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Research Article

Approximating sums by integrals only: multiple sums and sums over lattice polytopes

IOSIF PINELIS*

ABSTRACT. The Euler–Maclaurin (EM) summation formula is used in many theoretical studies and numerical calculations. It approximates the sum $\sum_{k=0}^{n-1} f(k)$ of values of a function f by a linear combination of a corresponding integral of f and values of its higher-order derivatives $f^{(j)}$. An alternative (Alt) summation formula was presented by the author, which approximates the sum by a linear combination of integrals only, without using derivatives of f. It was shown that the Alt formula will in most cases outperform the EM formula. In the present paper, a multiple-sum/multi-index-sum extension of the Alt formula is given, with applications to summing possibly divergent multi-index series and to sums over the integral points of integral lattice polytopes.

Keywords: Euler–Maclaurin summation formula, alternative summation formula, multiple sums, multi-index series, approximation, lattice polytopes.

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1. INTRODUCTION

The Euler–Maclaurin (EM) summation formula can be written as follows (see e.g. [16]):

$$\sum_{k=0}^{n-1} f(k) \approx \int_0^{n-1} \mathrm{d}x\, f(x) + \frac{f(n-1)+f(0)}{2} + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}[f^{(2j-1)}(n-1) - f^{(2j-1)}(0)], \tag{1.1}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth enough function, B_j is the j-th Bernoulli number, and n and m are natural numbers. The EM approximation is exact when f is a polynomial of degree $< 2m+1$.

The EM formula has been used in a large number of theoretical studies and numerical calculations.

Clearly, to use the EM formula in a theoretical or computational study, one will usually need to have an antiderivative F of f and the derivatives $f^{(2j-1)}$ for $j = 1, \dots, m$ in tractable or, respectively, computable form.

In [19], an alternative summation formula (Alt) was offered, which approximates the sum $\sum_{k=0}^{n-1} f(k)$ by a linear combination of values of an antiderivative F of f only, without using values of any derivatives of f:

$$\sum_{k=0}^{n-1} f(k) \approx \sum_{j=1-m}^{m-1} \tau_{m,1+|j|} \int_{j/2-1/2}^{n-1/2-j/2} \mathrm{d}x\, f(x), \tag{1.2}$$

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*Corresponding author: Iosif Pinelis; ipinelis@mtu.edu
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where f is again a smooth enough function, the coefficients $\tau_{m,r}$ are certain rational numbers not depending on f and such that $\sum_{j=1-m}^{m-1} \tau_{m,1+|j|} = 1$, and n and m are natural numbers. Similarly to the case of the EM formula, the Alt approximation is exact when f is a polynomial of degree $< 2m$.

It was shown in [19] that the Alt formula should be usually expected to outperform the EM one.

Extensions of the EM formula to the multiple sums, including sums over the integral points of integral lattice polytopes, have been of significant interest; see e.g. [20, 8, 7, 13, 21, 14, 6, 3, 10, 22, 18, 4]. In the present paper, a multiple-sum/multi-index-sum extension of the Alt formula will be given. The main result of this paper, Theorem 2.1, is then extended to sums over the integral points of integral lattice polytopes as well.

The rest of this paper is organized as follows.

In Section 2, the multi-index Alt formula is stated, with discussion.

In Section 3, an application of the multi-index Alt formula to summing possibly divergent multi-index series is given. A shift trick then allows one to make the remainder in the Alt formula arbitrarily small.

In Section 4, the mentioned extension to sums over the integral points of integral lattice polytopes is presented.

The necessary proofs are deferred to Section 5.

At the end of this introduction, let us fix notation to be used in the rest of the paper:

Suppose that p and m are natural numbers and $f\colon \mathbb{R}^p \to \mathbb{R}$ is a $2m$-times continuously differentiable function, with partial derivatives $f^{(\boldsymbol{\alpha})}$, where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p) \in \mathbb{Z}_+^p$ and $\mathbb{Z}_+ := \mathbb{Z} \cap [0, \infty)$.

Generally, boldface letters will denote vectors in $\mathbb{R}^p$, in $\mathbb{Z}^p$, or in $\mathbb{Z}_+^p$, with the coordinates denoted by the corresponding non-boldface letters with the indices: $\mathbf{x} = (x_1, \dots, x_p) \in \mathbb{R}^p$, $\mathbf{y} = (y_1, \dots, y_p) \in \mathbb{R}^p$, $\mathbf{u} = (u_1, \dots, u_p) \in \mathbb{R}^p$, $\mathbf{v} = (v_1, \dots, v_p) \in \mathbb{R}^p$, $\mathbf{n} = (n_1, \dots, n_p) \in \mathbb{Z}_+^p$, $\mathbf{k} = (k_1, \dots, k_p) \in \mathbb{Z}_+^p$, $\mathbf{j} = (j_1, \dots, j_p) \in \mathbb{Z}_+^p$, $\mathbf{i} = (i_1, \dots, i_p) \in \mathbb{Z}_+^p$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p) \in \mathbb{Z}_+^p$, and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p) \in \mathbb{Z}^p$. Let $\mathrm{I}\{A\}$ denote the indicator of an assertion A; that is, $\mathrm{I}\{A\} := 1$ if A is true and $\mathrm{I}\{A\} := 0$ if A is false. Let $\|\boldsymbol{\alpha}\| := \|\boldsymbol{\alpha}\|_1 = \alpha_1 + \dots + \alpha_p$; $\boldsymbol{\alpha}! := \alpha_1! \cdots \alpha_p!$; $\mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_p^{\alpha_p}$; $|\boldsymbol{\beta}| := (|\beta_1|, \dots, |\beta_p|)$; $\mathbf{1} := (1, \dots, 1) \in \mathbb{Z}_+^p$; $\mathbf{0} := 0\mathbf{1}$; $\mathbf{j}\mathbf{v} := (j_1 v_1, \dots, j_p v_p)$;

$$\mathbf{j} \ge \mathbf{i} \overset{\text{def}}{\iff} \mathbf{i} \le \mathbf{j} \overset{\text{def}}{\iff} i_r \le j_r \text{ for all } r \in [p] := \{1, \dots, p\};$$

$$[\mathbf{u}, \mathbf{v}] := \prod_{r=1}^{p} [u_r, v_r]; \wedge\mathbf{x} := x_1 \wedge \dots \wedge x_p; \vee\mathbf{x} := x_1 \vee \dots \vee x_p;$$

$$\mathbf{u} \wedge \mathbf{v} := (u_1 \wedge v_1, \dots, u_p \wedge v_p); \quad \mathbf{u} \vee \mathbf{v} := (u_1 \vee v_1, \dots, u_p \vee v_p);$$

$$\sum_{\mathbf{i}=\mathbf{j}}^{\mathbf{k}} := \sum_{\mathbf{i} \in \mathbb{Z}_+^p \colon \mathbf{j} \le \mathbf{i} \le \mathbf{k}}; \qquad \int_{\mathbf{u}}^{\mathbf{v}} \mathrm{d}\mathbf{x}\, h(\mathbf{x}) := (-1)^{\sum_{r=1}^p \mathrm{I}\{u_r > v_r\}} \int_{[\mathbf{u} \wedge \mathbf{v}, \mathbf{u} \vee \mathbf{v}]} \mathrm{d}\mathbf{x}\, h(\mathbf{x});$$

$$\int_{\mathbf{u}}^{\mathbf{v}} := \int_{\mathbf{u}}^{\mathbf{v}} \mathrm{d}\mathbf{x}\, f(\mathbf{x}).$$

Let $\mathbb{R}_+^p := [0, \infty)^p$.

2. A multi-index alternative (Alt) to the EM formula

The following extension of [19, Theorem 3.1] to multiple sums is the main result of this paper:

Theorem 2.1. *One has*

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f(\mathbf{k})\Big[=\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_p=0}^{n_p-1} f(k_1,\dots,k_p)\Big]=A_m-R_m, \tag{2.3}$$

where

$$A_m := \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{\mathbf{i}-\mathbf{j}/2}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}} = \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}} \tag{2.4}$$

$$= \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+|\boldsymbol{\beta}|}\int_{\boldsymbol{\beta}/2-\mathbf{1}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2} = \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+|\boldsymbol{\beta}|}\int_{-\mathbf{1}/2-\boldsymbol{\beta}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2} \tag{2.5}$$

$$= \sum_{\boldsymbol{\alpha}=\mathbf{0}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+\boldsymbol{\alpha}}\sum_{\boldsymbol{\beta}\colon|\boldsymbol{\beta}|=\boldsymbol{\alpha}}\int_{\boldsymbol{\beta}/2-\mathbf{1}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2} = \sum_{\boldsymbol{\alpha}=\mathbf{0}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+\boldsymbol{\alpha}}\sum_{\boldsymbol{\beta}\colon|\boldsymbol{\beta}|=\boldsymbol{\alpha}}\int_{-\mathbf{1}/2-\boldsymbol{\beta}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2} \tag{2.6}$$

is the integral approximation to the sum $\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f(\mathbf{k})$,

$$\gamma_{m,\mathbf{j}} := \prod_{r=1}^{p}\gamma_{m,j_r},\quad \gamma_{m,j} := (-1)^{j-1}\frac{2}{j}\binom{2m}{m+j}\Big/\binom{2m}{m}, \tag{2.7}$$

$$\tau_{m,\mathbf{j}} := \prod_{r=1}^{p}\tau_{m,j_r},\quad \tau_{m,j} := \sum_{\beta=0}^{\lfloor m/2-j/2\rfloor}\gamma_{m,j+2\beta} = \sum_{\beta=0}^{\infty}\gamma_{m,j+2\beta}, \tag{2.8}$$

and R_m *is the remainder given by the formula*

$$R_m := \frac{m}{2^{2m+p-1}} \times \sum_{\|\boldsymbol{\alpha}\|=2m}\frac{1}{\boldsymbol{\alpha}!}\int_0^1 \mathrm{d}s\,(1-s)^{2m-1}\int_{-\mathbf{1}}^{\mathbf{1}}\mathrm{d}\mathbf{v}\,\mathbf{v}^{\boldsymbol{\alpha}}\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\,\mathbf{j}^{\boldsymbol{\alpha}+\mathbf{1}}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f^{(\boldsymbol{\alpha})}(\mathbf{k}+s\mathbf{j}\mathbf{v}/2). \tag{2.9}$$

The sum of all the coefficients of the integrals in each of the expressions (2.4), (2.5), *and* (2.6) *of* A_m *is*

$$\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}1 = \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\,\mathbf{j}^{\mathbf{1}} = \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tau_{m,\mathbf{1}+|\boldsymbol{\beta}|} = 1. \tag{2.10}$$

If M_{2m} *is a real number such that*

$$\Big|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f^{(\boldsymbol{\alpha})}(\mathbf{k}+\mathbf{u})\Big| \le M_{2m}\quad \textit{for all } \boldsymbol{\alpha} \textit{ with } \|\boldsymbol{\alpha}\|=2m \textit{ and all } \mathbf{u}\in(-m\mathbf{1}/2, m\mathbf{1}/2], \tag{2.11}$$

then the remainder R_m *can be bounded as follows:*

$$|R_m| \le \frac{M_{2m}}{2^{2m}}\sum_{\|\boldsymbol{\alpha}\|=2m}\frac{1}{(\boldsymbol{\alpha}+\mathbf{1})!}\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}|\gamma_{m,\mathbf{j}}|\,\mathbf{j}^{\boldsymbol{\alpha}+\mathbf{1}} \tag{2.12}$$

$$\le M_{2m}\frac{1.0331(\pi m)^{(p+1)/2}}{(2m+1)!}(\kappa p m)^{2m}, \tag{2.13}$$

where

$$\kappa := \sqrt{\frac{\Lambda_*}{4}} = 0.27754\ldots \tag{2.14}$$

and

$$\Lambda_* := \max_{0<t<1} \Lambda(t) = 0.3081\ldots, \qquad \Lambda(t) := (1-t)^{t-1}(1+t)^{-1-t}t^2.$$

If $m \geq 2$, then the factor 1.0331 in (2.13) can be replaced by 1.001.

