Turk. J. Math. Comput. Sci.
15(1)(2023) 104–109
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DOI : 10.47000/tjmcs.1105934

L^∞ Decay Estimate for the Klein-Gordon Equation in the Anti-de Sitter Model of the Universe

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Received: 19-04-2022 • Accepted: 14-02-2023

Abstract. We consider the Klein-Gordon equation with non-zero initial data in anti-de Sitter spacetime. L^∞ decay estimate is derived for the solutions to the linear Klein-Gordon equations in the anti-de Sitter spacetime without source term.

2010 AMS Classification: 35L05, 35Q75, 35L15

Keywords: Klein-Gordon eqution, anti-de sitter spacetime, L^∞ estimate.

1. Introduction

In this manuscript, we contribute Cauchy problem for the Klein-Gordon equation in anti-de Sitter spacetime:

$$\begin{aligned} &\Gamma_{tt} + n\Gamma_t - e^{2t}\Delta\Gamma + m^2\Gamma = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ &\Gamma(x,0) = \omega_0(x), \qquad \Gamma_t(x,0) = \omega_1(x), \qquad x \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where ω_0, ω_1 are in Sobolev space $W^{[n/2]+1,2}(\mathbb{R}^n)$, and $m > 0$. The anti-de Sitter model of the universe is to describe the spatial contraction. The derivation of the equation in (1.1) is given in [5] and [7]. For the sake of completeness, we give a brief summary that how the equation is deduced. In de Sitter spacetime describing the spatial expansion of the model of the universe the line element has the following form,

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\alpha^2 + \sin^2\alpha d\beta^2).$$

Here, R denotes the universe radius. If the spherical coordinates and Lemaitre-Robertson transformation are used as in [3], the line element has the following form

$$ds^2 = -dt^2 + e^{2Ht}\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{1.2}$$

where we put $H = 1/R$. In the n spatial dimensional case, we may write (1.2) as $ds^2 = -dt^2 + e^{2Ht}\left(dx_1^2 + ... + dx_n^2\right)$. For the sake of simplicity, we take $H = 1$. Hence we get the following diagonal matrix; $(g_{ij})_{0\le i,j\le n} := \text{diag}(-1, e^{2t}, ..., e^{2t})$ for the line element. If we apply the determinant $g := \det(g_{ik})_{0\le i,k\le n}$, and $(g^{ik})_{0\le i,k\le n}$, the inverse matrix of $(g_{ik})_{0\le i,k\le n}$ to the following equation

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ik}\frac{\partial\Gamma}{\partial x_k}\right) = m^2\Gamma,$$

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we get the following equation

$$\Gamma_{tt} + n\Gamma_t - e^{-2t}\Delta\Gamma + m^2\Gamma = 0 \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{1.3}$$

where $x_0 := t$. If we set $\psi = e^{\frac{n}{2}t}\Gamma$, the equation (1.3) has the following form in de Sitter spacetime

$$\psi_{tt} - e^{-2t}\Delta\psi + \tau^2\psi = 0, \tag{1.4}$$

where $\tau^2 := m^2 - n^2/4$ and $m > 0$. The equation (1.4) converts to the following equation

$$\psi_{tt} - e^{2t}\Delta\psi + \tau^2\psi = 0,$$

with the inverse transformation of the time from t to $-t$ that is regarded as the equation in anti-de Sitter spacetime (see e.g., [5]).

The following problem in de Sitter spacetime;

$$\Gamma_{tt} + n\Gamma_t + e^{-2t}\Delta\Gamma + m^2\Gamma = f, \quad \Gamma(x,0) = \omega_0(x), \quad \Gamma_t(x,0) = \omega_1(x), \tag{1.5}$$

where $f \in \mathbb{C}(\mathbb{R}^{n+1})$ has been extensively investigated. In [6], Yagdjian and Galstian showed $L^p - L^q$ decay estimate for the case $n/2 \le m$ to the solution of the problem (1.5);

