Fundamental Journal of Mathematics and Applications, 5 (4) (2022) 210-227
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://dx.doi.org/10.33401/fujma.1107963

On Some Fixed Point Theorems for $\mathscr{G}(\Sigma, \vartheta, \Xi)-$Contractions in Modular $b-$Metric Spaces

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Article Info

Keywords: *Common fixed point, $C-$class function, Modular $b-$metric space*
2010 AMS: *47H10, 54H25*
Received: *23 April 2022*
Accepted: *27 August 2022*
Available online: *26 October 2022*

Abstract

This article aims to specify a new $C-$class function endowed with altering distance and ultra altering distance function via generalized $\Xi-$contraction, which is called the $\mathscr{G}(\Sigma, \vartheta, \Xi)-$contraction in modular $b-$metric spaces. Regarding these new contraction type mappings, the study includes some existence and uniqueness theorems, and to indicate the usability and productivity of these results, some applications related to integral type contractions and an application to the graph structure.

1. Introduction and preliminaries

In this study, the set of all natural and non-negative real numbers will be symbolized by $\mathbf{N}$ and $\mathbf{R}^+$, respectively.
Fixed point theory is an active and popular area for researchers in nonlinear analysis. Especially metric fixed point theory is a cornerstone for this research area. Researchers working in this field are indebted to S. Banach [1]. The focal point of this topic is to achieve the best suitable conditions on mappings to guarantee the existence and the uniqueness of fixed points, mainly the Banach Fixed Point Theorem put forward by Banach in 1922. In particular, extensive progress has been made in improving and expanding these conditions over the past few decades.
However, the metric structure has been generalized in many directions. One of the crucial results defined in different periods by Bakhtin [2] and Czerwik [3, 4] is $b-$metric space, as noted below.

Definition 1.1. [3] *Let $\mathbf{S}$ be a non-void set and $\kappa \geq 1, (\kappa \in \mathbf{R})$. Presume that the function $\eta : \mathbf{S} \times \mathbf{S} \to \mathbf{R}^+$ provides the following terms: for every $\varpi, \xi, \rho \in \mathbf{S}$,*

(η_1) $\eta(\varpi, \xi) = 0 \Leftrightarrow \varpi = \xi,$
(η_2) $\eta(\varpi, \xi) = \eta(\xi, \varpi),$
(η_3) $\eta(\varpi, \xi) \leq \kappa[\eta(\varpi, \rho) + \eta(\rho, \xi)].$

The function η is entitled a $b-$metric on $\mathbf{S}$, and the pair $(\mathbf{S}, \eta)$ is a $b-$metric space.

In the case of $\kappa = 1$, the concept of $b-$metric and ordinary metric coincide. Also, unlike standard metrics, the $b-$metric is not continuous. Accordingly, the following lemma is valuable and exceptionally significant for a $b-$metric space.

Lemma 1.2. [5] *Let $(\mathbf{S}, \eta)$ be a $b-$metric space with $\kappa \geq 1$, the sequences $\{\varpi_q\}$ and $\{\xi_q\}$ be convergent to ϖ and ξ, respectively. So, we have*

$$\frac{1}{\kappa^2}\eta(\varpi, \xi) \leq \liminf_{q\to\infty} \eta(\varpi_q, \xi_q) \leq \limsup_{q\to\infty} \eta(\varpi_q, \xi_q) \leq \kappa^2 \eta(\varpi, \xi).$$

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Especially, if $\varpi = \xi$*, then we have* $\lim_{q\to\infty} \eta(\varpi_q, \xi_q) = 0$*. Also, for* $z \in \mathbf{S}$*, we have*

$$\frac{1}{\kappa}\eta(\varpi, z) \le \liminf_{q\to\infty} \eta(\varpi_q, z) \le \limsup_{q\to\infty} \eta(\varpi_q, z) \le \kappa\eta(\varpi, z).$$

In 2008, V. V. Chistyakov [6] proposed a new concept of modular metric space generated by $F-$modular and the theory of this space. Afterward, in 2010, V. V. Chistyakov [7] defined the modular metric space using a modular that identifies an arbitrary set.
Initially, let $\mathbf{S}$ be a non-empty set and $\upsilon : (0,\infty) \times \mathbf{S} \times \mathbf{S} \to [0,\infty]$ be a function. For brevity, we will write:

$$\upsilon_\lambda(\varpi, \xi) = \upsilon(\lambda, \varpi, \xi),$$

for all $\lambda > 0$ and $\varpi, \xi \in \mathbf{S}$.

Definition 1.3. [7] *Let* $\mathbf{S}$ *be a non-empty set and* $\upsilon : (0,\infty) \times \mathbf{S} \times \mathbf{S} \to [0,\infty]$ *be a function that admits the following axioms. Thereupon, we say that* υ *is named a modular metric for all* $\varpi, \xi, \rho \in \mathbf{S}$

(υ_1) $\upsilon_\lambda(\varpi, \xi) = 0$ *for all* $\lambda > 0$ *if and only if* $\varpi = \xi$*,*
(υ_2) $\upsilon_\lambda(\varpi, \xi) = \upsilon_\lambda(\xi, \varpi)$ *for all* $\lambda > 0$*,*
(υ_3) $\upsilon_{\lambda+\mu}(\varpi, \xi) \le \upsilon_\lambda(\varpi, \rho) + \upsilon_\mu(\rho, \xi)$ *for all* $\lambda, \mu > 0$*.*

If we only exchange the (υ_1) *with*

$(\upsilon_1{}')$ $\upsilon_\lambda(\varpi, \varpi) = 0$ *for all* $\lambda > 0$*,*

then υ *is said to be a (metric) pseudomodular on* $\mathbf{S}$*.*

For more detail, it refers to [6]-[10].
In 2018, M. E. Ege and C. Alaca [11] introduced modular $b-$metric spaces by combining the structures of $b-$metric and modular metrics and, besides, established some fixed point theorems in the new space setting.

Definition 1.4. [11] *Let* $\mathbf{S}$ *be a non-empty set and let* $\kappa \ge 1$ $(\kappa \in \mathbf{R})$*. A map* $\ell : (0,\infty) \times \mathbf{S} \times \mathbf{S} \to [0,\infty]$ *is entitled as modular* $b-$*metric, provided that the following circumstances satisfied for all* $\varpi, \xi, \rho \in \mathbf{S}$*,*

(ℓ_1) $\ell_\lambda(\varpi, \xi) = 0$ *for all* $\lambda > 0$ *if and only if* $\varpi = \xi$*,*
(ℓ_2) $\ell_\lambda(\varpi, \xi) = \ell_\lambda(\xi, \varpi)$ *for all* $\lambda > 0$*,*
(ℓ_3) $\ell_{\lambda+\mu}(\varpi, \xi) \le \kappa\left[\ell_\lambda(\varpi, \rho) + \ell_\mu(\rho, \xi)\right]$ *for all* $\lambda, \mu > 0$*.*

The pair $(\mathbf{S}, \ell)$ *is a modular* $b-$*metric space expressed in MbMS.*

In fact, for $\kappa = 1$, it can be seen that MbMS is an extension of the modular metric space.

Example 1.5. [11] *Let us consider the space*

$$l_p = \left\{(\varpi_j) \subset \mathbf{R} : \sum_{j=1}^{\infty} \left|\varpi_j\right|^p < \infty\right\} \quad 0 < p < 1.$$

For $\lambda \in (0,\infty)$ *if we define* $\ell_\lambda(\varpi, \xi) = \frac{m(\varpi,\xi)}{\lambda}$ *such that*

$$m(\varpi, \xi) = \left(\sum_{j=1}^{\infty} \left|\varpi_j - \xi_j\right|^p\right)^{\frac{1}{p}}, \quad \varpi = \varpi_j, \xi = \xi_j \in l_p$$

then we see that $(\mathbf{S}, \ell)$ *is an MbMS with* $\kappa = 2^{\frac{1}{p}}$*.*

Example 1.6. [12] *Let* $(\mathbf{S}, \upsilon)$ *be a modular metric space and let* $p \ge 1$ *be a real number. Take* $\ell_\lambda(\varpi, \xi) = (\upsilon_\lambda(\varpi, \xi))^p$*. Due to the fact that the function* $\Gamma(t) = t^p$ *is convex for* $t \ge 0$*, by Jensen inequality, we attain*

$$(\alpha + \beta)^p \le 2^{p-1}(\alpha^p + \beta^p)$$

for $\alpha, \beta \ge 0$*. Thus,* $(\mathbf{S}, \ell)$ *is an MbMS with* $\kappa = 2^{p-1}$*.*

Definition 1.7. [11] *Let* ℓ *be a modular* $b-$*metric on a set* $\mathbf{S}$*, and a modular set is identified by*

$$\mathbf{S}_\ell = \left\{\xi \in \mathbf{S} : \xi \overset{\ell}{\sim} \varpi\right\},$$

where the $\stackrel{\ell}{\sim}$ *is a binary relation on* $\mathbf{S}$ *defined by,*

$$\varpi \sim \xi \Leftrightarrow \lim_{\lambda\to\infty} \ell_\lambda(\varpi,\xi)=0,$$

for $\varpi,\xi\in\mathbf{S}$. *Also, note that the set*

$$\mathbf{S}^*_\ell=\mathbf{S}^*_\ell(\varpi_0)=\{\varpi\in\mathbf{S}:\exists\lambda=\lambda(\varpi)>0 \text{ such that } \ell_\lambda(\varpi,\varpi_0)<\infty\}\ (\varpi_0\in\mathbf{S})$$

is mentioned as a modular metric space (around ϖ_0*).*

In what follows, we recollect some basic topological properties of MbMS.

Definition 1.8. [11] *Let* $(\mathbf{S},\ell)$ *be an MbMS and* $(\varpi_q)_{q\in\mathbf{N}}$ *be a sequence in* $\mathbf{S}^*_\ell$.

(i) $(\varpi_q)_{q\in\mathbf{N}}$ *is called* $\ell-$*convergent to* $\varpi\in\mathbf{S}^*_\ell$ *if and only if* $\ell_\lambda(\varpi_q,\varpi)\to 0$, *as* $q\to\infty$ *for all* $\lambda>0$.
(ii) $(\varpi_q)_{q\in\mathbf{N}}$ *in* $\mathbf{S}^*_\ell$ *is named* $\ell-$*Cauchy sequence if* $\lim_{q,m\to\infty}\ell_\lambda(\varpi_q,\varpi_m)=0$ *for all* $\lambda>0$.
(iii) $\mathbf{S}^*_\ell$ *is called* $\ell-$*complete if any* $\ell-$*Cauchy sequence in* $\mathbf{S}^*_\ell$ *is* $\ell-$*convergent to the point of* $\mathbf{S}^*_\ell$.

In [13], A.H. Ansari presented a novel class of functions named $C-$class functions. This class has extended many results for metric fixed point theory, which contains almost all types of contractions.

Definition 1.9. [13] *Let* $\mathscr{G}:[0,\infty)\times[0,\infty)\to\mathbf{R}$ *be a function. If for all* $p,q\in[0,\infty)$, *the function* $\mathscr{G}$ *is continuous and satisfies the below circumstances, then we say that* $\mathscr{G}$ *is a* $C-$*class function.*

$(\mathscr{G}_1)$ $\mathscr{G}(p,q)\le p$;
$(\mathscr{G}_2)$ $\mathscr{G}(p,q)=p$ *implies that either* $p=0$ *or* $q=0$.

The $C-$*class functions are symbolized by* $\mathscr{C}$.

Example 1.10. [13] *The following ones from* $\mathscr{G}_1$ *to* $\mathscr{G}_5$ *are examples of* $\mathscr{G}\in\mathscr{C}$.

(i) $\mathscr{G}_1(p,q)=p-q$ *for all* $p,q\in[0,\infty)$,
(ii) $\mathscr{G}_2(p,q)=mp$ *for all for all* $p,q\in[0,\infty)$ *where* $0<m<1$,
(iii) $\mathscr{G}_3(p,q)=\frac{p}{(1+q)^r}$ *for all* $p,q\in[0,\infty)$, *where* $r\in(0,\infty)$,
(iv) $\mathscr{G}_4(p,q)=p\beta(p)$ *for all* $p,q\in[0,\infty)$, *where* $\beta:[0,\infty)\to[0,\infty)$ *and is continuous,*
(v) $\mathscr{G}_5(p,q)=\sqrt[n]{\ln(1+p^n)}$ *for all* $p,q\in[0,\infty)$.

Definition 1.11. [14] *The family* Ω *denotes all function* $\Sigma:[0,\infty)\to[0,\infty)$, *which is named altering distance function, if*

(Σ_1) Σ *is continuous and non-decreasing;*
(Σ_2) $\Sigma(\iota)=0$ *if and only if* $\iota=0$.

Definition 1.12. [13] *The family* Π *denotes all function* $\vartheta:[0,\infty)\to[0,\infty)$, *which is named ultra altering distance function if*

(ϑ_1) ϑ *is continuous;*
(ϑ_2) $\vartheta(t)>0$ *for all* $t>0$.