Recall the convention that the sum of an empty family is 0. In particular, if $\wedge\mathbf{n} = 0$, then $\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f(\mathbf{k}) = 0 = A_m = R_m$.

Also, it is clear that $R_m = 0$ if the function f is any polynomial of degree at most $2m-1$.

One may note here that, in each of the formulas (2.4), (2.5), and (2.6), the first expression is a linear combination of integrals of the form $\int_{-\boldsymbol{\lambda}}^{\mathbf{n}-\mathbf{1}+\boldsymbol{\lambda}}$ for some $\boldsymbol{\lambda} \in \mathbb{R}^p$ with $|\boldsymbol{\lambda}| \leq (m-2)\mathbf{1}/2$. So, provided that $\mathbf{n} \geq (m-1)\mathbf{1}$, each of these integrals equals the Lebesgue integral of the function f over the p-dimensional interval $[-\boldsymbol{\lambda}, \mathbf{n}-\mathbf{1}+\boldsymbol{\lambda}]$, symmetric about the point $(\mathbf{n}-\mathbf{1})/2$.

In contrast, the second expression in each of the formulas (2.4), (2.5), and (2.6) is a linear combination of integrals of the form $\int_{\boldsymbol{\lambda}}^{\mathbf{n}+\boldsymbol{\lambda}}$ for some $\boldsymbol{\lambda} \in \mathbb{R}^p$; so, each of these integrals equals the Lebesgue integral of the function f over the p-dimensional interval $[\boldsymbol{\lambda}, \mathbf{n}+\boldsymbol{\lambda}]$, whose endpoints differ by the vector $\mathbf{n}$. This observation holds whether the condition $\mathbf{n} \geq (m-1)\mathbf{1}$ holds or not.

Remark 2.1. *As in [19] in the special case of ordinary sums, here, instead of assuming that the function f is real-valued, one may assume, more generally, that f takes values in any normed space. In particular, one may allow f to take values in the q-dimensional complex space $\mathbb{C}^q$, for any natural q. An advantage of dealing with a vector-valued function (rather than separately with each of its coordinates) is that this way one has to compute the coefficients – say $\tau_{m,\boldsymbol{\beta}}$ in (2.6) – only once, for all the components of the vector function.* □

3. Application to summing (possibly divergent) multi-index series

Let us say that a function $F\colon \mathbb{R}^p \to \mathbb{R}$ is an antiderivative of the function f if

$$F^{(\mathbf{1})} = f;$$

that is, if F is differentiated once with respect to every one of the p arguments of the function F, then the result of this p-fold partial differentiation is the function f. It is assumed that this result does not depend on the order of the arguments with respect to which the partial derivatives are taken. Here and elsewhere in the paper, f and p are as set in Section 1. In particular, it follows that the function f is continuous. Clearly, this notion of an antiderivative is a generalization of the corresponding notion for functions on $\mathbb{R}$.

For each set $J \subseteq [p]$, let $|J|$ denote the cardinality of J, and also let

$$\mathbf{1}_J := (\mathrm{I}\{1 \in J\}, \ldots, \mathrm{I}\{p \in J\}).$$

In particular, $\mathbf{1}_{[p]} = \mathbf{1}$ and $\mathbf{1}_\emptyset = \mathbf{0}$.

Remark 3.2. *A function F on $\mathbb{R}^p$ is an antiderivative of the function f if and only if one has a representation of the form*

$$F(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} \mathrm{d}\mathbf{y}\, f(\mathbf{y}) + \sum_{j=1}^{p} c_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p)$$

for all $\mathbf{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p$, where $c_1, \ldots, c_p$ are functions on $\mathbb{R}^{p-1}$ such that, for each $j \in \{1, \ldots, p\}$ and all $(x_1, \ldots, x_p) \in \mathbb{R}^p$, the mixed partial derivative $\dfrac{\partial^{p-1} c_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p)}{\partial x_1 \cdots \partial x_{j-1} \partial x_{j+1} \cdots \partial x_p}$ exists and does not depend on the order of the arguments $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p$ with respect to which the partial derivatives are taken.

The "if" part of the above statement is obvious. The "only if" part of it follows from the multidimensional version of the fundamental theorem of calculus to be given by Lemma 5.1 in Section 5 (take there $\mathbf{0}$ *and* $\mathbf{x}$*, respectively, in place of* $\mathbf{u}$ *and* $\mathbf{v}$ *in Lemma 5.1, and note that then* $F(\mathbf{v}_{[p]}) = F(\mathbf{v}) = F(\mathbf{x})$*).*

In particular, the function F *on* $\mathbb{R}^p$ *given by the condition* $F(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} \mathrm{d}\mathbf{y}\, f(\mathbf{y})$ *for all* $\mathbf{x} \in \mathbb{R}^p$ *is clearly an antiderivative of* f*; thus, there always exists an antiderivative of the function* f *– still assuming, of course, that* f *is* $2m$*-times continuously differentiable for some natural* m*; in fact, just the continuity of* f *would be enough for the existence of an antiderivative of* f*.* □

The alternative summation formula presented in Theorem 2.1 can be used for summing (possibly divergent) multi-index series, as follows.

Theorem 3.2. *Let* m_0 *be a natural number, and suppose that* $m \ge m_0$*. Let* F *be any antiderivative of* f*. Suppose that*

$$F^{(\boldsymbol{\alpha})}(x) \underset{\vee\mathbf{x}\to\infty}{\longrightarrow} 0 \text{ for each } \boldsymbol{\alpha} \in \mathbb{Z}_+^p \text{ with } \|\boldsymbol{\alpha}\| = 2m_0 \tag{3.15}$$

and the series

$$\sum_{\mathbf{k}=\mathbf{0}}^{\infty\mathbf{1}} f^{(\boldsymbol{\alpha})}(\mathbf{k}+\mathbf{u}) \text{ converges uniformly in } \mathbf{u} \in [-m\mathbf{1}/2, m\mathbf{1}/2] \quad \text{for each } \boldsymbol{\alpha} \in \mathbb{Z}_+^p \text{ with } \|\boldsymbol{\alpha}\| = 2m, \tag{3.16}$$

in the sense that $\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f^{(\boldsymbol{\alpha})}(\mathbf{k}+\mathbf{u})$ *converges uniformly as* $\wedge\mathbf{n} \to \infty$*. Then*

$$\sum_{\mathbf{k}\ge\mathbf{0}}^{\mathsf{Alt}} f(\mathbf{k}) := \lim_{\wedge\mathbf{n}\to\infty} \Big(\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f(\mathbf{k}) - \tilde{A}_{m_0,F}(\mathbf{n}) \Big) = (-1)^p A_{m,F}^{\emptyset}(\mathbf{0}) - R_{m,f}(\infty), \tag{3.17}$$

where (cf. (2.4), (2.5)*, and* (2.6)*)*

$$\tilde{A}_{m,F}(\mathbf{n}) := \sum_{\emptyset \ne J \subseteq [p]} (-1)^{p-|J|} A_{m,F}^J(\mathbf{n}), \tag{3.18}$$

$$A_{m,F}^J(\mathbf{n}) := \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}} \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}} F(\mathbf{n}\mathbf{1}_J - \mathbf{1} + \mathbf{j}/2 - \mathbf{i}) \tag{3.19}$$

$$= \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+|\boldsymbol{\beta}|}\, F(\mathbf{n}\mathbf{1}_J - \mathbf{1}/2 - \boldsymbol{\beta}/2) \tag{3.20}$$

$$= \sum_{\boldsymbol{\alpha}=\mathbf{0}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}:\ |\boldsymbol{\beta}|=\boldsymbol{\alpha}} F(\mathbf{n}\mathbf{1}_J - \mathbf{1}/2 - \boldsymbol{\beta}/2), \tag{3.21}$$

and (cf. (2.9)*)*

$$R_{m,f}(\infty) := \frac{m}{2^{2m+p-1}} \times \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{1}{\boldsymbol{\alpha}!} \int_0^1 \mathrm{d}s\,(1-s)^{2m-1} \int_{-\mathbf{1}}^{\mathbf{1}} \mathrm{d}\mathbf{v}\, \mathbf{v}^{\boldsymbol{\alpha}} \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}}\, \mathbf{j}^{\boldsymbol{\alpha}+\mathbf{1}} \sum_{\mathbf{k}=\mathbf{0}}^{\infty\mathbf{1}} f^{(\boldsymbol{\alpha})}(\mathbf{k}+s\mathbf{j}\mathbf{v}/2).$$

If condition (2.11) *holds for all* $\mathbf{n} \in \mathbb{Z}_+^p$*, then one can replace* R_m *in* (2.12)–(2.13) *by* $R_{m,f}(\infty)$*, so that*

$$|R_{m,f}(\infty)| \le M_{2m} \frac{1.0331(\pi m)^{(p+1)/2}}{(2m+1)!} (\kappa p m)^{2m}. \tag{3.22}$$

Looking, say, at the expression of $A^J_{m,F}(\mathbf{n})$ in (3.21), one may note that

$$A^{\emptyset}_{m,F}(\mathbf{0}) = A^{\emptyset}_{m,F}(\mathbf{n}) = A^J_{m,F}(\mathbf{0}) = \sum_{\boldsymbol{\alpha}=\mathbf{0}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}\colon |\boldsymbol{\beta}|=\boldsymbol{\alpha}} F(\boldsymbol{\beta}/2 - \mathbf{1}/2) \tag{3.23}$$

for all $\mathbf{n} \in \mathbb{Z}^p_+$ and $J \subseteq [p]$.

The limit $\sum^{\mathsf{Alt}}_{\mathbf{k}\geq\mathbf{0}} f(\mathbf{k})$ in (3.17) may be referred to as the (generalized) sum of the possibly divergent multi-index series $\sum^{\infty\mathbf{1}}_{\mathbf{k}=\mathbf{0}} f(\mathbf{k})$ by means of the Alt formula (2.3).

Theorem 3.2 is a multi-index extension of Proposition 5.1 in [19].

To compute the generalized sum $\sum^{\mathsf{Alt}}_{\mathbf{k}\geq\mathbf{0}} f(\mathbf{k})$ effectively, one has to ensure that the remainder $R_{m,f}(\infty)$ can be made arbitrarily small. This can be done as follows.

For any function $h\colon \mathbb{R}^p \to \mathbb{R}$ and any $\mathbf{c} \in \mathbb{R}^p$, let $h_{\mathbf{c}}$ denote the $\mathbf{c}$-shift of h defined by the formula

$$h_{\mathbf{c}}(\mathbf{x}) := h(\mathbf{x} + \mathbf{c}) \tag{3.24}$$

for all $\mathbf{x} \in \mathbb{R}^p$. Note that, if F is an antiderivative of f, then $F_{\mathbf{c}}$ is an antiderivative of $f_{\mathbf{c}}$.