$$\begin{aligned}\left\|(-\Delta)^{-s}\Gamma(.,t)\right\|_{L^q(\mathbb{R}^n)} &\le Ce^{-\frac{n}{2}t}(1+t)^{1-\operatorname{sgn}N}(1-e^{-t})^{\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)}\left\{e^{\frac{t}{2}}\|\omega_0\|_{L^p(\mathbb{R}^n)} + (1-e^{-t})\|\omega_1\|_{L^p(\mathbb{R}^n)}\right\}\\ &+ Ce^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}e^b\left(e^{-b}-e^{-t}\right)^{\left(1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)}\|f(.,b)\|_{L^p(\mathbb{R}^n)}(1+t-b)^{1-\operatorname{sgn}N}db\end{aligned}$$

for $t > 0$ when $1 < p \le 2$, $\frac{1}{p}+\frac{1}{q} = 1$, $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q}) \le 2s \le n(\frac{1}{p}-\frac{1}{q}) < 2s+1$. In [8], Yagdjian proved the following $L^p - L^q$ decay estimate to the solution of (1.5);

$$\begin{aligned}\left\|(-\Delta)^{-s}\Gamma(.,t)\right\|_{L^q(\mathbb{R}^n)} &\le Ce^{(N-\frac{n}{2})t}(1-e^{-t})^{\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)}\left\{\|\omega_0\|_{L^p(\mathbb{R}^n)} + (1-e^{-t})\|\omega_1\|_{L^p(\mathbb{R}^n)}\right\}\\ &+ Ce^{-(\frac{n}{2}-N)t}\int_0^t e^{(\frac{n}{2}-N)b}e^{-b\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)}\|f(.,b)\|_{L^p(\mathbb{R}^n)}db\end{aligned}$$

for all $t > 0$ and $0 < m < \sqrt{n^2-1}/2$ provided that $1 < p \le 2$, $\frac{1}{p}+\frac{1}{q} = 1$, $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q}) \le 2s \le n(\frac{1}{p}-\frac{1}{q}) < 2s+1$. In [4], Nakamura showed the energy estimate for the initial value problem (1.5) for the case $m \ge n/2$. In [11], the solution of the problem (1.5) has the following L^∞ estimate

$$\|\Gamma(.,t)\|_{L^\infty(\mathbb{R}^n)} \le Ce^{(N-\frac{n}{2})t}\left\{\|\omega_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} + \|\omega_1\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}\right\} + Ce^{-(\frac{n}{2}-N)t}\int_0^t e^{(\frac{n}{2}-N)b}\|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}db, \tag{1.6}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, where $\omega_0, \omega_1 \in C_0(\mathbb{R}^n)$ for the case $m < \sqrt{n^2-1}/2$ and $n \ge 2$. Here, we have set $N = \sqrt{n^2/4 - m^2}$.

Here, we define the Sobolev space as $W^{k,s}(\mathbb{R}^n) = \{u \in L^s(\mathbb{R}^n) : \ D^\alpha u \in L^s(\mathbb{R}^n),\ |\alpha| \le k\}$, with the following norm

$$\|u\|_{W^{k,s}(\mathbb{R}^n)} = \left(\sum_{|\alpha|\le k}\int_{\mathbb{R}^n}|D^\alpha u|^s\right)^{1/s}, \quad (1 \le s < \infty),$$
$$\|u\|_{W^{k,\infty}(\mathbb{R}^n)} = \sum_{|\alpha|\le k}\operatorname*{ess\,sup}_{\mathbb{R}^n}|D^\alpha u|.$$

Galstian and Yagdjian [2] consider the initial value problem for

$$\Gamma_{tt} + n\Gamma_t - e^{-2t}A(x,\partial_x)\Gamma + m^2\Gamma = f, \qquad t > 0,\ x \in \mathbb{R}^n,$$

in the Besov space $B_p^{s,q}$, where $A(x,\partial_x) = \sum_{|\alpha|\le 2} a_\alpha(x)\partial_x^\alpha$ is a second-order negative elliptic differential operator with real coefficients $a_\alpha \in \mathcal{B}^\infty$ and $m \in (0, \sqrt{n^2-1}/2) \cup [n/2, \infty)$ to show the similar estimates. Here, $\mathcal{B}^\infty$ denotes the space which contains the functions with uniformly bounded derivatives of all orders in C^∞. For $m \in (\sqrt{n^2-1}/2, n/2)$, decay estimates are also considered by Yagdjian [9] in the Besov space.