Also, for $C-$class functions, it refers to [15]-[18].
In 2017, Fulga and Proca [19] introduced a new contraction mapping involving the following expression and proved a fixed point theorem on a complete metric space,

$$\Xi(\varpi,\xi)=m(\varpi,\xi)+|m(\varpi,\Gamma\varpi)-m(\xi,\Gamma\xi)|,$$

whenever $(\mathbf{S},m)$ is a complete metric space and $\varpi,\xi\in\mathbf{S}$. Subsequently, it is used as $\Xi-$contraction and appears in many articles, see, [20]-[22].
In [23], Proca specified a new expression of $\Xi-$contraction with the "max operator" and also, in [24] verified a fixed point theorem, as indicated below.

Theorem 1.13. [24] *Let* $\Gamma:\mathbf{S}\to\mathbf{S}$ *be a mapping on a complete metric space* $(\mathbf{S},m)$. Γ *admits a unique fixed point in* $\mathbf{S}$ *if there exists* $\alpha\in[0,1)$ *such that for all* $\varpi,\xi\in\mathbf{S}$

$$m(\Gamma\varpi,\Gamma\xi)\le\alpha(M^*(\varpi,\xi)),$$

where

$$\begin{aligned}M^*(\varpi,\xi)=\max\{&m(\varpi,\xi)+|m(\varpi,\Gamma\varpi)-m(\xi,\Gamma\xi)|;\,m(\varpi,\Gamma\varpi)+|m(\varpi,\xi)-m(\xi,\Gamma\xi)|;\\&m(\xi,\Gamma\xi)+|m(\varpi,\xi)-m(\varpi,\Gamma\varpi)|;\,\tfrac{1}{2}[m(\varpi,\Gamma\xi)+m(\xi,\Gamma\varpi)+|m(\varpi,\Gamma\varpi)-m(\xi,\Gamma\xi)|]\}.\end{aligned}$$

Furthermore, in [24], Proca has given an example to explain that $M^*(\varpi,\xi)$ is more general than the value of the maximum of Ciric type contraction [25].
The following notion will be used throughout the study.

Definition 1.14. [26] *Let* $(\mathbf{S},m)$ *be a metric space and* $\Gamma,\Upsilon:\mathbf{S}\to\mathbf{S}$ *be two mappings. Then,* Γ *and* Υ *are said to be weakly compatible if* $\Gamma\varpi=\Upsilon\varpi$ *implies* $\Gamma\Upsilon\varpi=\Upsilon\Gamma\varpi$ *for some* $\varpi\in\mathbf{S}$.

2. Main results

Owing to the fact that the concept of modular metrics does not have to be finite, the following requirements are essential to assuring the existence and uniqueness of fixed points of contraction mappings in modular metric and modular $b-$metric spaces.

(M_1) $\ell_\lambda(\varpi,\Gamma\varpi)<\infty$ for all $\lambda>0$ and $\varpi\in\mathbf{S}_\ell^*$,
(M_2) $\ell_\lambda(\varpi,\xi)<\infty$ for all $\lambda>0$ and $\varpi,\xi\in\mathbf{S}_\ell^*$.

In this section, we aim to characterize the concept of $\mathscr{G}(\Sigma,\vartheta,\Xi)-$contractions by considering the $C-$class function endowed with the functions Σ and ϑ, including generalized $\Xi-$contractions for four mappings in the framework of modular $b-$metric spaces. We also put forward some new results derived immediately from the main result.

Definition 2.1. *Let ℓ be a modular $b-$metric with $\kappa\geq 1$ on set $\mathbf{S}_\ell^*$, and let $\Gamma,\Upsilon,J,\zeta:\ \mathbf{S}_\ell^*\to\ \mathbf{S}_\ell^*$ be mappings. The mappings $\Gamma,\Upsilon,J,$ and ζ are called $\mathscr{G}(\Sigma,\vartheta,\Xi)$-contraction, if there exist $\mathscr{G}\in\mathscr{C}$, $\Sigma\in\Omega$, and $\vartheta\in\Pi$ such that*

$$\Sigma\left(\kappa^3\ell_\lambda(\Gamma\varpi,\Upsilon\xi)\right)\leq\mathscr{G}\left(\Sigma(\Xi(\varpi,\xi)),\vartheta(\Xi(\varpi,\xi))\right), \tag{2.1}$$

where

$$\begin{aligned}\Xi(\varpi,\xi)=\max\Big\{&\ell_\lambda(\zeta\varpi,J\xi)+|\ell_\lambda(\zeta\varpi,\Gamma\varpi)-\ell_\lambda(J\xi,\Upsilon\xi)|;\ \ell_\lambda(\zeta\varpi,\Gamma\varpi)+|\ell_\lambda(\zeta\varpi,J\xi)-\ell_\lambda(J\xi,\Upsilon\xi)|;\\&\ell_\lambda(J\xi,\Upsilon\xi)+|\ell_\lambda(\zeta\varpi,J\xi)-\ell_\lambda(\zeta\varpi,\Gamma\varpi)|;\ \frac{1}{2}\left[\frac{\ell_{2\lambda}(\zeta\varpi,\Upsilon\xi)+\ell_{2\lambda}(J\xi,\Gamma\varpi)}{\kappa}+|\ell_\lambda(\zeta\varpi,\Gamma\varpi)-\ell_\lambda(J\xi,\Upsilon\xi)|\right]\Big\},\end{aligned}$$

for all distinct $\varpi,\xi\in\mathbf{S}_\ell^*$, and all $\lambda>0$.

Theorem 2.2. *Let $\mathbf{S}_\ell^*$ be an $\ell-$complete MbMS with constant $\kappa\geq 1$. Assume that the following statements are ensured:*

(i) The mappings $\Gamma,\Upsilon,J,$ and ζ are a $\mathscr{G}(\Sigma,\vartheta,\Xi)-$contraction such that $\Gamma(\mathbf{S}_\ell^*)\subset J(\mathbf{S}_\ell^*)$ and $\Upsilon(\mathbf{S}_\ell^*)\subset\zeta(\mathbf{S}_\ell^*)$,
(ii) One of the sets $\Gamma(\mathbf{S}_\ell^*),J(\mathbf{S}_\ell^*),\Upsilon(\mathbf{S}_\ell^*)$ and $\zeta(\mathbf{S}_\ell^*)$ is a closed subset of $\mathbf{S}_\ell^*$,
(iii) The pairs $\{J,\Upsilon\}$ and $\{\zeta,\Gamma\}$ are weakly compatible.

If the condition (M_1) is satisfied, then Γ,Υ,J and ζ admit a common fixed point in $\mathbf{S}_\ell^*$. Moreover, the condition (M_2) is provided, the common fixed point of $\Gamma,\Upsilon,J,$ and ζ is unique.

Proof. Let $\varpi_0\in\mathbf{S}_\ell^*$ be an arbitrary point. If we take into account the condition (i), there is a point $\varpi_1\in\mathbf{S}_\ell^*$ such that $\xi_0=\Gamma\varpi_0=J\varpi_1$. In a similar way, one can find a point $\varpi_2\in\mathbf{S}_\ell^*$ such that $\xi_1=\Upsilon\varpi_1=\zeta\varpi_2$ as $\Upsilon(\mathbf{S}_\ell^*)\subseteq\zeta(\mathbf{S}_\ell^*)$. Following the above process, we acquire a sequence $\{\xi_q\}$ such that

$$\xi_{2q}=\Gamma\varpi_{2q}=J\varpi_{2q+1}\quad\textit{and}\quad\xi_{2q+1}=\Upsilon\varpi_{2q+1}=\zeta\varpi_{2q+2}.$$

Assume that $\xi_{q_0}\neq\xi_{q_0+1}$, because if we accept that $\xi_{q_0}=\xi_{q_0+1}$ for some q_0, the proof is evident. Therefore, we have $\ell_\lambda(\xi_{2q},\xi_{2q+1})>0$ for all $\lambda>0$. From (2.1), we procure

$$\Sigma\left(\kappa^3\ell_\lambda(\Gamma\varpi_{2q},\Upsilon\varpi_{2q+1})\right)\leq\mathscr{G}\left(\Sigma(\Xi(\varpi_{2q},\varpi_{2q+1})),\vartheta(\Xi(\varpi_{2q},\varpi_{2q+1}))\right),$$

where

$$\begin{aligned}\Xi(\varpi_{2q},\varpi_{2q+1})&=\max\Big\{\ell_\lambda(\zeta\varpi_{2q},J\varpi_{2q+1})+\left|\ell_\lambda(\zeta\varpi_{2q},\Gamma\varpi_{2q})-\ell_\lambda(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})\right|;\\&\qquad\ell_\lambda(\zeta\varpi_{2q},\Gamma\varpi_{2q})+\left|\ell_\lambda(\zeta\varpi_{2q},J\varpi_{2q+1})-\ell_\lambda(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})\right|;\\&\qquad\ell_\lambda(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})+\left|\ell_\lambda(\zeta\varpi_{2q},J\varpi_{2q+1})-\ell_\lambda(\zeta\varpi_{2q},\Gamma\varpi_{2q})\right|;\\&\qquad\frac{1}{2}\left[\frac{\ell_{2\lambda}(\zeta\varpi_{2q},\Upsilon\varpi_{2q+1})+\ell_{2\lambda}(J\varpi_{2q+1},\Gamma\varpi_{2q})}{\kappa}+\left|\ell_\lambda(\zeta\varpi_{2q},\Gamma\varpi_{2q})-\ell_\lambda(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})\right|\right]\Big\}\\&=\max\Big\{\ell_\lambda(\xi_{2q-1},\xi_{2q})+\left|\ell_\lambda(\xi_{2q-1},\xi_{2q})-\ell_\lambda(\xi_{2q},\xi_{2q+1})\right|;\\&\qquad\ell_\lambda(\xi_{2q-1},\xi_{2q})+\left|\ell_\lambda(\xi_{2q-1},\xi_{2q})-\ell_\lambda(\xi_{2q},\xi_{2q+1})\right|;\\&\qquad\ell_\lambda(\xi_{2q},\xi_{2q+1})+\left|\ell_\lambda(\xi_{2q-1},\xi_{2q})-\ell_\lambda(\xi_{2q-1},\xi_{2q})\right|;\\&\qquad\frac{1}{2}\left[\frac{\ell_{2\lambda}(\xi_{2q-1},\xi_{2q+1})+\ell_{2\lambda}(\xi_{2q},\xi_{2q})}{\kappa}+\left|\ell_\lambda(\xi_{2q-1},\xi_{2q})-\ell_\lambda(\xi_{2q},\xi_{2q+1})\right|\right]\Big\}.\end{aligned}$$

Now, if we assume that $\sigma_q=\ell_\lambda(\xi_{q-1},\xi_q)$ and use the triangle inequality

$$\ell_{2\lambda}(\xi_{2q-1},\xi_{2q+1})\leq\kappa\left[\ell_\lambda(\xi_{2q-1},\xi_{2q})+\ell_\lambda(\xi_{2q},\xi_{2q+1})\right],$$

we get that

$$\Sigma\left(\kappa^3\sigma_{2q+1}\right)\leq\mathscr{G}\left(\Sigma(\Xi(\varpi_{2q},\varpi_{2q+1})),\vartheta(\Xi(\varpi_{2q},\varpi_{2q+1}))\right), \tag{2.2}$$

where

$$\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)\leq\max\left\{\sigma_{2q}+\left|\sigma_{2q}-\sigma_{2q+1}\right|;\ \sigma_{2q}+\left|\sigma_{2q}-\sigma_{2q+1}\right|;\right.$$
$$\left.\sigma_{2q+1}+\left|\sigma_{2q}-\sigma_{2q}\right|;\ \frac{1}{2}\left[\frac{\kappa\left(\sigma_{2q}+\sigma_{2q+1}\right)}{\kappa}+\left|\sigma_{2q}-\sigma_{2q+1}\right|\right]\right\}.$$

If $\sigma_{2q+1}\geq\sigma_{2q}$, we achieve

$$\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)\leq\max\left\{\sigma_{2q+1},\frac{\sigma_{2q}+\sigma_{2q+1}+\sigma_{2q+1}-\sigma_{2q}}{2}\right\}=\sigma_{2q+1}.$$

From the above, it is concluded that

$$\Sigma\left(\sigma_{2q+1}\right)\leq\Sigma\left(\kappa^{3}\sigma_{2q+1}\right)\leq\mathscr{G}\left(\Sigma\left(\sigma_{2q+1}\right),\vartheta\left(\sigma_{2q+1}\right)\right)\leq\Sigma\left(\sigma_{2q+1}\right),$$

which means

$$\mathscr{G}\left(\Sigma\left(\sigma_{2q+1}\right),\vartheta\left(\sigma_{2q+1}\right)\right)=\Sigma\left(\sigma_{2q+1}\right).$$

From $(\mathscr{G}_2)$, either $\Sigma\left(\sigma_{2q+1}\right)=0$ or $\vartheta\left(\sigma_{2q+1}\right)=0$. Nevertheless, a contradictory situation arises in both cases due to our assumption. For $\sigma_{2q+1}<\sigma_{2q}$, we have $\left|\sigma_{2q}-\sigma_{2q+1}\right|=\sigma_{2q}-\sigma_{2q+1}$ and

$$\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)\leq\max\left\{2\sigma_{2q}-\sigma_{2q+1},\sigma_{2q+1}\frac{\sigma_{2q}+\sigma_{2q+1}+\sigma_{2q}-\sigma_{2q+1}}{2}\right\}.$$

As $\left[2\sigma_{2q}-\sigma_{2q+1}>\sigma_{2q}>\sigma_{2q+1}\right]$, we yield that

$$\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)=\max\left\{2\sigma_{2q}-\sigma_{2q+1},\sigma_{2q+1},\sigma_{2q}\right\}=2\sigma_{2q}-\sigma_{2q+1}.\tag{2.3}$$

Moreover, by repeating similar steps, we acquire that $\sigma_{2q}<\sigma_{2q-1}$. Then, it ensures $\sigma_{q+1}<\sigma_{q}$. So, we say $\left\{\sigma_{q}\right\}=\left\{\ell_{\lambda}\left(\xi_{q-1},\xi_{q}\right)\right\}$ is a non-increasing sequence of non-negative real numbers. Thereby, there exists $\tau\geq0$ such that $\lim\limits_{q\to\infty}\sigma_{q}=\tau$ for all $\lambda>0$. Now, we aim to show $\tau=0$.