Theorem 3.3. *Suppose that the conditions of Theorem 3.2 hold. Take any $\mathbf{c} \in \mathbb{Z}^p_+$. Then*

$$\sum^{\mathsf{Alt}}_{\mathbf{k}\geq\mathbf{0}} f(\mathbf{k}) = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{c}-\mathbf{1}} f(\mathbf{k}) - \tilde{A}_{m,F}(\mathbf{c}) - R_{m,f,\mathbf{c}}(\infty), \tag{3.25}$$

where

$$R_{m,f,\mathbf{c}}(\infty) := -\sum_{\emptyset\neq J\subseteq[p]} (-1)^{p-|J|} R_{m,f_{\mathbf{c}\mathbf{1}_J}}(\infty) \tag{3.26}$$

(cf. (3.18)).

Under the conditions of Theorem 3.2, the remainder $R_{m,f,\mathbf{c}}(\infty)$ can be made arbitrarily small by making $\wedge\mathbf{c}$ large enough. The price to pay for this will be the need to compute a possibly large partial sum $\sum^{\mathbf{c}-\mathbf{1}}_{\mathbf{k}=\mathbf{0}} f(\mathbf{k})$ of the series.

Theorem 3.3 is a multi-index extension of Corollary 5.6 in [19].

Example 3.1. *In Theorem 3.3, let $p = 2$ and take any 4-times continuously differentiable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that*

$$f(x, y) = (x + y + 2)\ln(x + y + 2)$$

for real $x, y \geq 0$. Such a function f exists, by Whitney's theorem [24]; however, only the values of f on $[0,\infty)^2$ will matter for the purposes of this example. Then it is straightforward to check by direct differentiation that for an antiderivative F of f and all real $x, y \geq 0$ one will have

$$F(x, y) = \tfrac{1}{6}\,(x + y + 2)^3\,\ln(x + y + 2) - \tfrac{5}{12}\,(x + 1)(y + 1)(x + y + 2).$$

It is also straightforward to verify conditions (3.15) and (3.16) of Theorem 3.2 with $m_0 = m = 2$.

It also follows that, for $\mathbf{n} = (n, n)$, the term $\tilde{A}_{m,F}(\mathbf{n}) = \tilde{A}_{2,F}\big((n, n)\big)$ (defined in (3.18)) is expressed as a linear combination of certain terms of the form $P(n)\ln(a + bn)$ or $P(n)$, where P is polynomial with real coefficients, a is a nonnegative real number, and b is a positive real number. Replacing, in that expression for $\tilde{A}_{m,F}(\mathbf{n})$, every instance of $\ln(a + bn)$ by its large-n asymptotics $\ln n + \ln b + \frac{a}{bn} - \frac{a^2}{2b^2n^2} + \frac{a^3}{3b^3n^3} + O(\frac{1}{n^4})$, after some rather heavy algebra we find

$$\tilde{A}_{m,F}(\mathbf{n}) = S_n + \delta_n,$$

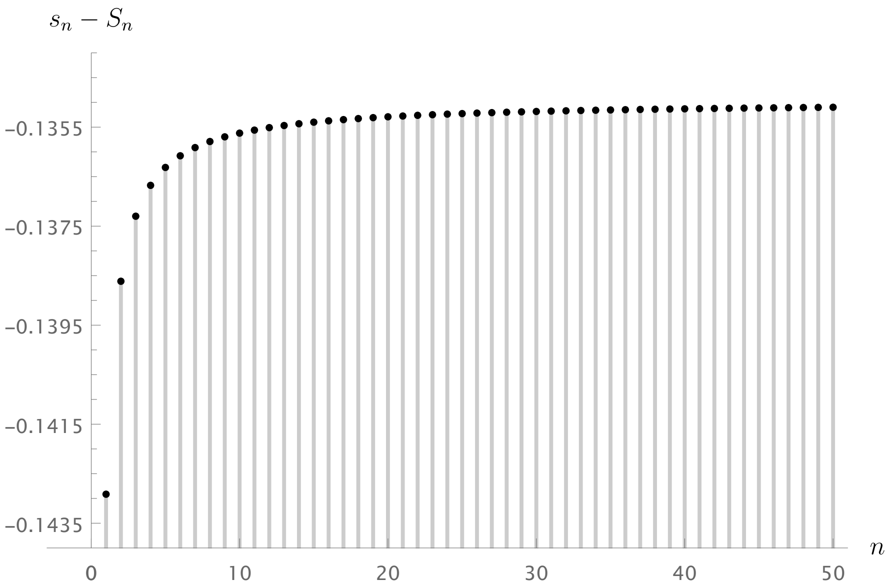

FIGURE 1. Graph $\{(n, s_n - S_n) \colon n \in \{1, \dots, 50\}\}$

where

$$S_n := n^3 \ln \frac{2^{4/3}\, n}{e^{5/6}} + n^2 \ln \frac{4n}{\sqrt{e}} + \frac{5}{6} n \ln 2 - \frac{1}{12} \ln \frac{en}{2} \tag{3.27}$$

and

$$\delta_n = O(1/n^2). \tag{3.28}$$

Thus, by Theorem 3.3,

$$s_n := \sum_{k=1}^{n} \sum_{l=1}^{n} (k+l) \ln(k+l) = S_n + L + r_n,$$

where

$$L := \sum_{\mathbf{k} \geq \mathbf{0}}^{\mathsf{Alt}} f(\mathbf{k}) = \lim_{n\to\infty} (s_n - S_n) \in \mathbb{R}$$

and

$$r_n := \delta_n + R_{2,f,(n,n)}(\infty) = O(1/n), \tag{3.29}$$

in view of (3.28), (3.26), (3.24), (3.22), (2.11), and (2.14); the universal positive real constant factor in $O(1/n)$ in (3.29) can be given explicitly. Note that the bound $O(1/n)$ on the error term r_n in (3.29) can be improved to $O(1/n^{m-1})$ by choosing the "approximation order" m in formula (3.25) to be any natural number greater than 2; of course, then the expression for S_n in (3.27) will have to be replaced by a more complicated expression.

The convergence of $s_n - S_n$ to the limit L is illustrated in Figure 1, which shows the discrete graph $\{(n, s_n - S_n) \colon n \in \{1, \dots, 50\}\}$. □

4. Application to sums over the integral points of integral lattice polytopes

Let P be an integral polytope in $\mathbb{R}^p$, that is, the convex hull of a finite subset of $\mathbb{Z}^p$.

Suppose that P is of full dimension, p. Let V denote the set of all vertices (that is, extreme points) of P.

By the main result of Haase [11], for each $\mathbf{v} \in V$ there exist a finite set $I_\mathbf{v}$, a map $I_\mathbf{v} \ni i \mapsto t_{\mathbf{v},i} \in \{0,1\}$, a map $I_\mathbf{v} \ni i \mapsto A_{\mathbf{v},i}$ into the set of all nonsingular $p \times p$ matrices over $\mathbb{Z}$, and a map $I_\mathbf{v} \ni i \mapsto J_{\mathbf{v},i}$ into the set of all subsets of the set $[p] = \{1, \dots, p\}$ such that

$$\llbracket P \rrbracket = \sum_{\mathbf{v}\in V} \sum_{i\in I_\mathbf{v}} (-1)^{t_{\mathbf{v},i}} \llbracket C_{\mathbf{v},i} \rrbracket, \tag{4.30}$$

where $\llbracket\cdot\rrbracket$ denotes the indicator/characteristic function,

$$C_{\mathbf{v},i} := \mathbf{v} + A_{\mathbf{v},i}\mathbb{R}^+_{J_{\mathbf{v},i}} = \{\mathbf{v} + A_{\mathbf{v},i}\mathbf{x} \colon \mathbf{x} \in \mathbb{R}^+_{J_{\mathbf{v},i}}\}, \tag{4.31}$$

$$\mathbb{R}^+_J := \prod_{j\in[p]} \mathbb{R}^+_{1-\llbracket J\rrbracket(j)} \quad \text{for} \quad J \subseteq [p],$$

and

$$\mathbb{R}^+_\varepsilon := \begin{cases} (0,\infty) & \text{if } \varepsilon = 0, \\ [0,\infty) & \text{if } \varepsilon = 1 \end{cases}$$

(so that the closure of $C_{\mathbf{v},i}$ is a polyhedral cone, for each pair $(\mathbf{v}, i)$). In the case when the polytope P is simple, decomposition (4.30) was obtained earlier by Lawrence [17]. To extend Lawrence's result, Haase used virtual infinitesimal deformations of vertices of P, identified with regular triangulations of the normal cones at the vertices.

Proposition 4.1. *Let A be any nonsingular $p \times p$ matrix over $\mathbb{Z}$, and let J be any subset of the set $[p]$. Then there exist a finite set I, a map $I \ni i \mapsto A_i$ into the set of all* unimodular *$p \times p$ matrices over $\mathbb{Z}$, and a map $I \ni i \mapsto J_i$ into the set of all subsets of the set $[p]$ such that*

$$\llbracket A\mathbb{R}^+_J \rrbracket = \sum_{i\in I} \llbracket A_i \mathbb{R}^+_{J_i} \rrbracket. \tag{4.32}$$

(Recall that a matrix is called unimodular if its determinant is 1 or -1.)

Thus, one can strengthen the statement on the decomposition (4.30) as follows:

Corollary 4.1. *One may assume that all the matrices $A_{\mathbf{v},i}$ in (4.30)–(4.31) are unimodular.*

A similar decomposition, but with polyhedral cones of lower dimensions, was obtained in [5].

The following corollary is almost immediate from Theorem 2.1 and Corollary 4.1.

Corollary 4.2. *Suppose that the function f is compactly supported. Then*

$$\sum_{\mathbf{k}\in P\cap\mathbb{Z}^p} f(\mathbf{k}) = A_m(f,P) - R_m(f,P), \tag{4.33}$$

where

$$A_m(f,P) := \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+|\boldsymbol{\beta}|} \sum_{\mathbf{v}\in V} (-1)^{t_\mathbf{v}} \sum_{i\in I_\mathbf{v}} \int_{C_{\mathbf{v},i}+A_{\mathbf{v},i}(\mathbf{1}_{J_{\mathbf{v},i}}-(\mathbf{1}+\boldsymbol{\beta})/2)} \mathrm{d}\mathbf{x}\, f(\mathbf{x}) \tag{4.34}$$

is the integral approximation to the sum $\sum\limits_{\mathbf{k}\in P\cap\mathbb{Z}^p} f(\mathbf{k})$ and $R_m(f,P)$ is the remainder given by the formula

$$R_m(f,P) := \frac{m}{2^{2m+p-1}} \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{1}{\boldsymbol{\alpha}!} \int_0^1 \mathrm{d}s\,(1-s)^{2m-1} \int_{-\mathbf{1}}^{\mathbf{1}} \mathrm{d}\mathbf{u}\,\mathbf{u}^{\boldsymbol{\alpha}}\,\Sigma_m(s\mathbf{u})$$

with

$$\Sigma_m(\mathbf{w}) := \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}}\,\mathbf{j}^{\boldsymbol{\alpha}+\mathbf{1}} \sum_{\mathbf{v}\in V} \sum_{i\in I_{\mathbf{v}}} (-1)^{t_{\mathbf{v}}} \sum_{\mathbf{k}\geq\mathbf{0}} g_{\mathbf{v},i}^{(\boldsymbol{\alpha})}(\mathbf{k}+\mathbf{1}_{J_{\mathbf{v},i}}+\mathbf{j}\mathbf{w}/2)$$

and

$$g_{\mathbf{v},i}(\mathbf{y}) := f(\mathbf{v}+A_{\mathbf{v},i}\mathbf{y})$$

for $\mathbf{y}\in\mathbb{R}^p$. *If* M_{2m} *is a real number such that*

$$\Big|\sum_{\mathbf{v}\in V}\sum_{i\in I_{\mathbf{v}}}(-1)^{t_{\mathbf{v}}}\sum_{\mathbf{k}\geq\mathbf{0}} g_{\mathbf{v},i}^{(\boldsymbol{\alpha})}(\mathbf{k}+\mathbf{u})\Big| \leq M_{2m} \quad \textit{whenever} \quad \|\boldsymbol{\alpha}\|=2m \quad \textit{and} \quad |\mathbf{u}|\leq(\tfrac{m}{2}+1)\mathbf{1},$$

then

$$|R_m(f,P)| \leq M_{2m}\,\frac{1.0331(\pi m)^{(p+1)/2}}{(2m+1)!}\,(\kappa p m)^{2m},$$

where κ *is as in* (2.14).