Returning to anti de-Sitter spacetime, the next theorem proved by Galstian [1] provide the inequality.

Theorem 1.1. *Let $\psi = \psi(x,t)$ be the solution of the initial value problem*

$$\psi_{tt} - e^{2t}\Delta\psi + \tau^2\psi = f, \quad \psi(x,0) = \omega_0, \quad \psi_t(x,0) = \omega_1 \tag{1.7}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, where $f \in C^\infty(\mathbb{R}^{n+1})$. Let $m \geq n/2$ and $n \geq 2$. Then, there exists a constant $C > 0$ such that

$$\left\|(-\Delta)^{-s}\psi(.,t)\right\|_{L^q(\mathbb{R}^n)} \leq C(e^t-1)^{\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)}\left\{\|\omega_0\|_{L^p(\mathbb{R}^n)} + (1-e^{-t})(1+t)^{1-\operatorname{sgn}\tau}\|\omega_1\|_{L^p(\mathbb{R}^n)}\right\}$$
$$+ Ce^{t\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)}\int_0^t \|f(.,b)\|_{L^p(\mathbb{R}^n)}(1+t-b)^{1-\operatorname{sgn}\tau}db$$

for all $t > 0$ and $\frac{n}{2} \leq m$ when $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)(\frac{1}{p} - \frac{1}{q}) \leq 2s \leq n(\frac{1}{p} - \frac{1}{q}) < 2s + 1$. Here, we have set $\tau = \sqrt{m^2 - \frac{n^2}{4}}$.

It is proved in [10] that for the solution of (1.7) with zero initial conditions, the following decay estimate holds with the case $n/2 \leq m$,

$$\|\psi(.,t)\|_{L^\infty(\mathbb{R}^n)} \leq C\int_0^t \|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}(1+t-b)^{1-\operatorname{sgn}\tau}db,$$

where $(x,t) \in \mathbb{R}^n \times (0,\infty)$, $f \in C^\infty(\mathbb{R}^{n+1})$, $n \geq 2$ and $\tau = \sqrt{m^2 - \frac{n^2}{4}}$.

The decay estimate is an essential technique for showing the global existence of nonlinear partial differential equations. Therefore, we consider the L^∞ decay estimate for the solution of (1.1). In this manuscript, we deal with the case of $m \geq n/2$ and prove the following theorem.

Theorem 1.2. *Let $\psi = \psi(x,t)$ be the solution of the initial value problem*

$$\psi_{tt} - e^{2t}\Delta\psi + \tau^2\psi = 0, \quad \psi(x,0) = \omega_0(x), \quad \psi_t(x,0) = \omega_1(x) \tag{1.8}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, where $\omega_0, \omega_1 \in C_0^\infty(\mathbb{R}^n)$. Then, there exists a constant $C > 0$ such that

$$\|\psi(.,t)\|_{L^\infty(\mathbb{R}^n)} \leq C\left\{\|\omega_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} + (1+t)^{1-\operatorname{sgn}\tau}(1-e^{-t})\|\omega_1\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}\right\} \tag{1.9}$$

for all $t > 0$. Here, we have set $\tau = \sqrt{m^2 - \frac{n^2}{4}}$.

Throughout this manuscript, the same letters C and C_M are used to indicate the positive constants that can vary.

2. The Kelin-Gordon Equation

In this section, we take into account the solutions in (1.1) which were given by Yagdjian-Galstian [7]. For $(x_0,t_0) \in \mathbb{R}^{n+1}$, the backward and forward light cones are denoted by

$$D_-(x_0,t_0) := \left\{(x,t) \in \mathbb{R}^{n+1} : \ |x-x_0| \leq e^{t_0} - e^t\right\},$$
$$D_+(x_0,t_0) := \left\{(x,t) \in \mathbb{R}^{n+1} : \ |x-x_0| \leq e^t - e^{t_0}\right\},$$

respectively. The function, introduced by Yagdjian-Galstian [7] and Galstian [1] is given as

