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Some Perturbed Trapezoid Inequalities for n-times Differentiable Strongly log-Convex Functions

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Abstract

The aim of this study is to introduce some inequalities for n-times differentiable strongly log-convex functions. The perturbed trapezoid inequality is used to establish the new inequalities. It is seen that these inequalities have a better upper bound than the inequalities obtained for log-convex functions. Besides, the mentioned inequalities for strongly log-convex functions are reduced to the ones given for log-convex functions with a suitable choice of the arbitrary constant.

Keywords: Convex function, log-convex function, strongly log-convex function, perturbed trapezoid inequalities.

2. Introduction

Convex functions have a great importance in different fields such as pure and applied mathematics, optimization theory, health, art, etc. Many researchers focus on convex functions and their expansions. One of the most important expansions and generalizations that adds innovation to this subject is the strongly convex functions. Polyak [1] were firstly investigated strongly convex functions. For minimizing a function, he established convergence of a gradient type algorithm via the strongly convex functions. Some applications of the strongly convex functions arise in solving linear systems, linear programming, dual formulations of linearly constrained convex problems [2], error analysis, complementarity problems, characterization of the inner product exponential stability of primal-dual gradient dynamics [3-6]. For several implementations related to strongly convex functions, see [7-13]. Merentes and Nikodem [14] introduced the Hermite-Hadamard type inequality and the integral Jensen-type inequality for strongly convex functions. Azcar et. al. [15] derived a convenient counterpart of the Fejér inequalities for strongly convex functions.

Recently, the strongly convexity has been frequently used in the convergence analysis of the approximate methods for solving variational inequalities and optimization theory [16]. It has concluded from many studies that some properties of strongly convex functions are merely stronger versions of properties of convex functions. Continuing to stay motivated in this evolving field, we discuss perturbed trapezoidal inequalities for strongly log-convex functions [17].

In this study, some basic definition and theorems are introduced in the second section. In the third section, the inequalities previously presented for log-convex functions will be extended for strongly log-convex functions.

3. Preliminaries

Definition 2.1 [18]: Let $\upsilon : I \to \mathbb{R}$, $\emptyset \neq I \subset \mathbb{R}$. If inequality

$$\upsilon\left(rx+(1-r)y\right) \le r\upsilon(x)+(1-r)\upsilon(y) \quad \textbf{(2.1)}$$

is satisfied for all $x, y \in I$ and $r \in [0,1]$, then the function υ is convex on I.

Definition 2.2 [1]: $\upsilon : I = [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ is defined as a strongly convex function with modulus c if

$$\upsilon\left(rx+(1-r)y\right) \le r\upsilon(x)+(1-r)\upsilon(y) - cr(1-r)\left\|y-x\right\|^2 \quad \textbf{(2.2)}$$

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where and $\forall x, y \in I,\ r \in [0,1]$ and $I \subseteq \mathbb{R}$ is an interval and c is a positive number. The trapezoid inequality is introduced for the numerical integration in the following:

Definition 2.3 [19]: The trapezoid inequality is defined as

$$\left|\int_{\alpha}^{\beta} \upsilon(x)dx - \frac{1}{2}(\beta-\alpha)(\upsilon(\alpha)+\upsilon(\beta))\right| \le \frac{1}{12}M_2(\beta-\alpha)^3 \tag{2.3}$$

where $\upsilon:[\alpha,\beta]\to\mathbb{R}$ is assumed to be twice differentiable on the interval (α,β), with the second derivative bounded on (α,β) by $M_2 = \sup_{x\in(\alpha,\beta)}|\upsilon''(x)| < +\infty$.

Dönmez Demir and Şanal [20] have introduced some perturbed trapezoid inequalities for n-times differentiable convex and log-convex functions. In this study, we present some inequalities for n-times differentiable strongly log-convex functions.