By using $(\mathscr{G}_1)$ and (2.3), contemplating the inequality (2.2), we get

$$\Sigma\left(\sigma_{2q+1}\right)\leq\Sigma\left(\kappa^{3}\sigma_{2q+1}\right)\leq\mathscr{G}\left(\Sigma\left(2\sigma_{2q}-\sigma_{2q+1}\right),\vartheta\left(2\sigma_{2q}-\sigma_{2q+1}\right)\right)\leq\Sigma\left(2\sigma_{2q}-\sigma_{2q+1}\right).$$

If we take the limit in the above inequality, we have

$$\Sigma\left(\tau\right)\leq\mathscr{G}\left(\Sigma\left(\tau\right),\vartheta\left(\tau\right)\right)\leq\Sigma\left(\tau\right),$$

and consequently, $\mathscr{G}\left(\Sigma\left(\tau\right),\vartheta\left(\tau\right)\right)=\Sigma\left(\tau\right)$. Then, from $(\mathscr{G}_2)$, either $\Sigma\left(\tau\right)=0$ or $\vartheta\left(\tau\right)=0$. This implies that $\tau=0$, i.e., for all $\lambda>0$

$$\ell_{\lambda}\left(\xi_{q-1},\xi_{q}\right)\to0\,,\quad\left(q\to\infty\right).\tag{2.4}$$

We need to show that $\left\{\xi_{q}\right\}$ is an $\ell-$Cauchy sequence. It is adequate to demonstrate that $\left\{\xi_{2q}\right\}$ is an $\ell-$Cauchy sequence. Presume on, by contrast; we will find $\varepsilon>0$ and also form two sequences $\{a_i\}$ and $\{b_i\}$ of positive integers fulfilling $b_i>a_i\geq i$ such that b_i is the smallest index for which

$$\ell_{\lambda}\left(\xi_{2a_i},\xi_{2b_i}\right)\geq\varepsilon\quad\text{and}\quad\ell_{\lambda}\left(\xi_{2a_i},\xi_{2b_i-2}\right)<\varepsilon,\quad\text{for all }\lambda>0.\tag{2.5}$$

From (2.5), we gain

$$\varepsilon\leq\ell_{\lambda}\left(\xi_{2a_i},\xi_{2b_i}\right)\leq\kappa\ell_{\frac{\lambda}{2}}\left(\xi_{2a_i},\xi_{2b_i+1}\right)+\kappa\ell_{\frac{\lambda}{2}}\left(\xi_{2b_i+1},\xi_{2b_i}\right).$$

Taking the limit superior in the above expression as $i\to\infty$ and by utilizing (2.4), we attain

$$\limsup_{q\to\infty}\ell_{\lambda}\left(\xi_{2a_i},\xi_{2b_i+1}\right)\geq\frac{\varepsilon}{\kappa}\,,\quad\text{for all }\lambda>0.\tag{2.6}$$

From (ℓ_3), we acquire

$$\ell_{\lambda}\left(\xi_{2a_i-1},\xi_{2b_i}\right)\leq\kappa\ell_{\frac{\lambda}{2}}\left(\xi_{2a_i-1},\xi_{2a_i}\right)+\kappa^{2}\ell_{\frac{\lambda}{2}}\left(\xi_{2a_i},\xi_{2b_i-2}\right)+\kappa^{3}\ell_{\frac{\lambda}{4}}\left(\xi_{2b_i-2},\xi_{2b_i-1}\right)+\kappa^{3}\ell_{\frac{\lambda}{4}}\left(\xi_{2b_i-1},\xi_{2b_i}\right).$$

Again, by taking the limit as $i\to\infty$ and taking the expressions (2.4) and (2.5) into account, the above inequality provides that

$$\limsup_{k\to\infty}\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)\le\kappa^2\varepsilon\,,\quad\text{for all }\lambda>0. \tag{2.7}$$

Thereby, by (2.1), we procure

$$\Sigma\left(\kappa^3\ell_\lambda\left(\Gamma\varpi_{2a_i},\Upsilon\varpi_{2b_i+1}\right)\right)\le\mathscr{G}\left(\Sigma\left(\Xi\left(\varpi_{2a_i},\varpi_{2b_i+1}\right)\right),\vartheta\left(\Xi\left(\varpi_{2a_i},\varpi_{2b_i+1}\right)\right)\right), \tag{2.8}$$

where

$$\begin{aligned}\Xi\left(\varpi_{2a_i},\varpi_{2b_i+1}\right)=\max\Big\{&\ell_\lambda\left(\zeta\varpi_{2a_i},J\varpi_{2b_i+1}\right)+\left|\ell_\lambda\left(\zeta\varpi_{2a_i},\Gamma\varpi_{2a_i}\right)-\ell_\lambda\left(J\varpi_{2b_i+1},\Upsilon\varpi_{2b_i+1}\right)\right|;\\
&\ell_\lambda\left(\zeta\varpi_{2a_i},\Gamma\varpi_{2a_i}\right)+\left|\ell_\lambda\left(\zeta\varpi_{2a_i},J\varpi_{2b_i+1}\right)-\ell_\lambda\left(J\varpi_{2b_i+1},\Upsilon\varpi_{2b_i+1}\right)\right|;\\
&\ell_\lambda\left(J\varpi_{2b_i+1},\Upsilon\varpi_{2b_i+1}\right)+\left|\ell_\lambda\left(\zeta\varpi_{2a_i},J\varpi_{2b_i+1}\right)-\ell_\lambda\left(\zeta\varpi_{2a_i},\Gamma\varpi_{2a_i}\right)\right|;\\
&\tfrac{1}{2}\left[\frac{\ell_{2\lambda}\left(\zeta\varpi_{2a_i},\Upsilon\varpi_{2b_i+1}\right)+\ell_{2\lambda}\left(J\varpi_{2b_i+1},\Gamma\varpi_{2a_i}\right)}{\kappa}+\left|\ell_\lambda\left(\zeta\varpi_{2a_i},\Gamma\varpi_{2a_i}\right)-\ell_\lambda\left(J\varpi_{2b_i+1},\Upsilon\varpi_{2b_i+1}\right)\right|\right]\Big\}\\
=\max\Big\{&\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)+\left|\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)-\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i+1}\right)\right|;\\
&\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)+\left|\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)-\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i+1}\right)\right|;\\
&\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i+1}\right)+\left|\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)-\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)\right|;\\
&\tfrac{1}{2}\left[\frac{\ell_{2\lambda}\left(\xi_{2a_i-1},\xi_{2b_i+1}\right)+\ell_{2\lambda}\left(\xi_{2b_i},\xi_{2a_i}\right)}{\kappa}+\left|\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)-\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i+1}\right)\right|\right]\Big\}.\end{aligned}$$

Also, by using (ℓ_3), we derive

$$\begin{aligned}\ell_{2\lambda}\left(\xi_{2a_i-1},\xi_{2b_i+1}\right)&\le\kappa\left[\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)+\ell_\lambda\left(\xi_{2b_i},\xi_{2b_i+1}\right)\right],\\
\ell_{2\lambda}\left(\xi_{2b_i},\xi_{2a_i}\right)&\le\kappa\left[\ell_\lambda\left(\xi_{2b_i},\xi_{2a_i-1}\right)+\ell_\lambda\left(\xi_{2a_i-1},\xi_{2a_i}\right)\right].\end{aligned} \tag{2.9}$$

Consequently, we combine the inequalities (2.8) and (2.9), we deduce $\Xi\left(\varpi_{2a_i},\varpi_{2b_i+1}\right)\le\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)$, thence, we get

$$\Sigma\left(\kappa^3\ell_\lambda\left(\xi_{2a_i},\xi_{2b_i+1}\right)\right)\le\mathscr{G}\left(\Sigma\left(\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)\right),\vartheta\left(\ell_\lambda\left(\xi_{2a_i-1},\xi_{2b_i}\right)\right)\right). \tag{2.10}$$

Now, by employing (2.6), (2.7), and $(\mathscr{G}_1)$, if we take the limit as $(i\to\infty)$ in (2.10), then we achieve

$$\Sigma\left(\kappa^3\frac{\varepsilon}{\kappa}\right)\le\mathscr{G}\left(\Sigma\left(\kappa^2\varepsilon\right),\vartheta\left(\kappa^2\varepsilon\right)\right)\le\Sigma\left(\kappa^2\varepsilon\right)$$

which stands for

$$\mathscr{G}\left(\Sigma\left(\kappa^2\varepsilon\right),\vartheta\left(\kappa^2\varepsilon\right)\right)=\Sigma\left(\kappa^2\varepsilon\right),$$

hence, it must be either $\Sigma\left(\kappa^2\varepsilon\right)=0$ or $\vartheta\left(\kappa^2\varepsilon\right)=0$. As $\kappa\ge1$ and $\varepsilon>0$, it is a contradiction, that is, $\left\{\xi_{2q}\right\}$ is an $\ell-$Cauchy sequence. Thus, $\left\{\xi_q\right\}$ is an $\ell-$Cauchy sequence in $\mathbf{S}_\ell^*$. Since $\mathbf{S}_\ell^*$ is an $\ell-$complete MbMS, there exists $c\in\mathbf{S}_\ell^*$ such that

$$\lim_{q\to\infty}\xi_q=c. \tag{2.11}$$

Now, we aim to show that $\Gamma c=\Upsilon c=Jc=\zeta c=c$. Primarily, we prove that $\Gamma c=\zeta c=c$, that is, c is a common fixed point for the maps Γ and ζ. The following statements are obvious.

$$\lim_{q\to\infty}\xi_{2q}=\lim_{q\to\infty}\Gamma\varpi_{2q}=\lim_{q\to\infty}J\varpi_{2q+1}=c,$$

$$\lim_{q\to\infty}\xi_{2q+1}=\lim_{q\to\infty}\Upsilon\varpi_{2q+1}=\lim_{q\to\infty}\zeta\varpi_{2q+2}=c.$$

Considering the hypothesis, let $\zeta\left(\mathbf{S}_\ell^*\right)$ be a closed subset of $\mathbf{S}_\ell^*$, there exists $u\in\mathbf{S}_\ell^*$ such that $c=\zeta u$. We claim that $\Gamma u=c$. Let us replace ϖ and ξ in expression (2.1) with u and ϖ_{2q+1}, respectively.

$$\Sigma\left(\kappa^3\ell_\lambda\left(\Gamma u,\Upsilon\varpi_{2q+1}\right)\right)\le\mathscr{G}\left(\Sigma\left(\Xi\left(u,\varpi_{2q+1}\right)\right),\vartheta\left(\Xi\left(u,\varpi_{2q+1}\right)\right)\right),$$

where

$$\begin{aligned}\Xi\left(u,\varpi_{2q+1}\right)=\max\Big\{&\ell_\lambda\left(\zeta u,J\varpi_{2q+1}\right)+\left|\ell_\lambda\left(\zeta u,\Gamma u\right)-\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)\right|;\\
&\ell_\lambda\left(\zeta u,\Gamma u\right)+\left|\ell_\lambda\left(\zeta u,J\varpi_{2q+1}\right)-\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)\right|;\\
&\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)+\left|\ell_\lambda\left(\zeta u,J\varpi_{2q+1}\right)-\ell_\lambda\left(\zeta u,\Gamma u\right)\right|;\\
&\tfrac{1}{2}\left[\frac{\ell_{2\lambda}\left(\zeta u,\Upsilon\varpi_{2q+1}\right)+\ell_{2\lambda}\left(J\varpi_{2q+1},\Gamma u\right)}{\kappa}+\left|\ell_\lambda\left(\zeta u,\Gamma u\right)-\ell_\lambda\left(J\varpi_{2q+1},\Upsilon\varpi_{2q+1}\right)\right|\right]\Big\}\\
=\max\Big\{&\ell_\lambda\left(c,\xi_{2q}\right)+\left|\ell_\lambda\left(c,\Gamma u\right)-\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)\right|;\\
&\ell_\lambda\left(c,\Gamma u\right)+\left|\ell_\lambda\left(c,\xi_{2q}\right)-\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)\right|;\\
&\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)+\left|\ell_\lambda\left(c,\xi_{2q}\right)-\ell_\lambda\left(c,\Gamma u\right)\right|;\\
&\tfrac{1}{2}\left[\frac{\ell_{2\lambda}\left(c,\xi_{2q+1}\right)+\ell_{2\lambda}\left(\xi_{2q},\Gamma u\right)}{\kappa}+\left|\ell_\lambda\left(c,\Gamma u\right)-\ell_\lambda\left(\xi_{2q},\xi_{2q+1}\right)\right|\right]\Big\}.\end{aligned}$$