$$\Xi(x,t;x_0,t_0) := (4e^{t_0+t})^{i\tau}\left((e^{t_0}+e^t)^2 - |x-x_0|^2\right)^{-\frac{1}{2}-i\tau} F\left(\frac{1}{2}+i\tau, \frac{1}{2}+i\tau; 1; \frac{(e^{t_0}-e^t)^2 - |x-x_0|^2}{(e^t+e^{t_0})^2 - |x-x_0|^2}\right),$$

for $(x,t) \in D_- \cup D_+$, where $\tau = \sqrt{m^2 - \frac{n^2}{4}}$ and $(x-x_0)^2$ is the inner product for $x, x_0 \in \mathbb{R}^n$. The function is crucial for the solution of (1.1). Here, $F(v,y;z;\zeta)$ describes the hypergeometric function denoted by the power series

$$F(v,y;z;\zeta) = \sum_{n=0}^\infty \frac{(v)_n(y)_n}{(z)_n}\frac{\zeta^n}{n!}, \qquad |\zeta| < 1,$$

where $v, y, z \in \mathbb{C}$ with $z \neq 0, -1, -2, ...$, and

$$\begin{cases} (v)_0 = 1, \\ (v)_n = v(v+1)...(v+n-1), \qquad n = 1,2,3,.... \end{cases}$$

The kernels $\aleph_0(z,t)$ and $\aleph_1(z,t)$ are set by Yagdjian-Galstian [7] and Yagdjian [1] as follows

$$\begin{aligned}\aleph_0(z,t) &:= -\left[\frac{\partial}{\partial b}\Xi(z,t;0,b)\right]_{b=0}\\ &= -(4e^t)^{i\tau}\left((1+e^t)^2-z^2\right)^{-i\tau-\frac{1}{2}}\left((1-e^t)^2-z^2\right)^{-1}\\ &\quad\times\Big[\left(e^t-1-i\tau(e^{2t}-1-z^2)\right)\\ &\quad\times F\left(\frac{1}{2}+i\tau,\frac{1}{2}+i\tau;1;\frac{(1-e^t)^2-z^2}{(1+e^t)^2-z^2}\right)\\ &\quad+(1-e^{2t}+z^2)\left(\frac{1}{2}-i\tau\right)\\ &\quad\times F\left(-\frac{1}{2}+i\tau,\frac{1}{2}+i\tau;1;\frac{(1-e^t)^2-z^2}{(1+e^t)^2-z^2}\right)\Big]\end{aligned}$$

and

$$\begin{aligned}\aleph_1(z,t) &:= \Xi(z,t;0,0)\\ &= (4e^t)^{i\tau}\left((1+e^t)^2-z^2\right)^{-\frac{1}{2}-i\tau}F\left(\frac{1}{2}+i\tau,\frac{1}{2}+i\tau;1;\frac{(e^t-1)^2-z^2}{(e^t+1)^2-z^2}\right),\end{aligned}$$

where $0\le z\le e^t-1$. Thus, the solution $\psi=\psi(x,t)$ of (1.8) is given by Yagdjian-Galstian [7] as follows

$$\psi(x,t)=e^{-\frac{t}{2}}\vartheta_{\omega_0}(x,\sigma(t))+2\int_0^1\vartheta_{\omega_0}(x,\sigma(t)s)\aleph_0(\sigma(t)s,t)\sigma(t)ds+2\int_0^1\vartheta_{\omega_1}(x,\sigma(t)s)\aleph_1(\sigma(t)s,t))\sigma(t)ds,\tag{2.1}$$

where $\sigma(t):=e^t-1$ with $t>0$. Here, for $\omega\in C_0^\infty(\mathbb{R}^n)$, $\vartheta_\omega(x,t)$ is the solution of the following wave equation with the initial conditions,

$$\vartheta_{tt}-\Delta\vartheta=0,\qquad \vartheta(x,0)=\omega(x),\quad \vartheta_t(x,0)=0,\qquad (x,t)\in\mathbb{R}^n\times(0,\infty).\tag{2.2}$$

3. L^∞ Estimate for the Klein-Gordon Equation

In this section, we derive the L^∞ estimate of the initial value problem for the Klein-Gordon equation in anti-de Sitter spacetime. We need to use the following two lemmas to show the estimate.