Definition 2.4 [21]: If

$$\varphi(\lambda x+(1-\lambda)y) \le \varphi^{\lambda}(x)\varphi^{1-\lambda}(y) \tag{2.4}$$

for all $x, y \in [\alpha,\beta]$ and $\lambda \in [0,1]$, it is said that the positive function φ is a log-convex function on a real interval $I=[\alpha,\beta]$. If φ is a positive log-concave function, then the inequality is inverted. Besides, if $\varphi>0$ and φ'' exists on I, then φ is log-convex if and only if $\varphi.\varphi''-(\varphi')^2 \ge 0.$

Definition 2.5 [10]: A positive function $f: I\subset\mathbb{R}\to(0,\infty)$ is strongly log-convex for $c>0$ if

$$f(\lambda x+(1-\lambda)y) \le [f(x)]^{\lambda}[f(y)]^{1-\lambda} - c\lambda(1-\lambda)(x-y)^2 \tag{2.5}$$

for all $x, y\in I$ and $\lambda\in(0,1)$. From the Definition 2.4, one obtains

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &\le [f(x)]^{\lambda}[f(y)]^{1-\lambda} - c\lambda(1-\lambda)(x-y)^2 \\ &\le \lambda f(x)+(1-\lambda)f(y) - c\lambda(1-\lambda)(x-y)^2 \\ &\le \max\{f(x), f(y)\} - c\lambda(1-\lambda)(x-y)^2 \end{aligned}$$

by the arithmetic-geometric mean inequality.

Lemma 2.1 [22]: Let's assume that $\upsilon: I\subseteq\mathbb{R}\to\mathbb{R}$ be n times differentiable mapping on I° such that I° represents interior of I, $\alpha,\beta\in I^{\circ}$ where $\alpha<\beta$ and n is even number. If $\upsilon^{(n)}\in L[\alpha,\beta]$, then one obtains

$$\begin{aligned} &\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\upsilon(x)dx - \frac{\upsilon(\alpha)+\upsilon(\beta)}{2} + \cdots \\ &- \frac{(\beta-\alpha)^{n-4}[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right] \\ &+ \frac{(\beta-\alpha)^{n-3}[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right] \\ &- \frac{(\beta-\alpha)^{n-2}[n.a_n+\cdots+2a_2]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right] \\ &+ \frac{(\beta-\alpha)^{n-1}[a_n+\cdots+a_1+2a_0]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right] \\ &= \frac{(\beta-\alpha)^n}{2.n!.a_n}\int_0^1 (a_n r^n+\cdots+a_1 r+a_0)\left[\upsilon^{(n)}(r\alpha+(1-r)\beta)+\upsilon^{(n)}(r\beta+(1-r)\alpha)\right]dr. \end{aligned} \tag{2.6}$$

4. Main Results

Theorem 3.1: Let $\upsilon: I\subset\mathbb{R}\to(0,\infty)$ be a positive function having n^{th} derivatives on I°, $\alpha,\beta\in I^{\circ}$ where $\alpha<\beta$ and n is even number. If $\left|\upsilon^{(n)}\right|$ is strongly log-convex on $[\alpha,\beta]$ with modulus $c>0$, then, the inequality in the following holds:

$$\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\upsilon(x)dx-\frac{\upsilon(\alpha)+\upsilon(\beta)}{2}+\cdots\right.$$
$$-\frac{(\beta-\alpha)^{n-4}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right]$$
$$+\frac{(\beta-\alpha)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right]$$
$$-\frac{(\beta-\alpha)^{n-2}\left[n.a_n+\cdots+2a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right] \quad \textbf{(3.1)}$$
$$\left.+\frac{(\beta-\alpha)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\right|$$
$$\leq\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left[\left|\upsilon^{(n)}(\beta)\sum_{i=0}^{n}\left[-\ln\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-\ln\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|\right)\right]\right|\right.$$
$$\left.+\left|\upsilon^{(n)}(\alpha)\sum_{i=0}^{n}\left[\ln\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,\ln\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|\right)\right]\right|-c(\beta-\alpha)^2\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right].$$