Indeed, for $J\subseteq[p]$, let

$$\mathbb{Z}_J^+ := \mathbb{Z}^p\cap\mathbb{R}_J^+ = \mathbb{Z}_+^p+\mathbf{1}_J,$$

where $\mathbb{Z}_+ := \mathbb{Z}\cap[0,\infty)$. Note that $A\mathbb{Z}^p=\mathbb{Z}^p$ for any unimodular matrix A over $\mathbb{Z}$. Now Corollary 4.2 follows by Corollary 4.1 and Theorem 2.1 because

$$\sum_{\mathbf{k}\in C_{\mathbf{v},i}\cap\mathbb{Z}^p} f(\mathbf{k}) = \sum_{\mathbf{q}\in\mathbb{Z}^+_{J_{\mathbf{v},i}}} f(\mathbf{v}+A_{\mathbf{v},i}\mathbf{q}) = \sum_{\mathbf{q}\geq\mathbf{0}} f(\mathbf{v}+A_{\mathbf{v},i}(\mathbf{q}+\mathbf{1}_{J_{\mathbf{v},i}})) = \sum_{\mathbf{q}\geq\mathbf{0}} g_{\mathbf{v},i}(\mathbf{q}+\mathbf{1}_{J_{\mathbf{v},i}})$$

and

$$\int_{[-\mathbf{1}/2-\boldsymbol{\beta}/2,\,\infty\mathbf{1})} \mathrm{d}\mathbf{y}\,g_{\mathbf{v},i}(\mathbf{y}+\mathbf{1}_{J_{\mathbf{v},i}}) = \int_{C_{\mathbf{v},i}+A_{\mathbf{v},i}(\mathbf{1}_{J_{\mathbf{v},i}}-(\mathbf{1}+\boldsymbol{\beta})/2)} \mathrm{d}\mathbf{x}\,f(\mathbf{x}).$$

The expression for $A_m(f,P)$ in (4.34) is based on the second expression for A_m in (2.5); of course, one can quite similarly use any one of the other 5 expressions in (2.4)–(2.6).

Notable differences between the Alt formula in Corollary 4.2 and the EM formula that is the main result of [14] (Theorem 2 therein) include the following: (i) in [14, Theorem 2], the summation is over all faces of the polytope P, whereas in (4.34) the corresponding summation is only over the vertices of P and (ii) instead of the plain summation $\sum_{\mathbf{k}\in P\cap\mathbb{Z}^p} f(\mathbf{k})$ in (4.33), in the corresponding sum in [14] the summands $f(\mathbf{k})$ are weighted (in accordance with the dimension of the relative interior of the face given that $\mathbf{k}$ belongs to that relative interior).

Note also that [14, Theorem 2] is obtained for simple polytopes. In [3], this result was extended to allow more general weights, and then further generalized to non-simple polytopes in [4].

The version of the EM formula for polytopes in [6] is given for polynomial functions f in terms of differential operators of infinite order, with the summation over all faces of the polytope.

It should be possible to extend Corollary 4.2 to the case when the function f is a so-called symbol in the sense of Hörmander [12] – cf. [14, Theorem 3], as well as conditions (3.15) and (3.16). (Recall that a function $f\in C^\infty(\mathbb{R}^p)$ is called a symbol of order N if for every $\boldsymbol{\alpha}\in\mathbb{Z}_+^p$ there is a real constant $C_{\boldsymbol{\alpha}}$ such that $|f^{(\boldsymbol{\alpha})}(\mathbf{x})|\leq C_{\boldsymbol{\alpha}}(1+\|\mathbf{x}\|)^{N-\|\boldsymbol{\alpha}\|}$ for all $\mathbf{x}\in\mathbb{R}^p$; here, as before, $\|\cdot\|:=\|\cdot\|_1$.) One way to attack this goal could be to show that, for any $\boldsymbol{\alpha}\in\mathbb{Z}_+^p$ such that $\boldsymbol{\alpha} \le (m-1)\mathbf{1}$, the essential support (except possibly for a set of Lebesgue measure 0) of the function

$$\sum_{\boldsymbol{\beta}\colon |\boldsymbol{\beta}|=\boldsymbol{\alpha}} \sum_{\mathbf{v}\in V} \sum_{i\in I_{\mathbf{v}}} (-1)^{t_{\mathbf{v},i}} \, [\![C_{\mathbf{v},i} + A_{\mathbf{v},i}(\mathbf{1}_{J_{\mathbf{v},i}} - (\mathbf{1}+\boldsymbol{\beta})/2)]\!]$$

is bounded, presumably being just a perturbed version of the indicator of the polytope P; cf. (4.34) and the equality in [14, formula (89)].

Moreover, in view of the results of Section 3, it appears not unlikely that Corollary 4.2 could be extended to general polyhedral sets.

5. Proofs

Proof of Theorem 2.1. Take any $\mathbf{k}$ (in $\mathbb{Z}_+^p$) such that $\mathbf{k} \le \mathbf{n} - \mathbf{1}$ and consider the Taylor expansion

$$f(\mathbf{x}) = \sum_{\|\boldsymbol{\alpha}\|\le 2m-1} \frac{f^{(\boldsymbol{\alpha})}(\mathbf{k})}{\boldsymbol{\alpha}!}\,\mathbf{u}^{\boldsymbol{\alpha}} + \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{2m}{\boldsymbol{\alpha}!}\,\mathbf{u}^{\boldsymbol{\alpha}} \int_0^1 \mathrm{d}s\,(1-s)^{2m-1} f^{(\boldsymbol{\alpha})}(\mathbf{k}+s\mathbf{u}) \tag{5.35}$$

for all $\mathbf{x} \in (\mathbf{k} - m\mathbf{1}/2,\, \mathbf{k} + m\mathbf{1}/2]$, where $\mathbf{u} := \mathbf{x} - \mathbf{k}$. Integrating both sides of this identity in $\mathbf{x} \in (\mathbf{k} - \mathbf{j}/2,\, \mathbf{k} + \mathbf{j}/2]$ (or, equivalently, in $\mathbf{u} \in (-\mathbf{j}/2,\, \mathbf{j}/2]$) for each $\mathbf{j}$ (in $\mathbb{Z}_+^p$) such that $\mathbf{j} \le m\mathbf{1}$, then multiplying by $\gamma_{m,\mathbf{j}}$, and then summing in $\mathbf{j}$, one has

$$A_{m,\mathbf{k}} = S_{m,\mathbf{k}} + R_{m,\mathbf{k}}, \tag{5.36}$$

where

$$A_{m,\mathbf{k}} := \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}} \int_{\mathbf{k}-\mathbf{j}/2}^{\mathbf{k}+\mathbf{j}/2} \mathrm{d}\mathbf{x}\, f(\mathbf{x}), \tag{5.37}$$

$$S_{m,\mathbf{k}} := \sum_{\|\boldsymbol{\alpha}\|\le m-1} \frac{f^{(2\alpha)}(\mathbf{k})}{(2\boldsymbol{\alpha}+1)!\,2^{2\|\boldsymbol{\alpha}\|}} \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}}\,\mathbf{j}^{2\boldsymbol{\alpha}+\mathbf{1}}, \tag{5.38}$$

$$\begin{aligned} R_{m,\mathbf{k}} &:= \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{2m}{\boldsymbol{\alpha}!} \int_0^1 \mathrm{d}s\,(1-s)^{2m-1} \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}} \int_{-\mathbf{j}/2}^{\mathbf{j}/2} \mathrm{d}\mathbf{u}\; \mathbf{u}^{\boldsymbol{\alpha}}\, f^{(\boldsymbol{\alpha})}(\mathbf{k}+s\mathbf{u}) \\ &= \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{2m}{\boldsymbol{\alpha}!} \int_0^1 \mathrm{d}s\,(1-s)^{2m-1} \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}}\,(\mathbf{j}/2)^{\boldsymbol{\alpha}+\mathbf{1}} \int_{-\mathbf{1}}^{\mathbf{1}} \mathrm{d}\mathbf{v}\; \mathbf{v}^{\boldsymbol{\alpha}}\, f^{(\boldsymbol{\alpha})}(\mathbf{k}+s\mathbf{j}\mathbf{v}/2); \end{aligned} \tag{5.39}$$

the latter equality is obtained by the change of variables $\mathbf{u} = \mathbf{j}\mathbf{v}$.

As noted before, in the special case $p = 1$ Theorem 2.1 turns into Theorem 3.1 of [19]. So, without loss of generality (w.l.o.g.) $p \ge 2$. Write

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} \int_{\mathbf{k}-\mathbf{j}/2}^{\mathbf{k}+\mathbf{j}/2} \mathrm{d}\mathbf{x}\, f(\mathbf{x}) = \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_p=0}^{n_p-1} \int_{k_p-j_p/2}^{k_p+j_p/2} \mathrm{d}x_p \cdots \int_{k_1-j_1/2}^{k_1+j_1/2} \mathrm{d}x_1\, f(\mathbf{x}). \tag{5.40}$$

In view of the multi-line display next after formula (7.7) in [19] (note, in particular, the penultimate expression there), the right-hand side of (5.40) can be rewritten as

$$\begin{aligned}&\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_{p-1}=0}^{n_{p-1}-1}\sum_{i_p=0}^{j_p-1}\int_{i_p-j_p/2}^{n_p-1+j_p/2-i_p}\mathrm{d}x_p\int_{k_{p-1}-j_{p-1}/2}^{k_{p-1}+j_{p-1}/2}\mathrm{d}x_{p-1}\cdots\int_{k_1-j_1/2}^{k_1+j_1/2}\mathrm{d}x_1\, f(\mathbf{x})\\
&=\sum_{i_p=0}^{j_p-1}\int_{i_p-j_p/2}^{n_p-1+j_p/2-i_p}\mathrm{d}x_p\sum_{k_1=0}^{n_1-1}\cdots\sum_{k_{p-1}=0}^{n_{p-1}-1}\int_{k_{p-1}-j_{p-1}/2}^{k_{p-1}+j_{p-1}/2}\mathrm{d}x_{p-1}\cdots\int_{k_1-j_1/2}^{k_1+j_1/2}\mathrm{d}x_1\, f(\mathbf{x})\\
&\ \vdots\\
&=\sum_{i_p=0}^{j_p-1}\int_{i_p-j_p/2}^{n_p-1+j_p/2-i_p}\mathrm{d}x_p\cdots\sum_{i_1=0}^{j_1-1}\int_{i_1-j_1/2}^{n_1-1+j_1/2-i_1}\mathrm{d}x_1\, f(\mathbf{x}).\end{aligned}$$

So,

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}\int_{\mathbf{k}-\mathbf{j}/2}^{\mathbf{k}+\mathbf{j}/2}\mathrm{d}\mathbf{x}\, f(\mathbf{x})=\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{\mathbf{i}-\mathbf{j}/2}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}\mathrm{d}\mathbf{x}\, f(\mathbf{x})$$

and hence, by (5.37),

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}A_{m,\mathbf{k}}=\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}\int_{\mathbf{k}-\mathbf{j}/2}^{\mathbf{k}+\mathbf{j}/2}\mathrm{d}\mathbf{x}\, f(\mathbf{x})=\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{\mathbf{i}-\mathbf{j}/2}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}\mathrm{d}\mathbf{x}\, f(\mathbf{x})=A_m. \tag{5.41}$$

Similarly, but using the last expression in the mentioned multi-line display next after formula (7.7) in [19] rather than the penultimate expression there, we have

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}}A_{m,\mathbf{k}}=\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}\mathrm{d}\mathbf{x}\, f(\mathbf{x}).$$

In particular, it follows that the two double sums in (2.4) are the same.