Lemma 3.1. *Let $\tau\ge 0$ and $\sigma(t)=e^t-1$. Then*

$$\int_0^1(1+\sigma(t)s)^{-\frac{n-1}{2}}\left|\aleph_1(\sigma(t)s,t)\right|\sigma(t)ds\le C_M(1+t)^{1-\operatorname{sgn}\tau}(1-e^{-t})\tag{3.1}$$

for all $t>0$.

Proof. If we set $r=1+\sigma(t)s$ and use the definition of $\aleph_1$, we obtain

$$\begin{aligned}\int_0^1(1+\sigma(t)s)^{-\frac{n-1}{2}}\left|\aleph_1(\sigma(t)s,t)\right|\sigma(t)ds &= \int_1^{e^t}r^{-\frac{n-1}{2}}((1+e^{-t})^2-(r-1)^2)^{-\frac{1}{2}}\\ &\quad\times\left|F\left(\frac{1}{2}+i\tau,\frac{1}{2}+i\tau;1;\frac{(1-e^t)^2-(r-1)^2}{(1+e^t)^2-(r-1)^2}\right)\right|dr\\ &\le C\int_0^{e^t-1}((e^t+1)^2-y^2)^{-\frac{1}{2}}\left|F\left(\frac{1}{2}+i\tau,\frac{1}{2}+i\tau;1;\frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\right)\right|dy,\end{aligned}$$

where $y=r-1$ variable changing is used in the last inequality. In [1], the integral in the last inequality is estimated as

$$\int_0^{e^t-1}((e^t+1)^2-y^2)^{-\frac{1}{2}}\left|F\left(\frac{1}{2}+i\tau,\frac{1}{2}+i\tau;1;\frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\right)\right|dy\le C_M(1+t)^{1-\operatorname{sgn}\tau}(e^t-1)(e^t+1)^{-1}$$

for $\tau\ge 0$. Hence, we have

$$\int_0^1(1+\sigma(t)s)^{-\frac{n-1}{2}}\left|\aleph_1(\sigma(t)s,t)\right|\sigma(t)ds\le C_M(1+t)^{1-\operatorname{sgn}\tau}(e^t-1)(e^t+1)^{-1},$$

which leads to (3.1). □

Lemma 3.2. *Let* $\tau \geq 0$ *and* $\sigma(t) = e^t - 1$. *Then,*

$$\int_0^1 (1+\sigma(t)s)^{-\frac{n-1}{2}} \left|\aleph_0(\sigma(t)s,t)\right| \sigma(t)ds \leq C_M(1-e^{-t})$$

for all $t > 0$.

Proof. Since the proof is similar to the previous one, we obtain

$$\begin{aligned} \int_0^1 (1+\sigma(t)s)^{-\frac{n-1}{2}} \left|\aleph_0(\sigma(t)s,t)\right| \sigma(t)ds \leq C \int_0^{e^t-1} &((e^t+1)^2-y^2)^{-\frac{1}{2}} \left((e^t-1)^2-y^2\right)^{-1} \\ &\times \Bigg\| \Big[(e^t-1-i\tau(e^{2t}-1-y^2))F(\frac{1}{2}+i\tau, \frac{1}{2}+i\tau; 1; \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}) \\ &+ (1-e^{2t}+y^2)(\frac{1}{2}-i\tau)F(-\frac{1}{2}+i\tau, \frac{1}{2}+i\tau; 1; \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2})\Big]\Bigg\| dy, \end{aligned} \tag{3.2}$$

where we have set $\sigma(t)s = y$. In [1], the estimate of the integral in the right hand side of (3.2) yields the following inequality

$$\begin{aligned} &\int_0^{e^t-1} ((e^t+1)^2-y^2)^{-\frac{1}{2}} \left((e^t-1)^2-y^2\right)^{-1} \\ &\quad\times \Bigg\| \Big[(e^t-1-i\tau(e^{2t}-1-y^2))F(\frac{1}{2}+i\tau, \frac{1}{2}+i\tau; 1; \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}) \\ &\quad+ (1-e^{2t}+y^2)(\frac{1}{2}-i\tau)F(-\frac{1}{2}+i\tau, \frac{1}{2}+i\tau; 1; \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2})\Big]\Bigg\| dy \\ &\leq C_M(1-e^{-t}) \end{aligned}$$

for all $t > 0$. □

Proof of Theorem 1.2. The solution of (1.8) in the case of $\omega_1 = 0$ is first considered. From (2.1), we have

$$\psi(x,t) = e^{-\frac{t}{2}}\vartheta_{\omega_0}(x,\sigma(t)) + 2\int_0^1 \vartheta_{\omega_0}(x,\sigma(t)s)\aleph_0(\sigma(t)s,t)\sigma(t)ds,$$