Proof 3.1: From Lemma 2.1 and Definition 2.5, it is concluded that

$$\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\upsilon(x)dx-\frac{\upsilon(\alpha)+\upsilon(\beta)}{2}+\cdots\right.$$
$$-\frac{(\beta-\alpha)^{n-4}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right]$$
$$+\frac{(\beta-\alpha)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right]$$
$$-\frac{(\beta-\alpha)^{n-2}\left[n.a_n+\cdots+2a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right]$$
$$\left.+\frac{(\beta-\alpha)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\right|$$
$$=\left|\frac{(\beta-\alpha)^n}{2.n!.a_n}\left\{\int_0^1\left(a_nr^n+\cdots+a_1r+a_0\right)\left[\upsilon^{(n)}\left(r\alpha+(1-r)\beta\right)+\upsilon^{(n)}\left(r\beta+(1-r)\alpha\right)\right]dr\right\}\right|$$
$$\leq\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left\{\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|\left[\left|\upsilon^{(n)}\left(r\alpha+(1-r)\beta\right)\right|+\left|\upsilon^{(n)}\left(r\beta+(1-r)\alpha\right)\right|\right]dr\right\}$$
$$\leq\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|\begin{bmatrix}\left|\upsilon^{(n)}(\alpha)\right|^r\left|\upsilon^{(n)}(\beta)\right|^{1-r}+\left|\upsilon^{(n)}(\beta)\right|^r\left|\upsilon^{(n)}(\alpha)\right|^{1-r}\\-c_1r(1-r)(\beta-\alpha)^2-c_2r(1-r)(\beta-\alpha)^2\end{bmatrix}dr$$
$$\leq\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\int_0^1\left(\left|a_nr^n\right|+\cdots+\left|a_1r\right|+\left|a_0\right|\right)\begin{bmatrix}\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|^r\left|\upsilon^{(n)}(\beta)\right|+\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|^r\left|\upsilon^{(n)}(\alpha)\right|\\-c_1r(1-r)(\beta-\alpha)^2-c_2r(1-r)(\beta-\alpha)^2\end{bmatrix}dr$$
$$\leq\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left[|a_n|\left|\upsilon^{(n)}(\beta)\right|\int_0^1r^n\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|^rdr+\cdots+|a_1|\left|\upsilon^{(n)}(\beta)\right|\int_0^1r\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|^rdr\right.$$

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$$+|a_0|\left|\upsilon^{(n)}(\beta)\right|\int_0^1\left|\frac{\upsilon^{(n)}(a)}{\upsilon^{(n)}(b)}\right|^r dr+|a_n|\left|\upsilon^{(n)}(\alpha)\right|\int_0^1 r^n\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|^r dr$$

$$\left.+\cdots+|a_1|\left|\upsilon^{(n)}(\alpha)\right|\int_0^1 r\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|^r dr+|a_0|\left|\upsilon^{(n)}(\alpha)\right|\int_0^1\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|^r dr-c(\beta-\alpha)^2\int_0^1 r(1-r)\left(\left|a_n r^n\right|+\cdots+\left|a_1 r\right|+\left|a_0\right|\right)dr\right]$$

$$\leq\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left[\left|\upsilon^{(n)}(\beta)\sum_{i=0}^{n}\left[-\ln\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-\ln\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|\right)\right]\right|\right.$$

$$\left.+\left|\upsilon^{(n)}(\alpha)\sum_{i=0}^{n}\left[\ln\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,\ln\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|\right)\right]\right|-c(\beta-\alpha)^2\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right].$$

Corollary 3.1: Under conditions of Theorem 3.1, it reduced to Theorem 1 given for $\log$-convex functions included in Ref. [20] when c is equal to zero.

Theorem 3.2: Let $\upsilon: I\subset\mathbb{R}\to(0,\infty)$ be a positive function having n^{th} derivatives on I°, $\alpha,\beta\in I^{\circ}$ where $\alpha<\beta$ and $p>1$ such that $1/p+1/q=1$ and n is even number. If the mapping $\left|\upsilon^{(n)}\right|^q$ is strongly $\log$-convex on $[\alpha,\beta]$, with modulus $c_1,c_2>0$, then, we obtain:

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta\upsilon(x)dx-\frac{\upsilon(\alpha)+\upsilon(\beta)}{2}+\cdots\right.$$
$$-\frac{(\beta-\alpha)^{n-4}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right]$$
$$+\frac{(\beta-\alpha)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right]$$
$$-\frac{(\beta-\alpha)^{n-2}\left[n.a_n+\cdots+2a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right]\tag{3.2}$$
$$\left.+\frac{(\beta-\alpha)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\right|$$

$$\leq\frac{(\beta-\alpha)^n}{n!.|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}\right]$$
$$\times\left\{\left[\left|\upsilon^{(n)}(\beta)\right|^q\frac{\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|^q-1}{q\ln\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|}-\frac{c_1(\beta-\alpha)^2}{6}\right]^{\frac{1}{q}}+\left[\left|\upsilon^{(n)}(\alpha)\right|^q\frac{\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|^q-1}{q\ln\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|}-\frac{c_2(\beta-\alpha)^2}{6}\right]^{\frac{1}{q}}\right\}$$

Proof 3.2: Using Lemma 2.1, Definition 2.5 and Hölder's integral inequality [22] and Minkowski's integral inequality [23], we construct

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta\upsilon(x)dx-\frac{\upsilon(\alpha)+\upsilon(\beta)}{2}+\cdots\right.$$

$$
\begin{aligned}
&-\frac{(\beta-\alpha)^{n-4}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right]\\
&+\frac{(\beta-\alpha)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right]\\
&-\frac{(\beta-\alpha)^{n-2}\left[n.a_n+\cdots+2a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right]\\
&+\frac{(\beta-\alpha)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\Bigg|
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
&=\left|\frac{(\beta-\alpha)^n}{2.n!.a_n}\int_0^1\left(a_nr^n+\cdots+a_1r+a_0\right)\left[\upsilon^{(n)}(r\alpha+(1-r)\beta)+\upsilon^{(n)}(r\beta+(1-r)\alpha)\right]dr\right|\\
&\le\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left[\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|\left|\upsilon^{(n)}(r\alpha+(1-r)\beta)\right|dr+\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|\left|\upsilon^{(n)}(r\beta+(1-r)\alpha)\right|dr\right]\\
&\le\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\\
&\times\left[\left(\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|^p dr\right)^{\frac{1}{p}}\left(\int_0^1\left|\upsilon^{(n)}(r\alpha+(1-r)\beta)\right|^q dr\right)^{\frac{1}{q}}+\left(\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|^p dr\right)^{\frac{1}{p}}\left(\int_0^1\left|\upsilon^{(n)}(r\beta+(1-r)\alpha)\right|^q dr\right)^{\frac{1}{q}}\right]\\
&\le\frac{(\beta-\alpha)^n}{n!.|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{(ip+1)^{\frac{1}{p}}}\right]\\
&\times\left\{\left[\left|\upsilon^{(n)}(\beta)\right|^q\frac{\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|^q-1}{q\ln\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|}-\frac{c_1(\beta-\alpha)^2}{6}\right]^{\frac{1}{q}}+\left[\left|\upsilon^{(n)}(\alpha)\right|^q\frac{\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|^q-1}{q\ln\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|}-\frac{c_2(\beta-\alpha)^2}{6}\right]^{\frac{1}{q}}\right\}
\end{aligned}
$$

such that $1/p+1/q=1$. Considering the strongly $\log$-convexity of $\left|\upsilon^{(n)}\right|^q$, then we find

$$
\begin{aligned}
\int_0^1\left|\upsilon^{(n)}(\alpha r+(1-r)\beta)\right|^q dr&\le\int_0^1\left[\left|\upsilon^{(n)}(\alpha)\right|^{rq}\left|\upsilon^{(n)}(\beta)\right|^{(1-r)q}-c_1r(1-r)(\beta-\alpha)^2\right]dr\\
&=\left|\upsilon^{(n)}(\beta)\right|^q\frac{\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|^q-1}{q\ln\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|}-\frac{c_1(\beta-\alpha)^2}{6}
\end{aligned} \tag{3.4}
$$

$$\int_0^1 \left|\upsilon^{(n)}\left(\beta r+(1-r)\alpha\right)\right|^q dr \le \int_0^1 \left[\left|\upsilon^{(n)}(\beta)\right|^{rq}\left|\upsilon^{(n)}(\alpha)\right|^{(1-r)q} - c_2 r(1-r)(\beta-\alpha)^2\right]dr$$

$$= \left|\upsilon^{(n)}(\alpha)\right|^q \frac{\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|^q - 1}{q\ln\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|} - \frac{c_2(\beta-\alpha)^2}{6} \quad \textbf{(3.5)}$$