Suppose now that some $\mathbf{i}$ and $\mathbf{j}$ in $\mathbb{Z}_+^p$ and some $\boldsymbol{\beta}\in\mathbb{Z}^p$ are related by the condition $\boldsymbol{\beta}=2\mathbf{i}-\mathbf{j}+\mathbf{1}$. Then the condition $\mathbf{1}\le\mathbf{j}\le m\mathbf{1}\ \&\ \mathbf{0}\le\mathbf{i}\le\mathbf{j}-\mathbf{1}$ is equivalent to the condition

$$(1-m)\mathbf{1}\le\boldsymbol{\beta}\le(m-1)\mathbf{1}\ \&\ \mathbf{1}+|\boldsymbol{\beta}|\le\mathbf{j}\le m\mathbf{1}\ \&\ (\mathbf{j}-\mathbf{1}-|\boldsymbol{\beta}|)/2\in\mathbb{Z}_+^p.$$

So,

$$\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{\mathbf{i}-\mathbf{j}/2}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}=\sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tilde{\tau}_{m,\mathbf{1}+|\boldsymbol{\beta}|}\int_{\boldsymbol{\beta}/2-\mathbf{1}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2} \tag{5.42}$$

and

$$\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}}\gamma_{m,\mathbf{j}}\sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{j}-\mathbf{1}}\int_{-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}^{\mathbf{n}-\mathbf{1}+\mathbf{j}/2-\mathbf{i}}=\sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}}\tilde{\tau}_{m,\mathbf{1}+|\boldsymbol{\beta}|}\int_{-\mathbf{1}/2-\boldsymbol{\beta}/2}^{\mathbf{n}-\mathbf{1}/2-\boldsymbol{\beta}/2}, \tag{5.43}$$

where

$$\begin{aligned}\tilde{\tau}_{m,\mathbf{1}+|\boldsymbol{\beta}|} &:= \sum_{\mathbf{j}=\mathbf{1}+|\boldsymbol{\beta}|}^{m\mathbf{1}} \gamma_{m,\mathbf{j}}\ \mathrm{I}\left\{(\mathbf{j}-\mathbf{1}-|\boldsymbol{\beta}|)/2 \in \mathbb{Z}_+^p\right\} \\ &= \sum_{j_1=1+|\beta_1|}^{m} \cdots \sum_{j_p=1+|\beta_p|}^{m} \prod_{r=1}^{p} \left(\gamma_{m,j_r}\ \mathrm{I}\left\{(j_r-1-|\beta_r|)/2 \in \mathbb{Z}_+\right\}\right) \\ &= \prod_{r=1}^{p} \sum_{j_r=1+|\beta_r|}^{m} \left(\gamma_{m,j_r}\ \mathrm{I}\left\{(j_r-1-|\beta_r|)/2 \in \mathbb{Z}_+\right\}\right) \\ &= \prod_{r=1}^{p} \tau_{m,1+|\beta_r|} = \tau_{m,\mathbf{1}+|\boldsymbol{\beta}|},\end{aligned}$$

in view of (2.7) and (2.8).

Thus, by (5.42) and (5.43), the first double sum in (2.4) equals the first sum in (2.5), and the second double sum in (2.4) equals the second sum in (2.5).

Also, it is obvious that the first sum in (2.6) equals the first sum in (2.5), and the second sum in (2.6) equals the second sum in (2.5).

Next, for any $\boldsymbol{\alpha}$ (in $\mathbb{Z}_+^p$) with $\|\boldsymbol{\alpha}\| \le m-1$,

$$\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} \gamma_{m,\mathbf{j}}\, \mathbf{j}^{2\boldsymbol{\alpha}+\mathbf{1}} = \sum_{j_1=1}^{m} \cdots \sum_{j_p=1}^{m} \prod_{r=1}^{p} \left(\gamma_{m,j_r} j_r^{2\alpha_r+1}\right) = \prod_{r=1}^{p} \sum_{j=1}^{m} \gamma_{m,j} j^{2\alpha_r+1} = \mathrm{I}\{\boldsymbol{\alpha}=\mathbf{0}\} \tag{5.44}$$

by formula (7.6) in [19]. So, by (5.38),

$$S_{m,\mathbf{k}} = f(\mathbf{k}). \tag{5.45}$$

Also, the case $\boldsymbol{\alpha} = \mathbf{0}$ in (5.44) shows that the first two sums in (2.10), involving the $\gamma_{m,\mathbf{j}}$'s, are equal to 1. The second equality in (2.10) follows from the equality of the first sums in (2.4) and (2.5) to each other by taking there $\mathbf{n} = m\mathbf{1}$ and $f(\mathbf{x}) \equiv \mathrm{I}\{(m/2-1)\mathbf{1} \le \mathbf{x} \le m\mathbf{1}/2\}$; then each of the integrals in (2.4)–(2.6) equals 1.

By (5.39) and (2.9),

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} R_{m,\mathbf{k}} = R_m.$$

So, (2.3) follows immediately from (5.36), (5.41), and (5.45).

In view of (2.9) and (2.11),

$$|R_m| \le \tilde{R}_m := M_{2m} \frac{m}{2^{2m+p-1}} \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{1}{\boldsymbol{\alpha}!} \int_0^1 \mathrm{d}s\,(1-s)^{2m-1} \int_{-\mathbf{1}}^{\mathbf{1}} \mathrm{d}\mathbf{v}\,|\mathbf{v}|^{\boldsymbol{\alpha}} \sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} |\gamma_{m,\mathbf{j}}|\,\mathbf{j}^{\boldsymbol{\alpha}+\mathbf{1}}.$$

Computing the integrals here, it is easy to check that $\tilde{R}_m$ equals the upper bound in (2.12). On the other hand, using the multinomial formula, the definition of $\gamma_{m,\mathbf{j}}$ in (2.7), and the Hölder inequality $\Big(\sum_{r=1}^p |v_r j_r|\Big)^{2m} \le p^{2m-1}\sum_{r=1}^p |v_r j_r|^{2m}$, we see that

$$
\begin{aligned}
\tilde{R}_m =& \frac{M_{2m}}{2^{2m}(2m)!}\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} |\gamma_{m,\mathbf{j}}|\,\mathbf{j}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} \mathrm{d}\mathbf{v} \sum_{\|\boldsymbol{\alpha}\|=2m} \frac{(2m)!}{\boldsymbol{\alpha}!}(\mathbf{v}\mathbf{j})^{\boldsymbol{\alpha}} \\
=& \frac{M_{2m}}{2^{2m}(2m)!}\sum_{\mathbf{j}=\mathbf{1}}^{m\mathbf{1}} |\gamma_{m,\mathbf{j}}|\,\mathbf{j}^{\mathbf{1}} \int_{\mathbf{0}}^{\mathbf{1}} \mathrm{d}\mathbf{v} \Big(\sum_{r=1}^p v_r j_r\Big)^{2m} \\
\le& \frac{M_{2m}p^{2m-1}}{2^{2m}(2m)!}\Big(\sum_{j_1=1}^m \cdots \sum_{j_p=1}^m |\gamma_{m,j_1}|j_1 \dots |\gamma_{m,j_p}|j_p\Big)\sum_{r=1}^p j_r^{2m}\int_{\mathbf{0}}^{\mathbf{1}} v_r^{2m}\,\mathrm{d}\mathbf{v} \\
=& \frac{M_{2m}p^{2m}}{2^{2m}(2m+1)!}\sum_{j=1}^m |\gamma_{m,j}|j^{2m+1}\Big(\sum_{j=1}^m |\gamma_{m,j}|j\Big)^{p-1}.
\end{aligned}
\tag{5.46}
$$

By Proposition 4.4 in [19],

$$
\sum_{j=1}^m |\gamma_{m,j}|j^{2m+1} \le 1.0331\pi\Lambda_*^m m^{2m+1}, \tag{5.47}
$$

and for $m \ge 2$ the factor 1.0331 can be replaced by 1.001.

It follows from [23] that $\Gamma(x+1)/\Gamma(x+1/2) > \sqrt{x+1/\pi}$ for real $x > 0$. For $x = m \in \mathbb{N}$, this inequality can be rewritten as $2^{2m}\Big/\binom{2m}{m} < \sqrt{\pi m + 1}$. So, in view of (2.7),

$$
\sum_{j=1}^m |\gamma_{m,j}|j = 2^{2m}\Big/\binom{2m}{m} - 1 < \sqrt{\pi m}. \tag{5.48}
$$

Collecting (5.46), (5.47), (5.48), and (2.14), we obtain (2.13).

Theorem 2.1 is now completely proved. □

To prove Theorem 3.2, we shall need the following multidimensional generalization of the fundamental theorem of calculus (FTC).

Lemma 5.1. (Multidimensional FTC) *Let F be any antiderivative of f. Take any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^p$. Then*

$$
\int_{\mathbf{u}}^{\mathbf{v}} \mathrm{d}\mathbf{x}\, f(\mathbf{x}) = \sum_{J\subseteq[p]} (-1)^{p-|J|} F(\mathbf{v}_J), \tag{5.49}
$$

where $\mathbf{v}_J := \mathbf{u}\mathbf{1}_{[p]\setminus J} + \mathbf{v}\mathbf{1}_J = \mathbf{u} + (\mathbf{v}-\mathbf{u})\mathbf{1}_J$.

For $p = 2$ and $\mathbf{u} \le \mathbf{v}$, formula (5.49) appears in the proof of Lemma 6.2 [9]; a version of it for general p seems to be implicit on page 515 in [15]. Related formulas were given in [2, (III.1)] and [1, Lemma 1]. The following simple proof – which is essentially just a p-fold application of the one-dimensional FTC, plus some organizing – will be given here for readers' convenience.

Proof of Lemma 5.1. This will be done by induction in p. For $p = 1$, (5.49) is the usual, one-dimensional FTC. Suppose that $p \ge 2$ and that (5.49) holds with $p-1$ in place of p.