Also, by using (2.11) and $(\mathscr{G}_1)$, if we take the limit as $q \to \infty$ in the above, and note that

$$\ell_{2\lambda}\left(\xi_{2q}, \Gamma u\right) \leq \kappa\left[\ell_\lambda\left(\xi_{2q}, \xi_{2q+1}\right) + \ell_\lambda\left(\xi_{2q+1}, \Gamma u\right)\right],$$

we conclude $\Xi\left(u, \varpi_{2q+1}\right) \leq \ell_\lambda\left(c, \Gamma u\right)$. Hence, we obtain

$$\Sigma\left(\ell_\lambda\left(\Gamma u, c\right)\right) \leq \Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma u, c\right)\right) \leq \mathscr{G}\left(\Sigma\left(\ell_\lambda\left(\Gamma u, c\right)\right), \vartheta\left(\ell_\lambda\left(\Gamma u, c\right)\right)\right) \leq \Sigma\left(\ell_\lambda\left(\Gamma u, c\right)\right),$$

which implies the following

$$\mathscr{G}\left(\Sigma\left(\ell_\lambda\left(\Gamma u, c\right)\right), \vartheta\left(\ell_\lambda\left(\Gamma u, c\right)\right)\right) = \Sigma\left(\ell_\lambda\left(\Gamma u, c\right)\right).$$

Then, from $(\mathscr{G}_2)$, either $\Sigma\left(\ell_\lambda\left(\Gamma u, c\right)\right) = 0$ or $\vartheta\left(\ell_\lambda\left(\Gamma u, c\right)\right) = 0$, which yields $\ell_\lambda\left(\Gamma u, c\right) = 0 \Leftrightarrow \Gamma u = c$. Therefore, $\Gamma u = \zeta u = c$. Since the mappings Γ and ζ are weakly compatible, we have $\Gamma c = \Gamma\zeta u = \zeta\Gamma u = \zeta c$. Next, we claim that $\Gamma c = c$. Again, from (2.1), we get

$$\Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma c, \Upsilon\varpi_{2q+1}\right)\right) \leq \mathscr{G}\left(\Sigma\left(\Xi\left(c, \varpi_{2q+1}\right)\right), \vartheta\left(\Xi\left(c, \varpi_{2q+1}\right)\right)\right),$$

where

$$\begin{aligned}
\Xi\left(c, \varpi_{2q+1}\right) = \max\Big\{ & \ell_\lambda\left(\zeta c, J\varpi_{2q+1}\right) + \left|\ell_\lambda\left(\zeta c, \Gamma c\right) - \ell_\lambda\left(J\varpi_{2q+1}, \Upsilon\varpi_{2q+1}\right)\right|; \\
& \ell_\lambda\left(\zeta c, \Gamma c\right) + \left|\ell_\lambda\left(\zeta c, J\varpi_{2q+1}\right) - \ell_\lambda\left(J\varpi_{2q+1}, \Upsilon\varpi_{2q+1}\right)\right|; \\
& \ell_\lambda\left(J\varpi_{2q+1}, \Upsilon\varpi_{2q+1}\right) + \left|\ell_\lambda\left(\zeta c, J\varpi_{2q+1}\right) - \ell_\lambda\left(\zeta c, \Gamma c\right)\right|; \\
& \frac{1}{2}\left[\frac{\ell_{2\lambda}\left(\zeta c, \Upsilon\varpi_{2q+1}\right) + \ell_{2\lambda}\left(J\varpi_{2q+1}, \Gamma c\right)}{\kappa} + \left|\ell_\lambda\left(\zeta c, \Gamma c\right) - \ell_\lambda\left(J\varpi_{2q+1}, \Upsilon\varpi_{2q+1}\right)\right|\right]\Big\} \\
= \max\Big\{ & \ell_\lambda\left(\Gamma c, \xi_{2q}\right) + \left|\ell_\lambda\left(\zeta c, \Gamma c\right) - \ell_\lambda\left(\xi_{2q}, \xi_{2q+1}\right)\right|; \\
& \ell_\lambda\left(\zeta c, \Gamma c\right) + \left|\ell_\lambda\left(\Gamma c, \xi_{2q}\right) - \ell_\lambda\left(\xi_{2q}, \xi_{2q+1}\right)\right|; \\
& \ell_\lambda\left(\xi_{2q}, \xi_{2q+1}\right) + \left|\ell_\lambda\left(\Gamma c, \xi_{2q}\right) - \ell_\lambda\left(\zeta c, \Gamma c\right)\right|; \\
& \frac{1}{2}\left[\frac{\ell_{2\lambda}\left(\Gamma c, \xi_{2q+1}\right) + \ell_{2\lambda}\left(\xi_{2q}, \Gamma c\right)}{\kappa} + \left|\ell_\lambda\left(\zeta c, \Gamma c\right) - \ell_\lambda\left(\xi_{2q}, \xi_{2q+1}\right)\right|\right]\Big\}.
\end{aligned}$$

Likewise, by utilizing (2.11), $(\mathscr{G}_1)$ and noting

$$\ell_{2\lambda}\left(\xi_{2q}, \Gamma c\right) \leq \kappa\left[\ell_\lambda\left(\xi_{2q}, \xi_{2q+1}\right) + \ell_\lambda\left(\xi_{2q+1}, \Gamma c\right)\right],$$

taking the limit as $q \to \infty$ in the above, we get

$$\Xi\left(c, \varpi_{2q+1}\right) \leq \ell_\lambda\left(\Gamma c, c\right)$$

and, so

$$\Sigma\left(\ell_\lambda\left(\Gamma c, c\right)\right) \leq \Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma c, c\right)\right) \leq \mathscr{G}\left(\Sigma\left(\Xi\left(\Gamma c, c\right)\right), \vartheta\left(\Xi\left(\Gamma c, c\right)\right)\right) \leq \Sigma\left(\Xi\left(\Gamma c, c\right)\right).$$

Thus, we have

$$\mathscr{G}\left(\Sigma\left(\ell_\lambda\left(\Gamma c, c\right)\right), \vartheta\left(\ell_\lambda\left(\Gamma c, c\right)\right)\right) = \Sigma\left(\ell_\lambda\left(\Gamma c, c\right)\right).$$

By $(\mathscr{G}_2)$, either $\Sigma\left(\ell_\lambda\left(\Gamma c, c\right)\right) = 0$ or $\vartheta\left(\ell_\lambda\left(\Gamma c, c\right)\right) = 0$. This shows that $\Gamma c = c$. The next step is to prove that c is the fixed point of Υ and J. Because $\Gamma\left(\mathbf{S}_\ell^*\right) \subset J\left(\mathbf{S}_\ell^*\right)$, there is an element v in $\mathbf{S}_\ell^*$ such that $\Gamma c = Jv$. Then, $\Gamma c = Jv = \zeta c = c$. We claim that $\Upsilon v = c$. From (2.1)

$$\Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma c, \Upsilon v\right)\right) \leq \mathscr{G}\left(\Sigma\left(\Xi\left(c, v\right)\right), \vartheta\left(\Xi\left(c, v\right)\right)\right),$$

where

$$\begin{aligned}
\Xi(c, v) = \max\Big\{ & \ell_\lambda\left(\zeta c, Jv\right) + \left|\ell_\lambda\left(\zeta c, \Gamma c\right) - \ell_\lambda\left(Jv, \Upsilon v\right)\right|;\ \ell_\lambda\left(\zeta c, \Gamma c\right) + \left|\ell_\lambda\left(\zeta c, Jv\right) - \ell_\lambda\left(Jv, \Upsilon v\right)\right|; \\
& \ell_\lambda\left(Jv, \Upsilon v\right) + \left|\ell_\lambda\left(\zeta c, Jv\right) - \ell_\lambda\left(\zeta c, \Gamma c\right)\right|;\ \frac{1}{2}\left[\frac{\ell_{2\lambda}\left(\zeta c, \Upsilon v\right) + \ell_{2\lambda}\left(Jv, \Gamma c\right)}{\kappa} + \left|\ell_\lambda\left(\zeta c, \Gamma c\right) - \ell_\lambda\left(Jv, \Upsilon v\right)\right|\right]\Big\} \\
= \max\Big\{ & \ell_\lambda\left(\zeta c, c\right) + \left|\ell_\lambda\left(\zeta c, c\right) - \ell_\lambda\left(c, \Upsilon v\right)\right|;\ \ell_\lambda\left(\zeta c, c\right) + \left|\ell_\lambda\left(\zeta c, c\right) - \ell_\lambda\left(c, \Upsilon v\right)\right|; \\
& \ell_\lambda\left(c, \Upsilon v\right) + \left|\ell_\lambda\left(\zeta c, c\right) - \ell_\lambda\left(\zeta c, c\right)\right|;\ \frac{1}{2}\left[\frac{\ell_{2\lambda}\left(c, \Upsilon v\right) + \ell_{2\lambda}\left(c, \Gamma c\right)}{\kappa} + \left|\ell_\lambda\left(c, \Gamma c\right) - \ell_\lambda\left(c, \Upsilon v\right)\right|\right]\Big\}.
\end{aligned}$$

Note that $\ell_{2\lambda}\left(c, \Upsilon v\right) \leq \kappa\left[\ell_\lambda\left(c, \xi_{2q}\right) + \ell_\lambda\left(\xi_{2q}, \Upsilon v\right)\right]$, we get $\Xi\left(c, v\right) \leq \ell_\lambda\left(c, \Upsilon v\right)$. Then, by using $(\mathscr{G}_1)$, we have

$$\begin{aligned}
\Sigma\left(\ell_\lambda\left(c, \Upsilon v\right)\right) \leq \Sigma\left(\kappa^3 \ell_\lambda\left(c, \Upsilon v\right)\right) &\leq \mathscr{G}\left(\Sigma\left(\ell_\lambda\left(c, \Upsilon v\right)\right), \vartheta\left(\ell_\lambda\left(c, \Upsilon v\right)\right)\right) \\
&\leq \Sigma\left(\ell_\lambda\left(c, \Upsilon v\right)\right),
\end{aligned}$$

which implies

$$\mathscr{G}\left(\Sigma\left(\ell_{\lambda}\left(c,\Upsilon v\right)\right),\vartheta\left(\ell_{\lambda}\left(c,\Upsilon v\right)\right)\right)=\Sigma\left(\ell_{\lambda}\left(c,\Upsilon v\right)\right).$$

So, similar to the above, by using $(\mathscr{G}_2)$, it is clear that $\Upsilon v = c$. By the weak compatibility of the mappings Υ and J, we achieve that $\Upsilon c = \Upsilon J v = JJv = Jc$.
Finally, we demand that $\Upsilon c = c$. Using from (2.1) we achieve

$$\Sigma\left(\kappa^{3}\ell_{\lambda}\left(\Gamma c,\Upsilon c\right)\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(c,c\right)\right),\vartheta\left(\Xi\left(c,c\right)\right)\right),$$

where

$$\begin{aligned}\Xi\left(c,c\right)=\max\{&\ell_{\lambda}\left(\zeta c,Jc\right)+\left|\ell_{\lambda}\left(\zeta c,\Gamma c\right)-\ell_{\lambda}\left(Jc,\Upsilon c\right)\right|;\ \ell_{\lambda}\left(\zeta c,\Gamma c\right)+\left|\ell_{\lambda}\left(\zeta c,Jc\right)-\ell_{\lambda}\left(Jc,\Upsilon c\right)\right|;\\ &\ell_{\lambda}\left(Jc,\Upsilon c\right)+\left|\ell_{\lambda}\left(\zeta c,Jc\right)-\ell_{\lambda}\left(\zeta c,\Gamma c\right)\right|;\ \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}\left(\zeta c,\Upsilon c\right)+\ell_{2\lambda}\left(Jc,\Gamma c\right)}{\kappa}+\left|\ell_{\lambda}\left(\zeta c,\Gamma c\right)-\ell_{\lambda}\left(Jc,\Upsilon c\right)\right|\right]\Big\}\\ =\max\{&\ell_{\lambda}\left(c,\Upsilon c\right)+\left|\ell_{\lambda}\left(c,c\right)-\ell_{\lambda}\left(\Upsilon c,\Upsilon c\right)\right|;\ \ell_{\lambda}\left(\zeta c,c\right)+\left|\ell_{\lambda}\left(\zeta c,c\right)-\ell_{\lambda}\left(c,\Upsilon c\right)\right|;\\ &\ell_{\lambda}\left(c,\Upsilon v\right)+\left|\ell_{\lambda}\left(\zeta c,c\right)-\ell_{\lambda}\left(\zeta c,c\right)\right|;\ \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}\left(c,\Upsilon c\right)+\ell_{2\lambda}\left(\Upsilon c,c\right)}{\kappa}+\left|\ell_{\lambda}\left(c,c\right)-\ell_{\lambda}\left(\Upsilon c,\Upsilon c\right)\right|\right]\Big\}.\end{aligned}$$