Using the Minkowski inequality, we have

$$\left(\int_0^1 \left|a_n r^n + \cdots + a_1 r + a_0\right|^p dr\right)^{\frac{1}{p}} \le \left(\int_0^1 |a_n|^p r^{np} dr\right)^{\frac{1}{p}} + \cdots + \left(\int_0^1 |a_1|^p r^p dr\right)^{\frac{1}{p}} + \left(\int_0^1 |a_0|^p dr\right)^{\frac{1}{p}}$$

$$\left(\int_0^1 \left|a_n r^n + \cdots + a_1 r + a_0\right|^p dr\right)^{\frac{1}{p}} \le \frac{|a_n|}{(np+1)^{1/p}} + \cdots + \frac{|a_1|}{(p+1)^{1/p}} + |a_0| = \sum_{i=0}^{n} \frac{|a_i|}{(ip+1)^{1/p}} \quad \textbf{(3.6)}$$

Substituting (3.4), (3.5) and (3.6) to (3.3) gives Eq. (3.2).

Corollary 3.2: If $c_1 = c_2 = 0$, then Theorem 2.2 is reduced to Theorem 2 given for $\log$-convex functions in Ref. [20].

Theorem 3.3: Let $\upsilon : I \subset \mathbb{R} \to (0,\infty)$ be a positive function having n^{th} derivatives on I°, $\alpha,\beta \in I^\circ$ where $a<b$ and $p>1$ with $1/p + 1/q = 1$ and n is even number. If the mapping $\left|\upsilon^{(n)}\right|^p$ is strongly $\log$-convex on $[\alpha,\beta]$, with modulus $c_1, c_2 > 0$, then the inequality in the following holds:

$$\Bigg|\frac{1}{\beta-\alpha}\int_\alpha^\beta \upsilon(x)dx - \frac{\upsilon(\alpha)+\upsilon(\beta)}{2} + \cdots$$
$$- \frac{(\beta-\alpha)^{n-4}\left[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right]$$
$$+ \frac{(\beta-\alpha)^{n-3}\left[n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right]$$
$$- \frac{(\beta-\alpha)^{n-2}\left[n.a_n + \cdots + 2a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right]$$
$$+ \frac{(\beta-\alpha)^{n-1}\left[a_n + \cdots + a_1 + 2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\Bigg| \quad \textbf{(3.7)}$$

$$\le \frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left[\sum_{i=0}^n \frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}$$
$$\times\left[\left|\upsilon^{(n)}(\beta)\right|\left(\sum_{i=0}^n |a_i|\left[-p\ln\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-p\ln\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|\right)\right] - c(\beta-\alpha)^2\sum_{i=0}^n \frac{|a_i|}{(i+2)(i+3)}\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\left|\upsilon^{(n)}(\alpha)\right|^p \sum_{i=0}^n \left[p\ln\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,p\ln\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|\right)\right] - c(\beta-\alpha)^2\sum_{i=0}^n \frac{|a_i|}{(i+2)(i+3)}\right)^{\frac{1}{p}}\right]$$

Proof 3.3: Using Lemma 2.1, Definition 2.5 and power mean integral inequality [24], one obtains