Introduce some notation, as follows. For $\mathbf{x} = (x_1, \dots, x_{p-1}, x_p) \in \mathbb{R}^p$, let $\tilde{\mathbf{x}} := (x_1, \dots, x_{p-1})$, and similarly define $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$. Also, for any $J \subseteq [p-1]$, define $\tilde{\mathbf{v}}_J$ similarly to $\mathbf{v}_J$, but based on $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ rather than on $\mathbf{u}$ and $\mathbf{v}$. For any function $h\colon \mathbb{R}^p \to \mathbb{R}$ and any real x_p, let h_{x_p} denote the "cross-section" function from $\mathbb{R}^{p-1}$ to $\mathbb{R}$ defined by the formula $h_{x_p}(\tilde{\mathbf{x}}) := h(\mathbf{x})$, again for $\mathbf{x} = (x_1, \dots, x_{p-1}, x_p) \in \mathbb{R}^p$. Note that, for each real x_p, the function $\big(F^{(\mathbf{1}_{\{p\}})}\big)_{x_p}$ is an antiderivative of the function f_{x_p}.

For real u and v, let $\Delta_{u,v} := \delta_v - \delta_u$, where δ_x is the Dirac measure at x. Consider the signed product measures

$$\Delta_{\mathbf{u},\mathbf{v}} := \Delta_{u_1,v_1} \otimes \cdots \otimes \Delta_{u_p,v_p} = \sum_{J\subseteq[p]} (-1)^{p-|J|}\delta_{\mathbf{v}_J}$$

and $\tilde{\Delta}_{\mathbf{u},\mathbf{v}} := \Delta_{u_1,v_1} \otimes \cdots \otimes \Delta_{u_{p-1},v_{p-1}}$, so that $\Delta_{\mathbf{u},\mathbf{v}} = \tilde{\Delta}_{\mathbf{u},\mathbf{v}} \otimes \Delta_{u_p,v_p}$.

Now, appropriately rewriting the right-hand side of (5.49) and then using the Fubini theorem and the induction hypothesis, we have

$$\begin{aligned}
\sum_{J\subseteq[p]} (-1)^{p-|J|} F(\mathbf{v}_J) &= \int_{\mathbb{R}^p} \mathrm{d}\Delta_{\mathbf{u},\mathbf{v}}\, F && \text{(rewriting)}\\
&= \int_{\mathbb{R}} \Delta_{u_p,v_p}(\mathrm{d}x_p) \int_{\mathbb{R}^{p-1}} \mathrm{d}\tilde{\Delta}_{\mathbf{u},\mathbf{v}}\, F_{x_p} && \text{(Fubini)}\\
&= \int_{\mathbb{R}} \Delta_{u_p,v_p}(\mathrm{d}x_p) \sum_{J\subseteq[p-1]} (-1)^{p-1-|J|} F_{x_p}(\tilde{\mathbf{v}}_J) && \text{(similar rewriting)}\\
&= \sum_{J\subseteq[p-1]} (-1)^{p-1-|J|} \int_{\mathbb{R}} \Delta_{u_p,v_p}(\mathrm{d}x_p)\, F_{x_p}(\tilde{\mathbf{v}}_J)\\
&= \sum_{J\subseteq[p-1]} (-1)^{p-1-|J|} \int_{u_p}^{v_p} \mathrm{d}x_p\, \frac{\mathrm{d}}{\mathrm{d}x_p} F_{x_p}(\tilde{\mathbf{v}}_J) && \text{(one-dimensional FTC)}\\
&= \int_{u_p}^{v_p} \mathrm{d}x_p \sum_{J\subseteq[p-1]} (-1)^{p-1-|J|} \frac{\mathrm{d}}{\mathrm{d}x_p} F_{x_p}(\tilde{\mathbf{v}}_J)\\
&= \int_{u_p}^{v_p} \mathrm{d}x_p \sum_{J\subseteq[p-1]} (-1)^{p-1-|J|} \big(F^{(\mathbf{1}_{\{p\}})}\big)_{x_p}(\tilde{\mathbf{v}}_J)\\
&= \int_{u_p}^{v_p} \mathrm{d}x_p \int_{\tilde{\mathbf{u}}}^{\tilde{\mathbf{v}}} \mathrm{d}\tilde{\mathbf{x}}\, f_{x_p}(\tilde{\mathbf{x}}) && \text{(induction)}\\
&= \int_{\mathbf{u}}^{\mathbf{v}} \mathrm{d}\mathbf{x} f(\mathbf{x}). && \text{(Fubini)}
\end{aligned}$$

This completes the proof of Lemma 5.1. □

Proof of Theorem 3.2. Let

$$R_{m,f}(\mathbf{n}) := R_m,$$

with R_m as defined in (2.9). Then, by (3.16),

$$R_{m,f}(\mathbf{n}) \underset{\wedge\mathbf{n}\to\infty}{\longrightarrow} R_{m,f}(\infty). \tag{5.50}$$

Let

$$A_{m,F}(\mathbf{n}) := \sum_{J\subseteq[p]} (-1)^{p-|J|} A^J_{m,F}(\mathbf{n}) = \tilde{A}_{m,F}(\mathbf{n}) + (-1)^p A^{\emptyset}_{m,F}(\mathbf{n}), \tag{5.51}$$

in view of (3.18). By (2.3)–(2.4), Lemma 5.1, (5.51), and (3.19),

$$
\begin{aligned}
&\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}-\mathbf{1}} f(\mathbf{k}) - A_{m_0,F}(\mathbf{n}) + R_{m,f}(\mathbf{n}) \\
&= A_{m,F}(\mathbf{n}) - A_{m_0,F}(\mathbf{n}) \\
&= \sum_{J\subseteq[p]} (-1)^{p-|J|}\big(A^J_{m,F}(\mathbf{n}) - A^J_{m_0,F}(\mathbf{n})\big) \\
&= \sum_{J\subseteq[p]} (-1)^{p-|J|}\big(A^J_{m,T_J}(\mathbf{n}) - A^J_{m_0,T_J}(\mathbf{n}) + A^J_{m,F-T_J}(\mathbf{n}) - A^J_{m_0,F-T_J}(\mathbf{n})\big),
\end{aligned}
\tag{5.52}
$$

where $T_J = T_{J,\mathbf{n},m_0,F}$ is the Taylor polynomial of order $2m_0 - 1$ for the function F at the point $\mathbf{n}\mathbf{1}_J - \mathbf{1}$, so that

$$
T_J(\mathbf{x}) = \sum_{\|\boldsymbol{\alpha}\|\le 2m_0-1} \frac{F^{(\boldsymbol{\alpha})}(\mathbf{n}\mathbf{1}_J - \mathbf{1})}{\boldsymbol{\alpha}!}(\mathbf{x} - \mathbf{n}\mathbf{1}_J + \mathbf{1})^{\boldsymbol{\alpha}}
$$

for $\mathbf{x} \in \mathbb{R}^p$.

Consider the monomial $P(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\alpha}}$ of degree $\|\boldsymbol{\alpha}\| \le 2m_0 - 1$, so that $P(\mathbf{x}) = \prod_{r=1}^p P_r(x)$, where $P_r(x) := x^{\alpha_r}$.

Take any $r = 1,\dots,p$ and any $J \subseteq [p]$, and let $n_{r,J} := n_r\, \mathrm{I}\{r \in J\}$. Following the lines of the proof of Proposition 5.1 in [19] for the case when $f = P_r'$ and $F = P_r$, so that the polynomial T therein coincides with $F = P_r$, we see from [19, (5.5) and (7.19)] that

$$
\begin{aligned}
\sum_{\beta=1-m}^{m-1} \tau_{m,1+|\beta|}\, P_r(n - 1/2 - \beta/2) = G_{m,P_r}(n) &= G_{m_0,P_r}(n) \\
&= \sum_{\beta=1-m_0}^{m_0-1} \tau_{m_0,1+|\beta|}\, P_r(n - 1/2 - \beta/2)
\end{aligned}
$$

for any $n \in \mathbb{Z}_+$. So, by (3.20) and (2.8),

$$
\begin{aligned}
A^J_{m,P}(\mathbf{n}) &= \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+|\boldsymbol{\beta}|}\, P(\mathbf{n}\mathbf{1}_J - \mathbf{1}/2 - \boldsymbol{\beta}/2) \\
&= \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}} \prod_{r=1}^{p} \big(\tau_{m,1+|\beta_r|}\, P_r(n_{r,J} - 1/2 - \beta_r/2)\big) \\
&= \prod_{r=1}^{p} \sum_{\beta=1-m}^{m-1} \big(\tau_{m,1+|\beta|}\, P_r(n_{r,J} - 1/2 - \beta/2)\big) \\
&= \prod_{r=1}^{p} \sum_{\beta=1-m_0}^{m_0-1} \big(\tau_{m_0,1+|\beta|}\, P_r(n_{r,J} - 1/2 - \beta/2)\big) = A^J_{m_0,P}(\mathbf{n}).
\end{aligned}
$$

Since T_J is a polynomial of degree $\le 2m_0 - 1$ and $A^J_{m,F}(\mathbf{n})$ is linear in F, we conclude that

$$
A^J_{m,T_J}(\mathbf{n}) - A^J_{m_0,T_J}(\mathbf{n}) = 0 \quad \text{for all } J \subseteq [p]. \tag{5.53}
$$

Further, the remainder $(F-T_J)(\mathbf{n1}_J-\mathbf{1}+\mathbf{u})$ at point $\mathbf{n1}_J-\mathbf{1}+\mathbf{u}$ of the Taylor approximation T_J of F at $\mathbf{n1}_J-\mathbf{1}$ equals (cf. (5.35))

$$\sum_{\|\boldsymbol{\alpha}\|=2m_0} \frac{2m_0}{\boldsymbol{\alpha}!}\,\mathbf{u}^{\boldsymbol{\alpha}} \int_0^1 \mathrm{d}s\,(1-s)^{2m_0-1}F^{(\boldsymbol{\alpha})}(\mathbf{n1}_J-\mathbf{1}+s\mathbf{u}),$$

which, by (3.15), goes to 0 as $\wedge\mathbf{n}\to\infty$ unless $J=\emptyset$. So, by (3.19),

$$A^J_{m,F-T_J}(\mathbf{n}) \underset{\wedge\mathbf{n}\to\infty}{\longrightarrow} 0 \quad\text{and}\quad A^J_{m_0,F-T_J}(\mathbf{n}) \underset{\wedge\mathbf{n}\to\infty}{\longrightarrow} 0 \quad \text{unless } J=\emptyset.$$

It follows now by (5.53), (3.23), the linearity of $A^J_{m,F}$ in F, and (again) (5.53) that the limit of the last expression in (5.52) as $\wedge\mathbf{n}\to\infty$ equals

$$\begin{aligned}
&(-1)^p\big(A^\emptyset_{m,F-T_\emptyset}(\mathbf{0})-A^\emptyset_{m_0,F-T_\emptyset}(\mathbf{0})\big)\\
=&(-1)^p\big(A^\emptyset_{m,F}(\mathbf{0})-A^\emptyset_{m_0,F}(\mathbf{0})\big)-(-1)^p\big(A^\emptyset_{m,T_\emptyset}(\mathbf{0})-A^\emptyset_{m_0,T_\emptyset}(\mathbf{0})\big)\\
=&(-1)^p\big(A^\emptyset_{m,F}(\mathbf{0})-A^\emptyset_{m_0,F}(\mathbf{0})\big).
\end{aligned}$$

Now (3.17) follows, in view of (5.50) and (the second equality in) (5.51).