Hence, this implies

$$\Xi\left(c,c\right)=\max\left\{\ell_{\lambda}\left(\Upsilon c,c\right),\frac{\ell_{2\lambda}\left(\Upsilon c,c\right)}{\kappa}\right\}=\ell_{\lambda}\left(\Upsilon c,c\right)$$

and from $(\mathscr{G}_1)$, we obtain

$$\Sigma\left(\ell_{\lambda}\left(c,\Upsilon c\right)\right)\leq\Sigma\left(\kappa^{3}\ell_{\lambda}\left(\Upsilon c,c\right)\right)\leq\mathscr{G}\left(\Sigma\left(\ell_{\lambda}\left(\Upsilon c,c\right)\right),\vartheta\left(\ell_{\lambda}\left(\Upsilon c,c\right)\right)\right)\leq\Sigma\left(\ell_{\lambda}\left(\Upsilon c,c\right)\right),$$

which yields

$$\mathscr{G}\left(\Sigma\left(\ell_{\lambda}\left(c,\Upsilon c\right)\right),\vartheta\left(\ell_{\lambda}\left(c,\Upsilon c\right)\right)\right)=\Sigma\left(\ell_{\lambda}\left(c,\Upsilon c\right)\right).$$

Then, from$(\mathscr{G}_2)$, one can conclude $\Gamma c = \Upsilon c = \zeta c = Jc = c$. Since $\Gamma\left(\mathbf{S}_{\ell}^{*}\right)\subset J\left(\mathbf{S}_{\ell}^{*}\right)$ and $\Upsilon\left(\mathbf{S}_{\ell}^{*}\right)\subset\zeta\left(\mathbf{S}_{\ell}^{*}\right)$, similar calculations can be done for the case in which $J\left(\mathbf{S}_{\ell}^{*}\right)$(or $\Gamma\left(\mathbf{S}_{\ell}^{*}\right)$, $\Upsilon\left(\mathbf{S}_{\ell}^{*}\right)$) is closed.
In conclusion, for the uniqueness of the common fixed point of Γ,Υ, J, and ζ, suppose that c^* is another common fixed point of our mappings, that is, $c^* = \Gamma c^* = \Upsilon c^* = Jc^* = \zeta c^*$ such that $c \neq c^*$. Then, from (2.1), we have

$$\Sigma\left(\kappa^{3}\ell_{\lambda}\left(\Gamma c,\Upsilon c^{*}\right)\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(c,c^{*}\right)\right),\vartheta\left(\Xi\left(c,c^{*}\right)\right)\right),$$

where

$$\begin{aligned}\Xi\left(c,c^{*}\right)=\max\{&\ell_{\lambda}\left(\zeta c,Jc^{*}\right)+\left|\ell_{\lambda}\left(\zeta c,\Gamma c\right)-\ell_{\lambda}\left(Jc^{*},\Upsilon c^{*}\right)\right|;\ \ell_{\lambda}\left(\zeta c,\Gamma c\right)+\left|\ell_{\lambda}\left(\zeta c,Jc^{*}\right)-\ell_{\lambda}\left(Jc^{*},\Upsilon c^{*}\right)\right|;\\ &\ell_{\lambda}\left(Jc^{*},\Upsilon c^{*}\right)+\left|\ell_{\lambda}\left(\zeta c,Jc^{*}\right)-\ell_{\lambda}\left(\zeta c,\Gamma c\right)\right|;\ \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}\left(\zeta c,\Upsilon c^{*}\right)+\ell_{2\lambda}\left(Jc^{*},\Gamma c\right)}{\kappa}+\left|\ell_{\lambda}\left(\zeta c,\Gamma c\right)-\ell_{\lambda}\left(Jc^{*},\Upsilon c^{*}\right)\right|\right]\Big\}\\ =\max\{&\ell_{\lambda}\left(c,c^{*}\right)+\left|\ell_{\lambda}\left(c,c\right)-\ell_{\lambda}\left(c^{*},c^{*}\right)\right|;\ \ell_{\lambda}\left(c,c\right)+\left|\ell_{\lambda}\left(c,c^{*}\right)-\ell_{\lambda}\left(c^{*},c^{*}\right)\right|;\\ &\ell_{\lambda}\left(c^{*},c^{*}\right)+\left|\ell_{\lambda}\left(c,c^{*}\right)-\ell_{\lambda}\left(c,c\right)\right|;\ \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}\left(c,c^{*}\right)+\ell_{2\lambda}\left(c^{*},c\right)}{\kappa}+\left|\ell_{\lambda}\left(c,c\right)-\ell_{\lambda}\left(c^{*},c^{*}\right)\right|\right]\Big\}\\ =\max&\left\{\ell_{\lambda}\left(c,c^{*}\right),\tfrac{\ell_{2\lambda}\left(c,c^{*}\right)}{\kappa}\right\}=\ell_{\lambda}\left(c,c^{*}\right).\end{aligned}$$

Thus, by $(\mathscr{G}_1)$, we get

$$\Sigma\left(\ell_{\lambda}\left(c,c^{*}\right)\right)\leq\Sigma\left(\kappa^{3}\ell_{\lambda}\left(c,c^{*}\right)\right)\leq\mathscr{G}\left(\Sigma\left(\ell_{\lambda}\left(c,c^{*}\right)\right),\vartheta\left(\ell_{\lambda}\left(c,c^{*}\right)\right)\right)\leq\Sigma\left(\ell_{\lambda}\left(c,c^{*}\right)\right),$$

which implies

$$\mathscr{G}\left(\Sigma\left(\ell_{\lambda}\left(c,c^{*}\right)\right),\vartheta\left(\ell_{\lambda}\left(c,c^{*}\right)\right)\right)=\Sigma\left(\ell_{\lambda}\left(c,c^{*}\right)\right).$$

Thereby, it is easy to show that $c = c^*$, which means that the common fixed point of Γ,Υ, J, and ζ is unique. □

Inferences drawn directly from the main result are presented below.

Corollary 2.3. *Let $\mathbf{S}_{\ell}^{*}$ be an ℓ-complete MbMS with constant $\kappa \geq 1$ and let $\Gamma,\Upsilon:\ \mathbf{S}_{\ell}^{*}\to\ \mathbf{S}_{\ell}^{*}$ be two self-mappings. Presume that the following statements are satisfied:*

(i) *There exist* $\mathscr{G} \in \mathscr{C}$, $\Sigma \in \Omega$, *and* $\vartheta \in \Pi$ *such that*

$$\Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma\varpi, \Upsilon\xi\right)\right) \leq \mathscr{G}\left(\Sigma\left(\Xi\left(\varpi,\xi\right)\right), \vartheta\left(\Xi\left(\varpi,\xi\right)\right)\right),$$

where

$$\begin{aligned}\Xi\left(\varpi,\xi\right) = \max\{&\ell_\lambda\left(\varpi,\xi\right) + \left|\ell_\lambda\left(\varpi,\Gamma\varpi\right) - \ell_\lambda\left(\xi,\Upsilon\xi\right)\right|; \ell_\lambda\left(\varpi,\Gamma\varpi\right) + \left|\ell_\lambda\left(\varpi,\xi\right) - \ell_\lambda\left(\xi,\Upsilon\xi\right)\right|;\\ &\ell_\lambda\left(\xi,\Upsilon\xi\right) + \left|\ell_\lambda\left(\varpi,\xi\right) - \ell_\lambda\left(\varpi,\Gamma\varpi\right)\right|; \frac{1}{2}\left[\frac{\ell_{2\lambda}\left(\varpi,\Upsilon\xi\right)+\ell_{2\lambda}\left(\xi,\Gamma\varpi\right)}{\kappa} + \left|\ell_\lambda\left(\varpi,\Gamma\varpi\right) - \ell_\lambda\left(\xi,\zeta\xi\right)\right|\right]\Big\},\end{aligned}$$

for all distinct $\varpi,\xi \in \mathbf{S}^*_\ell$ *and for all* $\lambda > 0$.

(ii) *The conditions* (M_1) *and* (M_2) *are provided.*

Then, Γ *and* Υ *admit a common unique fixed point in* $\mathbf{S}^*_\ell$.

If we distinguish $\Gamma = \Upsilon$ in Corollary 2.3, we yield the below one.

Corollary 2.4. *Let* $\mathbf{S}^*_\ell$ *be an* ℓ*-complete MbMS with coefficient* $\kappa \geq 1$ *and let* $\Gamma : \mathbf{S}^*_\ell \to \mathbf{S}^*_\ell$ *be a mapping. Presume that the following circumstances are satisfied:*

(i) *There exist* $\mathscr{G} \in \mathscr{C}$, $\Sigma \in \Omega$, *and* $\vartheta \in \Pi$ *such that*

$$\Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma\varpi, \Gamma\xi\right)\right) \leq \mathscr{G}\left(\Sigma\left(\Xi\left(\varpi,\xi\right)\right), \vartheta\left(\Xi\left(\varpi,\xi\right)\right)\right),$$

where

$$\begin{aligned}\Xi\left(\varpi,\xi\right) = \max\{&\ell_\lambda\left(\varpi,\xi\right) + \left|\ell_\lambda\left(\varpi,\Gamma\varpi\right) - \ell_\lambda\left(\xi,\Gamma\xi\right)\right|; \ell_\lambda\left(\varpi,\Gamma\varpi\right) + \left|\ell_\lambda\left(\varpi,\xi\right) - \ell_\lambda\left(\xi,\Gamma\xi\right)\right|;\\ &\ell_\lambda\left(\xi,\Gamma\xi\right) + \left|\ell_\lambda\left(\varpi,\xi\right) - \ell_\lambda\left(\varpi,\Gamma\varpi\right)\right|; \frac{1}{2}\left[\frac{\ell_{2\lambda}\left(\varpi,\Gamma\xi\right)+\ell_{2\lambda}\left(\xi,\Gamma\varpi\right)}{\kappa} + \left|\ell_\lambda\left(\varpi,\Gamma\varpi\right) - \ell_\lambda\left(\xi,\Gamma\xi\right)\right|\right]\Big\},\end{aligned} \tag{2.12}$$

for all distinct $\varpi,\xi \in \mathbf{S}^*_\ell$ *and for all* $\lambda > 0$.

(ii) *The conditions* (M_1) *and* (M_2) *are provided.*

Then Γ *possesses a unique fixed point in* $\mathbf{S}^*_\ell$.

Also, we get the following corollary if we choose $\mathscr{G}(p,q) = p - q$ for all $p,q \in [0,\infty)$ in Corollary 2.4.

Corollary 2.5. *Let* $\mathbf{S}^*_\ell$ *be an* ℓ*-complete MbMS with constant* $\kappa \geq 1$ *and let* $\Gamma : \mathbf{S}^*_\ell \to \mathbf{S}^*_\ell$ *be a mapping. Presume that the below statements hold:*

(i) *There exist* $\Sigma \in \Omega$ *and* $\vartheta \in \Pi$ *such that*

$$\Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma\varpi, \Gamma\xi\right)\right) \leq \Sigma\left(\Xi\left(\varpi,\xi\right)\right) - \vartheta\left(\Xi\left(\varpi,\xi\right)\right),$$

where $\Xi(\varpi,\xi)$ *is as in (2.12), for all distinct* $\varpi,\xi \in \mathbf{S}^*_\ell$ *and for all* $\lambda > 0$.

(ii) *The conditions* (M_1) *and* (M_2) *are provided.*

Then, Γ *holds a unique fixed point in* $\mathbf{S}^*_\ell$.

If we choose that $\mathscr{G}(p,q) = kp$, $k \in (0,1)$, for all $p \in [0,\infty)$, then we possess the below one.

Corollary 2.6. *Let* $\mathbf{S}^*_\ell$ *be an* ℓ*-complete MbMS with* $\kappa \geq 1$ *and let* $\Gamma : \mathbf{S}^*_\ell \to \mathbf{S}^*_\ell$ *be a mapping. Presume that the following conditions hold:*

(i) *There exist* $\Sigma \in \Omega$ *and* $k \in (0,1)$ *such that*

$$\Sigma\left(\kappa^3 \ell_\lambda\left(\Gamma\varpi, \Gamma\xi\right)\right) \leq k\Sigma\left(\Xi\left(\varpi,\xi\right)\right),$$

where $\Xi(\varpi,\xi)$ *is defined as in (2.12) and for all* $\varpi,\xi \in \mathbf{S}^*_\ell$ *and* $\lambda > 0$.

(ii) *The conditions* (M_1) *and* (M_2) *are provided.*

Thus, Γ *admits a unique fixed point in* $\mathbf{S}^*_\ell$.