Then, we get

$$\|\psi(.,t)\|_{L^\infty(\mathbb{R}^n)} \leq e^{-\frac{t}{2}}\|\vartheta_{\omega_0}(.,\sigma(t))\|_{L^\infty(\mathbb{R}^n)} + 2\int_0^1 \|\vartheta_{\omega_0}(.,\sigma(t)s)\|_{L^\infty(\mathbb{R}^n)} \left|\aleph_0(\sigma(t)s,t)\right| \sigma(t)ds.$$

Here, we remark from [12] that the solution $\vartheta(x,t)$ of (2.2) satisfies the estimate

$$\|\vartheta(.,t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n-1}{2}}\|\omega\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \tag{3.3}$$

for $t \geq 0$, if $n \geq 2$. For all $t \geq 0$, from (3.3) we have

$$\begin{aligned} e^{-\frac{t}{2}}\|\vartheta_{\omega_0}(.,\sigma(t)\|_{L^\infty(\mathbb{R}^n)} &\leq Ce^{-\frac{t}{2}}(1+\sigma(t))^{-\frac{n-1}{2}}\|\omega_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \\ &\leq C\|\omega_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}, \end{aligned} \tag{3.4}$$

where $\sigma(t) = e^t - 1$. On the other hand, from (3.3) we obtain

$$2\int_0^1 \|\vartheta_{\omega_0}(.,\sigma(t)s)\|_{L^\infty(\mathbb{R}^n)} \left|\aleph_0(\sigma(t)s,t)\right| \sigma(t)ds \leq C\|\omega_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \int_0^1 (1+\sigma(t)s)^{-\frac{n-1}{2}} \left|\aleph_0(\sigma(t)s,t)\right|\sigma(t)ds.$$

From Lemma 3.2, we have

$$\int_0^1 (1+\sigma(t)s)^{-\frac{n-1}{2}} \left|\aleph_0(\sigma(t)s,t)\right| \sigma(t)ds \leq C(1-e^{-t}). \tag{3.5}$$

Hence, from (3.4) and (3.5) we get

$$\|\psi(.,t)\|_{L^\infty(\mathbb{R}^n)} \leq C\,\|\omega_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \tag{3.6}$$

for $\omega_1 = 0$. If we consider the case $\omega_0 = 0$, from (2.1), we have

$$\psi(x,t) = 2\int_0^1 \vartheta_{\omega_1}(x,\sigma(t)s)\aleph_1(\sigma(t)s,t)\sigma(t)ds.$$

From (3.3) we obtain

$$2\int_0^1 \|\vartheta_{\omega_1}(.,\sigma(t)s)\|_{L^\infty(\mathbb{R}^n)} \left|\aleph_1(\sigma(t)s,t)\right| \sigma(t)ds \le C\|\omega_1\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \int_0^1 (1+\sigma(t)s)^{-\frac{n-1}{2}} |\aleph_1(\sigma(t)s,t)|\sigma(t)ds.$$

From Lemma 3.1, we have

$$\int_0^1 (1+\sigma(t)s)^{-\frac{n-1}{2}} \left|\aleph_1(\sigma(t)s,t)\right| \sigma(t)ds \le C(1+t)^{1-\operatorname{sgn}\tau}(1-e^{-t}).$$

Hence, we obtain

$$\|\psi(.,t)\|_{L^\infty(\mathbb{R}^n)} \le C(1+t)^{1-\operatorname{sgn}\tau}(1-e^{-t}) \|\omega_1\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} . \tag{3.7}$$

Thus, (3.6) and (3.7) leads to (1.9). □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contributions Statement

Author have read and agreed to the published version of the manuscript.

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