Inequality (3.22) follows immediately from (2.11) and (2.12)–(2.13).

Formula (3.23) follows immediately from (3.21).

Theorem 3.2 is completely proved. □

Proof of Theorem 3.3. Note that

$$\begin{aligned}
\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{c}-\mathbf{1}} f(\mathbf{k}) &= \sum_{\mathbf{k}\ge\mathbf{0}} f(\mathbf{k})\,\mathrm{I}\{\mathbf{k}\le\mathbf{c}-\mathbf{1}\}\\
&= \sum_{\mathbf{k}\ge\mathbf{0}} f(\mathbf{k}) \prod_{r=1}^{p}\big(\mathrm{I}\{k_r\le n_r+c_r-1\}-\mathrm{I}\{c_r\le k_r\le n_r+c_r-1\}\big)\\
&= \sum_{\mathbf{k}\ge\mathbf{0}} f(\mathbf{k}) \sum_{J\subseteq[p]} (-1)^{|J|}\,\mathrm{I}\{k_r\le n_r+c_r-1\ \forall r\in[p]\setminus J,\\
&\qquad\qquad c_r\le k_r\le n_r+c_r-1\ \forall r\in J\}\\
&= \sum_{J\subseteq[p]}(-1)^{|J|}\sum_{\mathbf{k}=\mathbf{c1}_J}^{\mathbf{n}+\mathbf{c}-\mathbf{1}} f(\mathbf{k}) = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}+\mathbf{c}-\mathbf{1}} f(\mathbf{k}) + \sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|}\sum_{\mathbf{k}=\mathbf{c1}_J}^{\mathbf{n}+\mathbf{c}-\mathbf{1}} f(\mathbf{k}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}+\mathbf{c}-\mathbf{1}} f(\mathbf{k}) - \tilde A_{m_0,F}(\mathbf{n}+\mathbf{c})\\
&= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{c}-\mathbf{1}} f(\mathbf{k}) - \sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|}\sum_{\mathbf{k}=\mathbf{c1}_J}^{\mathbf{n}+\mathbf{c}-\mathbf{1}} f(\mathbf{k}) - \tilde A_{m_0,F}(\mathbf{n}+\mathbf{c})\\
&= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{c}-\mathbf{1}} f(\mathbf{k}) - \sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|}\Big(\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}+\mathbf{c}-\mathbf{c1}_J-\mathbf{1}} f_{\mathbf{c1}_J}(\mathbf{k}) - \tilde A_{m_0,F_{\mathbf{c1}_J}}(\mathbf{n}+\mathbf{c}-\mathbf{c1}_J)\Big) + \mathcal{R},
\end{aligned}\tag{5.54}$$

where

$$\mathcal{R} := -\sum_{\emptyset\ne J\subseteq[p]}(-1)^{|J|}\tilde A_{m_0,F_{\mathbf{c1}_J}}(\mathbf{n}+\mathbf{c}-\mathbf{c1}_J) - \tilde A_{m_0,F}(\mathbf{n}+\mathbf{c}).$$

By Lemma 5.1 with $F = 1$ (and $f = 0$),

$$\sum_{J\subseteq[p]} (-1)^{|J|} = 0 \quad \text{and hence} \sum_{\emptyset\neq J\subseteq[p]} (-1)^{|J|} = -1. \tag{5.55}$$

Therefore and in view of (3.18) and (3.20),

$$\mathcal{R} = \sum_{\emptyset\neq J\subseteq[p]} (-1)^{|J|}\mathcal{R}_J,$$

where

$$\mathcal{R}_J :=\tilde{A}_{m_0,F}(\mathbf{n}+\mathbf{c}) - \tilde{A}_{m_0,F_{\mathbf{c}\mathbf{1}_J}}(\mathbf{n}+\mathbf{c}-\mathbf{c}\mathbf{1}_J) = \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+|\boldsymbol{\beta}|}\,\mathcal{R}_{J,\boldsymbol{\beta}},$$

$$\mathcal{R}_{J,\boldsymbol{\beta}} := \sum_{\emptyset\neq K\subseteq[p]} (-1)^{p-|K|}\big[H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K\big) - H\big(\mathbf{c}\mathbf{1}_J + (\mathbf{n}+\mathbf{c}-\mathbf{c}\mathbf{1}_J)\mathbf{1}_K\big)\big],$$

and $H(\mathbf{x}) := F(\mathbf{x}-\mathbf{1}/2-\boldsymbol{\beta}/2)$. Thus,

$$\mathcal{R} = \sum_{\boldsymbol{\beta}=(1-m)\mathbf{1}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+|\boldsymbol{\beta}|} \sum_{\emptyset\neq K\subseteq[p]} (-1)^{p-|K|}\mathcal{R}_{\boldsymbol{\beta},K}, \tag{5.56}$$

where

$$\begin{aligned}
\mathcal{R}_{\boldsymbol{\beta},K} &:= \sum_{\emptyset\neq J\subseteq[p]} (-1)^{|J|}\big[H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K\big) - H\big(\mathbf{c}\mathbf{1}_J + (\mathbf{n}+\mathbf{c}-\mathbf{c}\mathbf{1}_J)\mathbf{1}_K\big)\big] \\
&= \sum_{\emptyset\neq J\subseteq[p]} (-1)^{|J|}\big[H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K\big) - H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K + \mathbf{c}\mathbf{1}_{J\setminus K}\big)\big] \\
&= \sum_{L\in\mathcal{L}_K} \big[H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K\big) - H\big((\mathbf{n}+\mathbf{c})\mathbf{1}_K + \mathbf{c}\mathbf{1}_L\big)\big] \sum_{J\in\mathcal{J}_{K,L}} (-1)^{|J|},
\end{aligned} \tag{5.57}$$

$$\mathcal{L}_K := \{L\colon L\subseteq[p],\ L\neq\emptyset,\ L\cap K=\emptyset\}, \quad \mathcal{J}_{K,L} := \{J\colon \emptyset\neq J\subseteq[p],\ J\setminus K = L\}.$$

For any $K \subseteq [p]$ and any $L \in \mathcal{L}_K$, the map $J \mapsto I_J := J\cap K$ is a bijection of the set $\mathcal{J}_{K,L}$ onto the set $\{I\colon I\subseteq K\}$, and for any $J\in\mathcal{J}_{K,L}$ the set J is the disjoint union of the sets I_J and L, so that $|J| = |I_J| + |L|$. It follows by (5.55) that for any $K\subseteq[p]$ and any $L\in\mathcal{L}_K$ one has $\sum_{J\in\mathcal{J}_{K,L}}(-1)^{|J|} = \sum_{I\subseteq K}(-1)^{|I|}(-1)^{|L|} = 0$. Looking back at (5.57) and (5.56), we see that $\mathcal{R} = 0$.

Letting now $\wedge\mathbf{n} \to \infty$ and recalling (5.54), (3.17), the definition (3.26) of $R_{m,f,\mathbf{c}}(\infty)$, and formulas (3.23), (3.21), and (3.18), we have

$$\begin{aligned}
&\sum_{\mathbf{k}\ge\mathbf{0}}^{\mathsf{Alt}} f(\mathbf{k}) - \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{c}-\mathbf{1}} f(\mathbf{k}) \\
&= -\sum_{\emptyset\ne J\subseteq[p]} (-1)^{|J|} \sum_{\mathbf{k}\ge\mathbf{0}}^{\mathsf{Alt}} f_{\mathbf{c}\mathbf{1}_J}(\mathbf{k}) \\
&= -\sum_{\emptyset\ne J\subseteq[p]} (-1)^{|J|} \big[(-1)^p A^{\emptyset}_{m,F_{\mathbf{c}\mathbf{1}_J}}(\mathbf{0}) - R_{m,f_{\mathbf{c}\mathbf{1}_J}}(\infty)\big] \\
&= -R_{m,f,\mathbf{c}}(\infty) - \sum_{\emptyset\ne J\subseteq[p]} (-1)^{p-|J|} \sum_{\boldsymbol{\alpha}=\mathbf{0}}^{(m-1)\mathbf{1}} \tau_{m,\mathbf{1}+\boldsymbol{\alpha}} \sum_{\boldsymbol{\beta}:\ |\boldsymbol{\beta}|=\boldsymbol{\alpha}} F(\mathbf{c}\mathbf{1}_J + \boldsymbol{\beta}/2 - \mathbf{1}/2) \\
&= -R_{m,f,\mathbf{c}}(\infty) - \tilde{A}_{m,F}(\mathbf{c}),
\end{aligned}$$

which completes the proof of Theorem 3.3. $\square$

Proof of Proposition 4.1. Let $\mathbf{a}_1, \dots, \mathbf{a}_p$ denote the columns of the matrix A, so that $\mathbf{a}_i \in \mathbb{Z}^p$ for each $i \in [p]$ and

$$C := A\mathbb{R}^+_J = \sum_{i\in[p]} \mathbb{R}^+_{\varepsilon_i}\mathbf{a}_i, \quad \text{where} \quad \varepsilon_i := 1 - [\![J]\!](i).$$

If the matrix A is unimodular, there is nothing to prove. So, w.l.o.g., $|\det A| \ge 2$. Then there is a vector $\mathbf{w} \in \mathbb{Z}^p \setminus \{\mathbf{0}\}$ such that

$$\mathbf{w} = w_1\mathbf{a}_1 + \cdots + w_p\mathbf{a}_p \tag{5.58}$$

for some real numbers $w_1, \dots, w_p$ in the interval $[0,1)$ (in fact, there are exactly $|\det A| - 1$ such vectors $\mathbf{w}$). Thus, w.l.o.g. for some $k \in [p]$ one has

$$w_j > 0 \text{ for } j \in [k] \quad \text{and} \quad w_j = 0 \text{ for } j \in [p] \setminus [k]. \tag{5.59}$$

For each $i \in [k]$, let A_i be the (integral) matrix obtained from the matrix A by replacing its i-th column, $\mathbf{a}_i$, by $\mathbf{w}$; then $\det A_i = w_i \det A$ and hence

$$0 < |\det A_i| < |\det A|. \tag{5.60}$$

We shall see that (4.32) holds with $I = [k]$, the matrices A_i just defined, and some subsets $J_1, \dots, J_k$ of the set $[p]$.

Then, repeating the step described in the last paragraph – for each of the matrices $A_1, \dots, A_k$ in place of A, in view of (5.60) we shall eventually obtain (4.32) with unimodular $p \times p$ matrices A_i over $\mathbb{Z}$, as required. This step relies mainly on the following combinatorial lemma.