3. An application to graph theory

Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with $\kappa \geq 1$, and let consider: $\Lambda = \left\{(\varpi,\varpi) : \varpi \in \mathbf{S}_\ell^*\right\}$, which denotes the diagonal of the Cartesian product $\mathbf{S}_\ell^* \times \mathbf{S}_\ell^*$. Also, let H be a directed graph such that

- $V(H)$: vertices coincide with $\mathbf{S}_\ell^*$,
- $B(H)$: edges contain all loops such that $\Lambda \subseteq B(H)$.

The pair $(V(H), B(H))$ could be displayed as the graph H. The following set

$$B\left(H^{-1}\right) = \{(\varpi,\xi) \in \mathbf{S}_\ell^* \times \mathbf{S}_\ell^* | \quad (\xi,\varpi) \in B(H)\},$$

is identified where H^{-1} is obtained in the graph H by reversing the direction of edges and called conversion of H. The graph H can be called an undirected graph, which denotes $\tilde{H}$, in case of the direction is ignored and so, we get

$$B\left(\tilde{H}\right) = B(H) \cup B\left(H^{-1}\right).$$

Let K be a subgraph of a graph H such that $V(K) \subseteq V(H)$ and $B(K) \subseteq B(H)$. If ϖ and ξ be vertices in a graph H, then a path from ϖ to ξ of length $j \in \mathbf{N}$ is a sequence (ϖ_j), which has $j+1$ distinct vertices such that $\varpi = \varpi_0, \varpi_1, ..., \varpi_j$ and $(\varpi_{i-1}, \varpi_i) \in B(H)$ for $i = 1, \ldots, j$.

Also, H is called a "connected graph" if there is a path between two vertices. Moreover, H is a "weakly connected graph", provided that $\tilde{H}$ is connected. For more detail about the graph theory, see [27]-[29].

Definition 3.1. *Let $\Gamma : \mathbf{S} \to \mathbf{S}$ be a mapping on a metric space $(\mathbf{S}, m)$. Presume that the followings hold:*

(i) $(\varpi,\xi) \in B(H) \Rightarrow (\Gamma\varpi,\Gamma\xi) \in B(H)$, for all $\varpi,\xi \in \mathbf{S}$,
(ii) $m(\Gamma\varpi,\Gamma\xi) \leq \mu m(\varpi,\xi)$, for all $(\varpi,\xi) \in B(H)$, *and* $\mu \in (0,1)$.

Then, Γ is called a Banach H-contraction mapping on $\mathbf{S}$.

Let $\mathbf{S}_\ell^*$ be an MbMS endowed with a graph H and $\Gamma : \mathbf{S}_\ell^* \to \mathbf{S}_\ell^*$. We set

$$\mathbf{S}_\ell^{*\Gamma} = \{\varpi \in \mathbf{S}_\ell^* | (\varpi,\Gamma\varpi) \in B(H)\}.$$

We present a new concept using the graph structure entitled $\mathscr{G}(\Sigma,\vartheta,\Xi)$-graphic contraction, as noted below.

Definition 3.2. *Let $\mathbf{S}_\ell^*$ be an MbMS endowed with a graph H. Presume that the following statements are provided:*

(i) *Γ preserves edges of H, i.e.,*

$$(\varpi,\xi) \in B(H) \quad \Rightarrow \quad (\Gamma\varpi,\Gamma\xi) \in B(H)$$

for all $\varpi\xi \in \mathbf{S}_\ell^*$.

(ii) *There exists a $\mathscr{G} \in \mathscr{C}$, $\Sigma \in \Omega$, and $\vartheta \in \Pi$ such that*

$$\Sigma\left(\kappa^3 \ell_\lambda(\Gamma\varpi,\Gamma\xi)\right) \leq \mathscr{G}(\Sigma(\Xi(\varpi,\xi)), \vartheta(\Xi(\varpi,\xi))), \tag{3.1}$$

where

$$\begin{aligned}\Xi(\varpi,\xi) = \max\{&\ell_\lambda(\varpi,\xi) + |\ell_\lambda(\varpi,\Gamma\varpi) - \ell_\lambda(\xi,\Gamma\xi)|; \ell_\lambda(\varpi,\Gamma\varpi) + |\ell_\lambda(\varpi,\xi) - \ell_\lambda(\xi,\Gamma\xi)|;\\ &\ell_\lambda(\xi,\Gamma\xi) + |\ell_\lambda(\varpi,\xi) - \ell_\lambda(\varpi,\Gamma\varpi)|; \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\varpi,\Gamma\xi)+\ell_{2\lambda}(\xi,\Gamma\varpi)}{\kappa} + |\ell_\lambda(\varpi,\Gamma\varpi) - \ell_\lambda(\xi,\Gamma\xi)|\right]\right\},\end{aligned}$$

for all $\varpi,\xi \in B(H)$, and $\lambda > 0$.

Then, Γ is called a $\mathscr{G}(\Sigma,\vartheta,\Xi)$-graphic contraction on $\mathbf{S}_\ell^*$.

Theorem 3.3. *Let $\mathbf{S}_\ell^*$ be an $\ell-$complete MbMS endowed with a graph H and $\Gamma : \mathbf{S}_\ell^* \to \mathbf{S}_\ell^*$ be a mapping. Assume that the circumstances below hold:*

(i) *There exists $\varpi_0 \in \mathbf{S}_\ell^{*\Gamma}$,*
(ii) *Γ is a $\mathscr{G}(\Sigma,\vartheta,\Xi)-$graphic contraction,*
(iii) *If $\{\varpi_k\}$ is a sequence in $\mathbf{S}_\ell^*$ such that $\lim_{k\to\infty} \ell_\lambda(\varpi_k,\varpi^*) = 0$ and $(\varpi_k,\varpi_{k+1}) \in B(H)$, then there exists a subsequence $\{\varpi_{k_s}\}$ of $\{\varpi_k\}$ such that $(\varpi_{k_s},\varpi^*) \in B(H)$,*
(iv) *H is a weakly connected graph.*

Then, by the conditions (M_1) and (M_2), Γ holds a unique fixed point in $\mathbf{S}_\ell^*$.

Proof. Define the sequence $\{\varpi_k\}$ in $\mathbf{S}_\ell^*$ by $\varpi_{k+1} = \Gamma\varpi_k$ for all $k \in \mathbf{N}$. From (i), since $\varpi_0 \in \mathbf{S}_\ell^{*\Gamma}$; we have $(\varpi_0, \Gamma\varpi_0) \in B(H)$. If we denote $\varpi_1 = \Gamma\varpi_0$, then

$$(\varpi_0, \Gamma\varpi_0) = (\varpi_0, \varpi_1) \in B(G).$$

Because Γ preserves the edges of H, the following expression is provided:

$$(\varpi_0, \varpi_1) \in B(H) \quad \Rightarrow \quad (\Gamma\varpi_0, \Gamma\varpi_1) \in B(H).$$

Continuing this way, we procure

$$(\varpi_k, \varpi_{k+1}) \in B(H).$$

So, from Corollary 2.4, we get $\{\varpi_k\}$ is an $\ell-$Cauchy sequence in $\mathbf{S}_\ell^*$. Because $\mathbf{S}_\ell^*$ is an $\ell-$complete space, there exists $\varpi^* \in \mathbf{S}_\ell^*$ such that

$$\lim_{k\to\infty} \ell_\lambda(\varpi_k, \varpi^*) = 0. \tag{3.2}$$

Now, we show that ϖ^* is a fixed point of Γ. Using (iii), we have $(\varpi_{k_s}, \varpi^*) \in B(H)$. Then, from (3.1), we obtain

$$\Sigma\left(\kappa^3 \ell_\lambda(\Gamma\varpi_{k_s}, \Gamma\varpi^*)\right) \le \mathscr{G}\left(\Sigma(B(\varpi_{k_s}, \varpi^*)), \vartheta(B(\varpi_{k_s}, \varpi^*))\right), \tag{3.3}$$

where

$$\begin{aligned}
\Xi(\varpi_{k_s}, \varpi^*) = \max\{&\ell_\lambda(\varpi_{k_s}, \varpi^*) + |\ell_\lambda(\varpi_{k_s}, \Gamma\varpi_{k_s}) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|;\\
&\ell_\lambda(\varpi_{k_s}, \Gamma\varpi_{k_s}) + |\ell_\lambda(\varpi_{k_s}, \varpi^*) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|;\\
&\ell_\lambda(\varpi^*, \Gamma\varpi^*) + |\ell_\lambda(\varpi_{k_s}, \varpi^*) - \ell_\lambda(\varpi_{k_s}, \Gamma\varpi_{k_s})|;\\
&\tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\varpi_{k_s}, \Gamma\varpi^*) + \ell_{2\lambda}(\varpi^*, \Gamma\varpi_{k_s})}{\kappa} + |\ell_\lambda(\varpi_{k_s}, \Gamma\varpi_{k_s}) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|\right]\Big\}\\
= \max\{&\ell_\lambda(\varpi_{k_s}, \varpi^*) + |\ell_\lambda(\varpi_{k_s}, \varpi_{k_s-1}) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|;\\
&\ell_\lambda(\varpi_{k_s}, \varpi_{k_s-1}) + |\ell_\lambda(\varpi_{k_s}, \varpi^*) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|;\\
&\ell_\lambda(\varpi^*, \Gamma\varpi^*) + |\ell_\lambda(\varpi_{k_s}, \varpi^*) - \ell_\lambda(\varpi_{k_s}, \varpi_{k_s-1})|;\\
&\tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\varpi_{k_s}, \Gamma\varpi^*) + \ell_{2\lambda}(\varpi^*, \varpi_{k_s-1})}{\kappa} + |\ell_\lambda(\varpi_{k_s}, \varpi_{k_s-1}) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|\right]\Big\}.
\end{aligned} \tag{3.4}$$

Taking the limit as $s \to \infty$ in (3.3) and (3.4) and by employing (3.2) and $(\mathscr{G}_1)$, we deduce that

$$\Sigma(\ell_\lambda(\varpi^*, \Gamma\varpi^*)) \le \Sigma\left(\kappa^3 \ell_\lambda(\varpi^*, \Gamma\varpi^*)\right) \le \mathscr{G}\left(\Sigma(\ell_\lambda(\varpi^*, \Gamma\varpi^*)), \vartheta(\ell_\lambda(\varpi^*, \Gamma\varpi^*))\right) \le \Sigma(\ell_\lambda(\varpi^*, \Gamma\varpi^*)),$$

and as a consequence, the subsequent term is found.

$$\mathscr{G}\left(\Sigma(\ell_\lambda(\varpi^*, \Gamma\varpi^*)), \vartheta(\ell_\lambda(\varpi^*, \Gamma\varpi^*))\right) = \Sigma(\ell_\lambda(\varpi^*, \Gamma\varpi^*)).$$

Using the properties of $\mathscr{G}$, we say that $\varpi^* = \Gamma\varpi^*$, that is, ϖ^* is a fixed point of Γ.
Next, we show that ϖ^* is a unique fixed point of Γ. Let w be another fixed point of Γ, i.e., $w = \Gamma w$, there exists $r \in \mathbf{S}_\ell^*$ such that $(\varpi^*, r) \in B(H)$ and $(r, w) \in B(H)$. Using (iv), we have $(\varpi^*, w) \in B(H)$.
Thence, from (3.2), we achieve

$$\Sigma\left(\kappa^3 \ell_\lambda(\Gamma\varpi^*, \Gamma w)\right) \le \mathscr{G}\left(\Sigma(\Xi(\varpi^*, w)), \vartheta(\Xi(\varpi^*, w))\right), \tag{3.5}$$

where

$$\begin{aligned}
\Xi(\varpi^*, w) = \max\{&\ell_\lambda(\varpi^*, w) + |\ell_\lambda(\varpi^*, \Gamma\varpi^*) - \ell_\lambda(w, \Gamma w)|; \ell_\lambda(\varpi^*, \Gamma\varpi^*) + |\ell_\lambda(\varpi^*, w) - \ell_\lambda(w, \Gamma w)|;\\
&\ell_\lambda(w, \Gamma w) + |\ell_\lambda(\varpi^*, w) - \ell_\lambda(\varpi^*, \Gamma\varpi^*)|; \tfrac{1}{2}\left[\tfrac{\ell_{2\lambda}(\varpi^*, \Gamma w) + \ell_{2\lambda}(w, \Gamma\varpi^*)}{\kappa} + |\ell_\lambda(\varpi^*, \Gamma\varpi^*) - \ell_\lambda(w, \Gamma w)|\right]\Big\}.
\end{aligned} \tag{3.6}$$

Together with (3.5) and (3.6), also applying the features of the functions $\mathscr{G}, \Sigma, \vartheta$, we yield that $\varpi^* = w$, hence ϖ^* is a unique fixed point of Γ. □

4. Application to integral type contractions

This section consists of a common fixed point theorem, including integral type contraction and some consequences, which can be obtained by applying particular expressions in the main result.

