedure a local solution is computed for the biharmonic equation in plane co-ordinates (r,θ) of the form

$$\Psi = \sum_{p} r^{p} f_{p}(\theta). \tag{25}$$

The boundary value problem to be solved in the corner region has boundary conditions,

$$\Psi = 0, \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = S \text{ at } \theta = 0,$$

$$\Psi = 0, \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = 0 \text{ at } \theta = \frac{-\pi}{2}.$$

The lowest order solution is $\Psi_1 = rf_1(\theta)$ given by

$$\Psi_1 = \frac{r}{\pi^2 - 4}(4\theta\cos\theta + 2\pi\theta\sin\theta - \pi^2\sin\theta).$$

The lower order describes the asymptotic behaviour at the corners. The convergence of the coefficients of series (25) shows the same behaviour as we found in our computations whereas the convergence of the derivative of streamfuction (13) is decreased near the corners. However, as the number of terms, n, used in the series $\frac{\partial \psi}{\partial y}$ is increased, even the convergence in these regions improves. For example, for any value of A, $\frac{\partial \psi}{\partial y}$ is correct to 2 decimal places when $n = 100$ terms whereas when $n=200$ terms 3-4 decimal places, see table 3.

Table 3. A comparison of the accuracy of the horizontal component of liquid velocity given by the series evaluated after n terms for $A = 0.1$ at $S = -1/2$ and 0.

$S = -1/2$ and $A = 0.1$				
y	n	actual values of ψ		
		$x = -0.7$	$x = -0.3$	$x = 0.0$
$y = A$	50	-0.507	-0.503	-0.496
	100	-0.496	-0.498	-0.498
	200	-0.498	-0.499	-0.499
$y = A/2$	50	0.34504269983902	0.34373097945617	0.34376688894477
	100	0.34505909029467	0.34374991319534	0.34375000713707
	200	0.34505907515618	0.34374991531351	0.34375000010319
	300	0.34505907515618	0.34374990531351	0.34375000010319
$y = 0$	50	-0.12500287386101	-0.12500000777502	-0.12499999302999
	100	-0.12500285981042	-0.12500000000094	-0.12500000000001
	200	-0.12500285981040	-0.12500000000095	-0.12500000000000
	300	-0.12500285981040	-0.12500000000095	-0.12500000000000
$y = -A/2$	50	-0.40754876310628	-0.40624844239106	-0.40625144451786
	100	-0.40755695832287	-0.40624990925386	-0.40625000362014
	200	-0.40755695075362	-0.40624990531291	-0.40625000010320
	300	-0.40755695075362	-0.40624990531291	-0.40625000010320
$y = -A$	50	1.014	1.007	0.993
	100	0.992	0.996	0.996
	200	0.9964	0.9982	0.9984
$S = -1.0$ and $A = 1.0$				
y	n	actual values of ψ		
		$x = -0.7$	$x = -0.3$	$x = 0.0$
$y = 1.0$	10	-1.068	-1.035	-0.968
	20	-0.965	-0.982	-0.984
	50	-1.014	-1.007	-0.993
	100	-0.992	-0.996	-0.996
$y = 0.5$	10	-0.069175427893891	-0.27044200038033	-0.32634509494049
	20	-0.069175472292034	-0.27044205430477	-0.32634504155722
	50	-0.069175472292028	-0.27044205430476	-0.32634504155721
	100	-0.069175472292028	-0.27044205430476	-0.32634504155721
$y = -0.5$	10	0.069175427893891	0.27044200038033	0.32634509494049
	20	0.069175472292034	0.27044205430477	0.32634504155722
	50	0.069175472292028	0.27044205430476	0.32634504155721
	100	0.069175472292028	0.27044205430476	0.32634504155721
$y = -0.5$	10	1.068	1.035	0.968
	20	0.965	0.982	0.984
	50	1.014	1.007	0.993
	100	0.992	0.996	0.996

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