Lemma 5.2. *Let $\mathbf{a}_1, \dots, \mathbf{a}_p$, C, $\mathbf{w}$, and k be as described above. For each $i \in [k]$, let*

$$C_i := \mathbb{R}^+_{\varepsilon_{ii}}\mathbf{w} + \sum_{j\in[p]\setminus\{i\}} \mathbb{R}^+_{\varepsilon_{ij}}\mathbf{a}_j, \tag{5.61}$$

where the ε_{ij}'s are any numbers in the set $\{0,1\}$ satisfying the following conditions:

- (i) $\varepsilon_{ij} = \varepsilon_j$ *for* $i \in [k]$ *and* $j \in [p] \setminus [k]$;
- (ii) $\varepsilon_{ii} = \varepsilon_i$ *for* $i \in [k]$;
- (iii) $\varepsilon_{ij} + \varepsilon_{ji} = 1$ *for any distinct* i *and* j *in* $[k]$;
- (iv) *for each* $i \in [k]$, *the condition* $\varepsilon_i = 1$ *implies* $\varepsilon_{ij} \le \varepsilon_j$ *for all* $j \in [k]$;

(v) *for each nonempty subset J of the set $[k]$, there is some $i \in J$ such that for all $j \in J \setminus \{i\}$ one has $\varepsilon_{ij} = 1$.*

Then

$$[\![C]\!] = \sum_{i\in[k]} [\![C_i]\!]. \tag{5.62}$$

We also have

Lemma 5.3. *Take any $\varepsilon_1, \dots, \varepsilon_p$ in $\{0,1\}$ and any $k \in [p]$. Then there exist numbers ε_{ij} in the set $\{0,1\}$ satisfying all the conditions (i)–(v) in Lemma 5.2.*

We shall prove these two lemmas in a moment.

Letting now $J_i = \{j \in [p] \colon \varepsilon_{ij} = 0\}$ for each $i \in [k]$ (so that $\varepsilon_{ij} = 1 - [\![J_i]\!](j)$ for all $i \in [k]$ and $j \in [p]$), we will have $C_i = A_i \mathbb{R}^+_{J_i}$ for $i \in [k]$, which will complete the step described in the paragraph containing formulas (5.58)–(5.60). Thus, to complete the entire proof of Proposition 4.1, it remains to prove Lemmas 5.2 and 5.3.

Proof of Lemma 5.2. Take any $\mathbf{x} \in \mathbb{R}^p$. Let $(y_1, \dots, y_p) = (y_1(\mathbf{x}), \dots, y_p(\mathbf{x}))$ denote the p-tuple of the coordinates of the vector $\mathbf{x}$ in the basis $(\mathbf{a}_1, \dots, \mathbf{a}_p)$ of $\mathbb{R}^p$, so that

$$\mathbf{x} = \sum_{j\in[p]} y_j \mathbf{a}_j. \tag{5.63}$$

Also, for each $i \in [k]$, let $(y_{i1}, \dots, y_{ip}) = (y_{i1}(\mathbf{x}), \dots, y_{ip}(\mathbf{x}))$ denote the p-tuple of the coordinates of the vector $\mathbf{x}$ in the basis $(\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{w}, \mathbf{a}_{i+1}, \dots, \mathbf{a}_p)$ of $\mathbb{R}^p$, so that

$$\mathbf{x} = y_{ii}\mathbf{w} + \sum_{j\in[p]\setminus\{i\}} y_{ij}\mathbf{a}_j = y_{ii}w_i\mathbf{a}_i + \sum_{j\in[p]\setminus\{i\}} (y_{ii}w_j + y_{ij})\mathbf{a}_j.$$

In view of (5.58) and (5.59),

$$y_{ij} = y_j \quad \text{for } i \in [k],\ j \in [p] \setminus [k]. \tag{5.64}$$

As for i and j in $[k]$, we have $y_i = y_{ii}w_i$ and $y_j = y_{ii}w_j + y_{ij} = \frac{y_i}{w_i} w_j + y_{ij}$ if $j \neq i$, which can be rewritten as

$$\forall (i,j) \in [k] \times [k] \ \Big(y_{ii}w_i = y_i \quad \text{and} \quad j \neq i \implies \frac{y_{ij}}{w_j} = r_j - r_i \Big), \tag{5.65}$$

where

$$r_j := r_j(\mathbf{x}) := \frac{y_j}{w_j}.$$

Note that (5.62) means precisely that C is the disjoint union of the C_i's. Thus, the proof of Lemma 5.2 will be completed in the following three steps.

Step 1: checking that $C_i \subseteq C$ for each $i \in [k]$. Take indeed any $i \in [k]$, and then take any $\mathbf{x} \in C_i$, so that, by (5.61), $y_{ij} \in \mathbb{R}^+_{\varepsilon_{ij}}$ for all $j \in [p]$. Then $y_{ii} \in \mathbb{R}^+_{\varepsilon_{ii}} = \mathbb{R}^+_{\varepsilon_i}$ by condition (ii) of Lemma 5.2 and hence $y_i = y_{ii}w_i \in \mathbb{R}^+_{\varepsilon_i}$. Also, by (5.64) and condition (i) of Lemma 5.2, $y_j = y_{ij} \in \mathbb{R}^+_{\varepsilon_{ij}} = \mathbb{R}^+_{\varepsilon_j}$ for $j \in [p] \setminus [k]$.

If $y_i > 0$, then $y_j = \frac{y_i}{w_i} w_j + y_{ij} > y_{ij} \geq 0$ for all $j \in [k] \setminus \{i\}$, whence $y_j > 0$ for all $j \in [k]$. So, by (5.63), $\mathbf{x} \in \sum_{j\in[k]} \mathbb{R}^+_0 \mathbf{a}_j + \sum_{j\in[p]\setminus[k]} \mathbb{R}^+_{\varepsilon_j} \mathbf{a}_j \subseteq \sum_{j\in[p]} \mathbb{R}^+_{\varepsilon_j} \mathbf{a}_j = C$.

If now $y_i = 0$, then the mentioned condition $y_i \in \mathbb{R}^+_{\varepsilon_i}$ implies $\varepsilon_i = 1$. So, by condition (iv) of Lemma 5.2, for all $j \in [k]$ we have $\varepsilon_{ij} \leq \varepsilon_j$ and hence $\mathbb{R}^+_{\varepsilon_{ij}} \subseteq \mathbb{R}^+_{\varepsilon_j}$, which yields $y_j = \frac{y_i}{w_i} w_j + y_{ij} = y_{ij} \in \mathbb{R}^+_{\varepsilon_{ij}} \subseteq \mathbb{R}^+_{\varepsilon_j}$. So, in this case as well, $\mathbf{x} \in C$.

Step 2: checking that the C_i's are disjoint. Take any distinct i and j in $[k]$, and then take any $\mathbf{x} \in C_i \cap C_j$. Then $y_{ij} \in \mathbb{R}^+_{\varepsilon_{ij}}$, whence, by (5.65), $r_j - r_i = y_{ij}/w_j \in \mathbb{R}^+_{\varepsilon_{ij}}$. Similarly, $r_i - r_j \in \mathbb{R}^+_{\varepsilon_{ji}}$, that is, $r_j - r_i \in -\mathbb{R}^+_{\varepsilon_{ji}} = \mathbb{R} \setminus \mathbb{R}^+_{\varepsilon_{ij}}$, by condition (iii) of Lemma 5.2. Thus, $r_j - r_i \in \mathbb{R}^+_{\varepsilon_{ij}} \cap \big(\mathbb{R} \setminus \mathbb{R}^+_{\varepsilon_{ij}}\big) = \emptyset$, which is a contradiction.

Step 3: checking that $C \subseteq \bigcup_{i\in[k]} C_i$. Take any $\mathbf{x} \in C$, so that $y_j \in \mathbb{R}^+_{\varepsilon_j}$ for all $j \in [p]$. Let

$$J_{\mathbf{x}} := \{i \in [k] \colon r_i(\mathbf{x}) \le r_j(\mathbf{x})\ \forall j \in [k]\}.$$

Then, by (5.65), $y_{ij} \ge 0$ for all $i \in J_{\mathbf{x}}$ and $j \in [k]$. Moreover, $r_j(\mathbf{x}) > r_i(\mathbf{x})$ for all $i \in J_{\mathbf{x}}$ and $j \in [k] \setminus J_{\mathbf{x}}$.

So, again by (5.65), for all $i \in J_{\mathbf{x}}$ and $j \in [k] \setminus J_{\mathbf{x}}$ we have $y_{ij} > 0$, so that $y_{ij} \in \mathbb{R}^+_0 \subseteq \mathbb{R}^+_{\varepsilon_{ij}}$. Note that $J_{\mathbf{x}} \ne \emptyset$. So, by condition (v) of Lemma 5.2, there is some $i_{\mathbf{x}} \in J_{\mathbf{x}}$ such that for all $j \in J_{\mathbf{x}} \setminus \{i_{\mathbf{x}}\}$ one has $\varepsilon_{i_{\mathbf{x}}j} = 1$, so that $y_{i_{\mathbf{x}}j} \in \mathbb{R}^+_{\varepsilon_{i_{\mathbf{x}}j}}$. Thus, $y_{i_{\mathbf{x}}j} \in \mathbb{R}^+_{\varepsilon_{i_{\mathbf{x}}j}}$ for all $j \in [k] \setminus \{i_{\mathbf{x}}\}$. Also, $y_{i_{\mathbf{x}}i_{\mathbf{x}}} \in \mathbb{R}^+_{\varepsilon_{i_{\mathbf{x}}i_{\mathbf{x}}}}$ – in view of the first equality in (5.65), the condition $y_i \in \mathbb{R}^+_{\varepsilon_i}$ for all $i \in [p]$, and condition (ii) of Lemma 5.2. Moreover, $y_{i_{\mathbf{x}}j} \in \mathbb{R}^+_{\varepsilon_{i_{\mathbf{x}}j}}$ for all $j \in [p] \setminus [k]$ – in view of (5.64), the condition $y_j \in \mathbb{R}^+_{\varepsilon_j}$ for all $j \in [p]$, and condition (i) of Lemma 5.2. We conclude that $y_{i_{\mathbf{x}}j} \in \mathbb{R}^+_{\varepsilon_{i_{\mathbf{x}}j}}$ for all $j \in [p]$, that is, $\mathbf{x} \in C_{i_{\mathbf{x}}} \subseteq \bigcup_{i\in[k]} C_i$.

Lemma 5.2 is now proved. □

Proof of Lemma 5.3. For $i \in [k]$ and $j \in [p] \setminus [k]$, let $\varepsilon_{ij} := \varepsilon_j$, in accordance with condition (i) of Lemma 5.2.

Similarly, let $\varepsilon_{ii} := \varepsilon_i$ for $i \in [k]$, in accordance with condition (ii) of Lemma 5.2.

Next, w.l.o.g. ε_j is nondecreasing in $j \in [k]$. Let then $\varepsilon_{ij} := 1$ and $\varepsilon_{ji} := 0$ for all i and j in $[k]$ with $i < j$.

It is now straightforward to check that all the conditions (i)–(v) in Lemma 5.2 hold. In particular, concerning condition (iv), note that, if $\varepsilon_i = 1$ and $\varepsilon_{ij} = 1$ for some distinct i and j in $[k]$, then $i < j$ and hence $1 = \varepsilon_i \le \varepsilon_j$, so that $\varepsilon_j = 1$. Concerning condition (v), for each nonempty subset J of the set $[k]$, let $i := \min J$; then for all $j \in J \setminus \{i\}$ one has $i < j$ and hence $\varepsilon_{ij} = 1$. Lemma 5.3 is now proved. □

The entire proof of Proposition 4.1 is thus complete. □

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IOSIF PINELIS
MICHIGAN TECHNOLOGICAL UNIVERSITY
DEPARTMENT OF MATHEMATICAL SCIENCES
1400 TOWNSEND DRIVE, HOUGHTON, MICHIGAN 49931, USA
ORCID: 0000-0003-4742-5789
E-mail address: ipinelis@mtu.edu