Definition 4.1. *Let* $\Theta := \{\mu : \mathbf{R}^+ \to \mathbf{R}^+ |\ \mu$ Lebesgue integrable mapping$\}$ *be a class of mappings satisfying the followings:*

(μ_1) *μ is non-negative and summable function;*
(μ_2) *for all $\varepsilon > 0$*

$$\int_0^{\varepsilon} \mu(\rho)\, d\rho > 0.$$

In what follows, Branciari [30] demonstrated a fixed point theorem regarding a contractive condition of integral type.

Theorem 4.2. [30] *Let Γ be a self-mapping on a complete metric space $(\mathbf{S}, m)$, and there exists $k \in (0,1)$ and $\mu \in \Theta$. If for $\varpi, \xi \in \mathbf{S}$*

$$\int_0^{m(\Gamma\varpi,\Gamma\xi)} \mu(\rho)\, d\rho \leq k \int_0^{m(\varpi,\xi)} \mu(\rho)\, d\rho$$

is satisfied, then Γ possesses a unique fixed point in $(\mathbf{S}, m)$.

Subsequently, numerous studies have been done about the consequence of Branciari with some known properties. In [31], Azadifar et al. verified that a common fixed point theorem was satisfying a contractive condition of integral type in the sense of modular metric spaces.
Now, we construct our main result of this section by defining the $\mathscr{G}(\Sigma, \vartheta, \Xi)-$contraction of integral type, as indicated below.

Definition 4.3. *Let $\mathbf{S}_\ell^*$ be an MbMS with the coefficient $\kappa \geq 1$ and let $\Gamma, \Upsilon, J, \zeta : \mathbf{S}_\ell^* \to \mathbf{S}_\ell^*$ be mappings. The mappings Γ, Υ, J and ζ are called $\mathscr{G}(\Sigma, \vartheta, \Xi)$-contraction of integral type, if there exist $\mathscr{G} \in \mathscr{C}$, $\Sigma \in \Omega$, $\vartheta \in \Pi$, and $\mu \in \Theta$ such that*

$$\Sigma\left(\kappa^3 \int_0^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi,\Upsilon\xi)} \mu(\rho)\, d\rho\right) \leq \mathscr{G}\left(\Sigma\left(\Xi(\varpi,\xi)\right), \vartheta\left(\Xi(\varpi,\xi)\right)\right), \tag{4.1}$$

where

$$\Xi(\varpi,\xi) = \max\left\{ \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,J\xi)} \mu(\rho)\, d\rho + \left| \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,\Gamma\varpi)} \mu(\rho)\, d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(J\xi,\Upsilon\xi)} \mu(\rho)\, d\rho \right|; \right.$$

$$\int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,\Gamma\varpi)} \mu(\rho)\, d\rho + \left| \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,J\xi)} \mu(\rho)\, d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(J\xi,\Upsilon\xi)} \mu(\rho)\, d\rho \right|;$$

$$\int_0^{\ell_{\frac{\lambda}{l}}(J\xi,\Upsilon\xi)} \mu(\rho)\, d\rho + \left| \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,J\xi)} \mu(\rho)\, d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,\Gamma\varpi)} \mu(\rho)\, d\rho \right|;$$

$$\left. \frac{1}{2}\left[\frac{\int_0^{\ell_{\frac{2\lambda}{l}}(\zeta\varpi,\Upsilon\xi)} \mu(\rho)d\rho + \int_0^{\ell_{\frac{2\lambda}{l}}(J\xi,\Gamma\varpi)} \mu(\rho)d\rho}{\kappa} + \left| \int_0^{\ell_{\frac{\lambda}{l}}(\zeta\varpi,\Gamma\varpi)} \mu(\rho)\, d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(J\xi,\Upsilon\xi)} \mu(\rho)\, d\rho \right| \right] \right\}$$

for all distinct $\varpi, \xi \in \mathbf{S}_\ell^*$, for $c, l \in \mathbf{R}^+$, $c > l$ and for all $\lambda > 0$.

Theorem 4.4. *Let $\mathbf{S}_\ell^*$ be an $\ell-$complete MbMS with coefficient $\kappa \geq 1$. Assume that the following statements hold:*

(i) Γ, Υ, J and ζ be a $\mathscr{G}(\Sigma, \vartheta, \Xi)-$contraction such that $\Gamma(\mathbf{S}_\ell^*) \subset J(\mathbf{S}_\ell^*)$ and $\Upsilon(\mathbf{S}_\ell^*) \subset \zeta(\mathbf{S}_\ell^*)$,
(ii) One of the sets $\Gamma(\mathbf{S}_\ell^*), J(\mathbf{S}_\ell^*), \Upsilon(\mathbf{S}_\ell^*)$ and *$\zeta(\mathbf{S}_\ell^*)$ is a closed subset of $\mathbf{S}_\ell^*$,*
(iii) The pairs $\{J, \Upsilon\}$ and $\{\zeta, \Gamma\}$ are weakly compatible.

Under the conditions (M_1) and (M_2), the mappings Γ, Υ, J, and ζ admit a unique common fixed point in $\mathbf{S}_\ell^*$.

Proof. Let $\varpi_0 \in \mathbf{S}_\ell^*$ be an arbitrary point and similar to the proof of Theorem 2.2, we construct a sequence $\{\xi_q\}$ in $\mathbf{S}_\ell^*$ such that

$$\xi_{2q} = \Gamma\varpi_{2q} = J\varpi_{2q+1}, \quad \xi_{2q+1} = \Upsilon\varpi_{2q+1} = \zeta\varpi_{2q+2}.$$

From (4.1), we get

$$\Sigma\left(\kappa^{3}\int_{0}^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi_{2q},\Upsilon\varpi_{2q+1})}\mu(\rho)\,d\rho\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)\right),\vartheta\left(\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)\right)\right),$$

where

$$\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)=\max\left\{\begin{array}{l}\int_{0}^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},J\varpi_{2q+1})}\mu(\rho)\,d\rho+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},\Gamma\varpi_{2q})}\mu(\rho)\,d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})}\mu(\rho)\,d\rho\right|;\\ \int_{0}^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},\Gamma\varpi_{2q})}\mu(\rho)\,d\rho+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},J\varpi_{2q+1})}\mu(\rho)\,d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})}\mu(\rho)\,d\rho\right|;\\ \int_{0}^{\ell_{\frac{\lambda}{l}}(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})}\mu(\rho)\,d\rho+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},J\varpi_{2q+1})}\mu(\rho)\,d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},\Gamma\varpi_{2q})}\mu(\rho)\,d\rho\right|;\\ \frac{1}{2}\left[\frac{\int_{0}^{\ell_{\frac{2\lambda}{l}}(\zeta\varpi_{2q},\Upsilon\varpi_{2q+1})}\mu(\rho)d\rho+\int_{0}^{\ell_{\frac{2\lambda}{l}}(J\varpi_{2q+1},\Gamma\varpi_{2q})}\mu(\rho)d\rho}{\kappa}+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}(\zeta\varpi_{2q},\Gamma\varpi_{2q})}\mu(\rho)\,d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}(J\varpi_{2q+1},\Upsilon\varpi_{2q+1})}\mu(\rho)\,d\rho\right|\right]\end{array}\right\}$$

$$=\max\left\{\begin{array}{l}\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{2q},\xi_{2q+1})}\mu(\rho)\,d\rho\right|;\\ \int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{2q},\xi_{2q+1})}\mu(\rho)\,d\rho\right|;\\ \int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{2q},\xi_{2q+1})}\mu(\rho)\,d\rho+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho\right|;\\ \frac{1}{2}\left[\frac{\int_{0}^{\ell_{\frac{2\lambda}{l}}(\xi_{2q-1},\xi_{2q+1})}\mu(\rho)d\rho+\int_{0}^{\ell_{\frac{2\lambda}{l}}(\xi_{2q},\xi_{2q})}\mu(\rho)d\rho}{\kappa}+\left|\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{2q-1},\xi_{2q})}\mu(\rho)\,d\rho-\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{2q},\xi_{2q+1})}\mu(\rho)\,d\rho\right|\right]\end{array}\right\}.$$

Now, we accept $\sigma_q=\int_{0}^{\ell_{\frac{\lambda}{c}}(\xi_{q-1},\xi_q)}\mu(\rho)\,d\rho$, and $\sigma_q^{*}=\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{q-1},\xi_q)}\mu(\rho)\,d\rho$ with $c>l$ and suppose that $\sigma_{2q+1}^{*}\geq\sigma_{2q}^{*}$. Again, similar to the proof of Theorem 2.2 and by using ($\mathscr{G}_1$), we obtain

$$\Sigma\left(\sigma_{2q+1}^{*}\right)\leq\Sigma\left(\sigma_{2q+1}\right)\leq\Sigma\left(\kappa^{3}\sigma_{2q+1}\right)\leq\mathscr{G}\left(\Sigma\left(\sigma_{2q+1}^{*}\right),\vartheta\left(\sigma_{2q+1}^{*}\right)\right)\leq\Sigma\left(\sigma_{2q+1}^{*}\right).$$

From the properties of $\mathscr{G}$, we have a contradiction. Then, we get $\sigma_{2q+1}^{*}<\sigma_{2q}^{*}$ such that $\Xi\left(\varpi_{2q},\varpi_{2q+1}\right)=2\sigma_{2q}^{*}-\sigma_{2q+1}^{*}$. Also, in a similar way, we yield that $\sigma_{2q}^{*}<\sigma_{2q-1}^{*}$. So, it ensures that $\sigma_{q+1}^{*}<\sigma_{q}^{*}$ such that

$$\sigma_q^{*}=\left\{\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{q-1},\xi_q)}\mu(\rho)\,d\rho\right\}$$

is a non-increasing sequence of non-negative real numbers and so the following sequence

$$\left\{\int_{0}^{\ell_{\frac{\lambda}{l}}(\xi_{q-1},\xi_q)}\mu(\rho)\,d\rho\right\}$$

converges to a non-negative number r. We shall prove that $r=0$. By the same argument, we conclude that

$$\Sigma\left(\sigma_{2q+1}\right) \leq \Sigma\left(\kappa^3\sigma_{2q+1}\right) \leq \mathscr{G}\left(\Sigma\left(2\sigma^*_{2q}-\sigma^*_{2q+1}\right), \vartheta\left(2\sigma^*_{2q}-\sigma^*_{2q+1}\right)\right) \leq \Sigma\left(2\sigma^*_{2q}-\sigma^*_{2q+1}\right).$$

If we take the limit above, we gain

$$\Sigma(r) \leq \mathscr{G}\left(\Sigma(r), \vartheta(r)\right) \leq \Sigma(r),$$

and consequently, $\mathscr{G}\left(\Sigma(r), \vartheta(r)\right)=\Sigma(r)$. Then, from ($\mathscr{G}_2$), either $\Sigma(r)=0$ or $\vartheta(r)=0$. This implies that $r=0$, that is, $\int_0^{\ell_\lambda\left(\xi_{q-1},\xi_q\right)} \mu(\rho)\,d\rho \to 0$, as $q\to\infty$. It follows that

$$\ell_\lambda\left(\xi_{q-1},\xi_q\right) \to 0\,, \quad (q\to\infty).$$

In the next step, we demonstrate that $\left\{\xi_q\right\}$ is an $\ell-$Cauchy sequence. It is enough to achieve that $\left\{\xi_{2q}\right\}$ is an $\ell-$Cauchy sequence. If it is not, then we can find $\varepsilon>0$ such that there exist two sequences $\{a_i\}$ and $\{b_i\}$ of positive integers satisfying $b_i>a_i\geq i$ such that b_i is the smallest index for which

$$\ell_{\frac{\lambda}{l}}\left(\xi_{2a_i},\xi_{2b_i}\right) \geq \varepsilon, \quad \text{and} \quad \ell_{\frac{\lambda}{l}}\left(\xi_{2a_i},\xi_{2b_i-2}\right) < \varepsilon, \quad \text{for all } \lambda>0.$$

Note that $\varepsilon \leq \ell_{\frac{\lambda}{l}}\left(\xi_{2a_i},\xi_{2b_i}\right) \leq \ell_{\frac{\lambda}{c}}\left(\xi_{2a_i},\xi_{2b_i}\right)$ and continuing as in the proof of Theorem 2.2, we deduce that

$$\begin{aligned}
\Sigma\left(\int_0^{\ell_{\frac{\lambda}{c}}\left(\xi_{2a_i},\xi_{2b_i+1}\right)} \mu(\rho)\,d\rho\right) &\leq \Sigma\left(\kappa^3\int_0^{\ell_{\frac{\lambda}{c}}\left(\xi_{2a_i},\xi_{2b_i+1}\right)} \mu(\rho)\,d\rho\right) \\
&\leq \mathscr{G}\left(\Sigma\left(\int_0^{\ell_{\frac{\lambda}{l}}\left(\xi_{2a_i-1},\xi_{2b_i}\right)} \mu(\rho)\,d\rho\right), \vartheta\left(\int_0^{\ell_{\frac{\lambda}{l}}\left(\xi_{2a_i-1},\xi_{2b_i}\right)} \mu(\rho)\,d\rho\right)\right) \\
&\leq \Sigma\left(\int_0^{\ell_{\frac{\lambda}{l}}\left(\xi_{2a_i-1},\xi_{2b_i}\right)} \mu(\rho)\,d\rho\right).
\end{aligned}$$