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \upsilon(x)dx-\frac{\upsilon(\alpha)+\upsilon(\beta)}{2}+\cdots\right.$$
$$-\frac{(\beta-\alpha)^{n-4}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}\left[\upsilon^{(n-4)}(\alpha)+\upsilon^{(n-4)}(\beta)\right]$$
$$+\frac{(\beta-\alpha)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-3)}(\beta)-\upsilon^{(n-3)}(\alpha)\right]$$
$$-\frac{(\beta-\alpha)^{n-2}\left[n.a_n+\cdots+2a_2\right]}{2.n!.a_n}\left[\upsilon^{(n-2)}(\alpha)+\upsilon^{(n-2)}(\beta)\right]$$
$$\left.+\frac{(\beta-\alpha)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}\left[\upsilon^{(n-1)}(\beta)-\upsilon^{(n-1)}(\alpha)\right]\right|$$
$$=\left|\frac{(\beta-\alpha)^n}{2.n!.a_n}\int_0^1\left(a_nr^n+\cdots+a_1r+a_0\right)\left[\upsilon^{(n)}(r\alpha+(1-r)\beta)+\upsilon^{(n)}(r\beta+(1-r)\alpha)\right]dr\right|$$
$$\le\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left\{\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|\left|\upsilon^{(n)}(r\alpha+(1-r)\beta)\right|dr+\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|\left|\upsilon^{(n)}(r\beta+(1-r)\alpha)\right|dr\right\}$$
$$\le\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left(\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|dr\right)^{1-\frac{1}{p}}$$
$$\times\left\{\left(\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|\left|\upsilon^{(n)}(r\alpha+(1-r)\beta)\right|^p dr\right)^{\frac{1}{p}}+\left(\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|\left|\upsilon^{(n)}(r\beta+(1-r)\alpha)\right|^p dr\right)^{\frac{1}{p}}\right\}$$
$$\le\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left(|a_n|\int_0^1 r^n dr+\cdots+|a_1|\int_0^1 r\,dr+\int_0^1|a_0|dr\right)^{1-\frac{1}{p}}$$
$$\times\left[\left(\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|\left[\left|\upsilon^{(n)}(\alpha)\right|^{rp}\left|\upsilon^{(n)}(\beta)\right|^{(1-r)p}-c_1r(1-r)(\beta-\alpha)^2\right]dr\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|\left[\left|\upsilon^{(n)}(\beta)\right|^{rp}\left|\upsilon^{(n)}(\alpha)\right|^{(1-r)p}-c_2r(1-r)(\beta-\alpha)^2\right]dr\right)^{\frac{1}{p}}\right]$$
$$\le\frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left(|a_n|\int_0^1 r^n dr+\cdots+|a_1|\int_0^1 rdr+\int_0^1|a_0|dr\right)^{1-\frac{1}{p}}$$
$$\times\left[\left(\begin{array}{l}|a_n|\left|\upsilon^{(n)}(\beta)\right|^p\int_0^1 r^n\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|^{tp}dr+\cdots+|a_1|\left|\upsilon^{(n)}(\beta)\right|^p\int_0^1 r\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|^{tp}dr\\+|a_0|\left|\upsilon^{(n)}(\beta)\right|^p\int_0^1\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|^{rp}dr-c_1(\beta-\alpha)^2\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|r(1-r)dr\end{array}\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\begin{array}{l}|a_n|\left|\upsilon^{(n)}(\alpha)\right|^p\int_0^1 r^n\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|^{rp}dr+\cdots+|a_1|\left|\upsilon^{(n)}(\alpha)\right|^p\int_0^1 r\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|^{rp}dr\\+|a_0|\left|\upsilon^{(n)}(\alpha)\right|^p\int_0^1\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|^{rp}dr-c_2(\beta-\alpha)^2\int_0^1\left|a_nr^n+\cdots+a_1r+a_0\right|r(1-r)dr\end{array}\right)^{\frac{1}{p}}\right]$$

$$\leq \frac{(\beta-\alpha)^n}{2.n!.|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}$$

$$\times\left[\left(\sum_{i=0}^{n}\left|\upsilon^{(n)}(\beta)\right|^p |a_i|\left[-p\ln\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,-p\ln\left|\frac{\upsilon^{(n)}(\alpha)}{\upsilon^{(n)}(\beta)}\right|\right)\right]-c_1(\beta-\alpha)^2\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\left|\upsilon^{(n)}(\alpha)\right|^p\sum_{i=0}^{n}\left[p\ln\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|\right]^{-i-1}\left[\Gamma(i+1)-\Gamma\left(i+1,p\ln\left|\frac{\upsilon^{(n)}(\beta)}{\upsilon^{(n)}(\alpha)}\right|\right)\right]\right)^{\frac{1}{p}}-c_2(\beta-\alpha)^2\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right]$$

The proof is completed.

Corollary 3.3: Under conditions of Theorem 3.3, if $c_1=c_2=0$, it reduced to Theorem 3 given for $\log$-convex functions in Ref. [20].

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Ethics

There are no ethical issues after the publication of this manuscript.

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