Nevertheless, the above inequality causes a contradiction, that is, $\left\{\xi_{2q}\right\}$ is an $\ell-$Cauchy sequence. Thus, $\left\{\xi_q\right\}$ is an $\ell-$Cauchy sequence in $\mathbf{S}^*_\ell$. Since $\mathbf{S}^*_\ell$ is $\ell-$complete MbMS, there exists $z\in\mathbf{S}^*_\ell$ such that

$$\lim_{q\to\infty}\xi_q = z.$$

Now, we shall prove that $\Gamma z=\Upsilon z=Jz=\zeta z=z$. Indeed, we only need to show that $\Gamma z=\zeta z=z$. Also, similar to Theorem 2.2, it is clear that z is a common fixed point of Υ and J. Assuming that $\zeta\left(\mathbf{S}^*_\ell\right)$ is a closed subset of $\mathbf{S}^*_\ell$, there is $u\in\mathbf{S}^*_\ell$ such that $z=\zeta u$. We claim that $\Gamma u=z$. From (4.1), we have

$$\Sigma\left(\kappa^3\int_0^{\ell_{\frac{\lambda}{c}}\left(\Gamma u,\Upsilon\varpi_{2q+1}\right)} \mu(\rho)\,d\rho\right) \leq \mathscr{G}\left(\Sigma\left(\Xi\left(u,\varpi_{2q+1}\right)\right), \vartheta\left(\Xi\left(u,\varpi_{2q+1}\right)\right)\right),$$

where

$$\Xi\left(u,\varpi_{2q+1}\right)=\max\left\{\int_0^{\ell_{\frac{\lambda}{l}}\left(z,\xi_{2q}\right)}\mu\left(\rho\right)d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}\left(z,\Gamma u\right)}\mu\left(\rho\right)d\rho-\int_0^{\ell_{\frac{\lambda}{l}}\left(\xi_{2q},\xi_{2q+1}\right)}\mu\left(\rho\right)d\rho\right|;\right.$$

$$\int_0^{\ell_{\frac{\lambda}{l}}\left(z,\Gamma u\right)}\mu\left(\rho\right)d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}\left(z,\xi_{2q}\right)}\mu\left(\rho\right)d\rho-\int_0^{\ell_{\frac{\lambda}{l}}\left(\xi_{2q},\xi_{2q+1}\right)}\mu\left(\rho\right)d\rho\right|;$$

$$\int_0^{\ell_{\frac{\lambda}{l}}\left(\xi_{2q},\xi_{2q+1}\right)}\mu\left(\rho\right)d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}\left(z,\xi_{2q}\right)}\mu\left(\rho\right)d\rho-\int_0^{\ell_{\frac{\lambda}{l}}\left(z,\Gamma u\right)}\mu\left(\rho\right)d\rho\right|;$$

$$\left.\frac{1}{2}\left[\frac{\int_0^{\ell_{\frac{2\lambda}{l}}\left(z,\xi_{2q+1}\right)}\mu(\rho)d\rho+\int_0^{\ell_{\frac{2\lambda}{l}}\left(\xi_{2q},\Gamma u\right)}\mu(\rho)d\rho}{\kappa}+\left|\int_0^{\ell_{\frac{\lambda}{l}}\left(z,\Gamma u\right)}\mu\left(\rho\right)d\rho-\int_0^{\ell_{\frac{\lambda}{l}}\left(\xi_{2q},\xi_{2q+1}\right)}\mu\left(\rho\right)d\rho\right|\right]\right\}.$$

Step by step, similar to the proof of Theorem 2.2, we obtain

$$\begin{aligned}\Sigma\left(\int_0^{\ell_{\frac{\lambda}{l}}(\Gamma u,z)}\mu\left(\rho\right)d\rho\right)&\leq\Sigma\left(\int_0^{\ell_{\frac{\lambda}{c}}(\Gamma u,z)}\mu\left(\rho\right)d\rho\right)\leq\Sigma\left(\kappa^3\int_0^{\ell_{\frac{\lambda}{c}}(\Gamma u,z)}\mu\left(\rho\right)d\rho\right)\\&\leq\mathscr{G}\left(\Sigma\left(\int_0^{\ell_{\frac{\lambda}{l}}(\Gamma u,z)}\mu\left(\rho\right)d\rho\right),\vartheta\left(\int_0^{\ell_{\frac{\lambda}{l}}(\Gamma u,z)}\mu\left(\rho\right)d\rho\right)\right)\\&\leq\Sigma\left(\int_0^{\ell_{\frac{\lambda}{l}}(\Gamma u,z)}\mu\left(\rho\right)d\rho\right).\end{aligned}$$

Hence, from ($\mathscr{G}_2$), we achieve that $\Gamma u=\zeta u=z$. Since the maps Γ and ζ are weakly compatible, we have $\Gamma z=\Gamma\zeta u=\zeta\Gamma u=\zeta z$. In another step, we show that $\Gamma z=z$. However, it can be shown similar to the above proof. So $\Gamma z=\zeta z=z$ is procured. Finally, for the uniqueness, we assume that w be another common fixed point, i.e., $\Gamma w=\Upsilon w=Jw=\zeta w$ such that $z\neq w$. Then, from (4.1), we get

$$\Sigma\left(\kappa^3\int_0^{\ell_{\frac{\lambda}{c}}(\Gamma z,\Upsilon w)}\mu\left(\rho\right)d\rho\right)\leq\mathscr{G}\left(\Sigma\left(\Xi\left(z,w\right)\right),\vartheta\left(\Xi\left(z,w\right)\right)\right),$$

and it is straightforward that $\Xi(z,w)=\int_0^{\ell_{\frac{\lambda}{l}}(z,w)}\mu\left(\rho\right)d\rho$. Thereupon, we get

$$\begin{aligned}\Sigma\left(\int_0^{\ell_{\frac{\lambda}{l}}(z,w)}\mu\left(\rho\right)d\rho\right)&\leq\Sigma\left(\int_0^{\ell_{\frac{\lambda}{c}}(z,w)}\mu\left(\rho\right)d\rho\right)\leq\Sigma\left(\kappa^3\int_0^{\ell_{\frac{\lambda}{c}}(z,w)}\mu\left(\rho\right)d\rho\right)\\&\leq\mathscr{G}\left(\Sigma\left(\int_0^{\ell_{\frac{\lambda}{l}}(z,w)}\mu\left(\rho\right)d\rho\right),\vartheta\left(\int_0^{\ell_{\frac{\lambda}{l}}(z,w)}\mu\left(\rho\right)d\rho\right)\right)\\&\leq\Sigma\left(\int_0^{\ell_{\frac{\lambda}{l}}(z,w)}\mu\left(\rho\right)d\rho\right).\end{aligned}$$

Hence, from the properties of the function $\mathscr{G}$, $z=w$ is gained as a unique common fixed point of Γ, Υ, J and ζ. □

Some conclusions can be drawn from the main result of this section are given.

Corollary 4.5. *Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with constant $\kappa \geq 1$ and let Γ and Υ be self mappings in $\mathbf{S}_\ell^*$. The following statements hold:*

(i) *There exist $\mathscr{G} \in \mathscr{C}$, $\Sigma \in \Omega$, $\vartheta \in \Pi$, and $\mu \in \Theta$ such that*

$$\Sigma\left(\kappa^3 \int_0^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi,\Upsilon\xi)} \mu(\rho)\,d\rho\right) \leq \mathscr{G}\left(\Sigma\left(\Xi(\varpi,\xi)\right),\vartheta\left(\Xi(\varpi,\xi)\right)\right),$$

where

$$\Xi(\varpi,\xi)=\max\left\{\begin{array}{l} \int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)} \mu(\rho)\,d\rho + \left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)} \mu(\rho)\,d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Upsilon\xi)} \mu(\rho)\,d\rho\right|; \\ \int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)} \mu(\rho)\,d\rho + \left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)} \mu(\rho)\,d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Upsilon\xi)} \mu(\rho)\,d\rho\right|; \\ \int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Upsilon\xi)} \mu(\rho)\,d\rho + \left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)} \mu(\rho)\,d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)} \mu(\rho)\,d\rho\right|; \\ \frac{1}{2}\left[\frac{\int_0^{\ell_{\frac{2\lambda}{l}}(\varpi,\Upsilon\xi)} \mu(\rho)d\rho + \int_0^{\ell_{\frac{2\lambda}{l}}(\xi,\Gamma\varpi)} \mu(\rho)d\rho}{\kappa} + \left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)} \mu(\rho)\,d\rho - \int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Upsilon\xi)} \mu(\rho)\,d\rho\right|\right] \end{array}\right\},$$

for all distinct $\varpi,\xi \in \mathbf{S}_\ell^*$, for $c,l \in \mathbf{R}^+$, $c>l$ and for all $\lambda>0$.

(ii) *(M_1) and (M_2) are provided.*

The mappings Γ and Υ admit a unique common fixed point in $\mathbf{S}_\ell^*$.

By choosing $\Gamma=\Upsilon$ in Corollary 4.5, we obtain the following one.

Corollary 4.6. *Let $\mathbf{S}_\ell^*$ be an ℓ-complete MbMS with coefficient $\kappa \geq 1$ and let $\Gamma:\ \mathbf{S}_\ell^* \to \mathbf{S}_\ell^*$ be a mapping. Suppose that the following circumstances are satisfied:*

(i) *There exists $\mathscr{G} \in \mathscr{C}$, $\Sigma \in \Omega$, $\vartheta \in \Pi$ and $\mu \in \Theta$ such that*

$$\Sigma\left(\kappa^3 \int_0^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi,\Gamma\xi)} \mu(\rho)\,d\rho\right) \leq \mathscr{G}\left(\Sigma\left(\Xi(\varpi,\xi)\right),\vartheta\left(\Xi(\varpi,\xi)\right)\right),$$

where

$$
\begin{aligned}
\Xi(\varpi,\xi)=\max\Bigg\{ & \int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Gamma\xi)}\mu(\rho)\,d\rho\right|;\\
& \int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Gamma\xi)}\mu(\rho)\,d\rho\right|;\\
& \int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Gamma\xi)}\mu(\rho)\,d\rho+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\xi)}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)}\mu(\rho)\,d\rho\right|;\\
& \frac{1}{2}\left[\frac{\int_0^{\ell_{\frac{2\lambda}{l}}(\varpi,\Gamma\xi)}\mu(\rho)d\rho+\int_0^{\ell_{\frac{2\lambda}{l}}(\xi,\Gamma\varpi)}\mu(\rho)d\rho}{\kappa}+\left|\int_0^{\ell_{\frac{\lambda}{l}}(\varpi,\Gamma\varpi)}\mu(\rho)\,d\rho-\int_0^{\ell_{\frac{\lambda}{l}}(\xi,\Gamma\xi)}\mu(\rho)\,d\rho\right|\right]\Bigg\},
\end{aligned}
\tag{4.2}
$$

for all distinct $\varpi,\xi\in\mathbf{S}_\ell^*$, *for* $c,l\in\mathbf{R}^+$, $c>l$ *and for all* $\lambda>0$.

(ii) *The conditions* (M_1) *and* (M_2) *are provided.*

The mapping Γ *holds a unique fixed point in* $\mathbf{S}_\ell^*$.

Besides, we attain the following consequence if we perceive $\mathscr{G}(p,q)=p-q$ for all $p,q\in[0,\infty)$ in Corollary 4.6.

Corollary 4.7. *Let* $\mathbf{S}_\ell^*$ *be an* ℓ*-complete MbMS with coefficient* $\kappa\geq 1$ *and let* $\Gamma:\ \mathbf{S}_\ell^*\to\mathbf{S}_\ell^*$ *be a mapping. Assume that the following statements hold:*

(i) *There exist* $\Sigma\in\Omega$, $\vartheta\in\Pi$ *and* $\mu\in\Theta$ *such that*

$$
\Sigma\left(\kappa^3\int_0^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi,\Gamma\xi)}\mu(\rho)\,d\rho\right)\leq\Sigma(\Xi(\varpi,\xi))-\vartheta(\Xi(\varpi,\xi)),
$$

where $\Xi(\varpi,\xi)$ *is defined as in (4.2) and for all distinct* $\varpi,\xi\in\mathbf{S}_\ell^*$, *for* $c,l\in\mathbf{R}^+$, $c>l$ *and for all* $\lambda>0$.

(ii) *The condition* (M_1) *and* (M_2) *are provided.*

Then, Γ *admits a unique fixed point in* $\mathbf{S}_\ell^*$.

If we constitute $\mathscr{G}(p,q)=kp,\quad k\in(0,1)$ for all $p\in[0,\infty)$, then we get the below one.

Corollary 4.8. *Let* $\mathbf{S}_\ell^*$ *be an* ℓ*-complete MbMS with* $\kappa\geq 1$ *and let* $\Gamma:\ \mathbf{S}_\ell^*\to\mathbf{S}_\ell^*$ *be a mapping. Assume that the following ones hold:*

(i) *There exists* $\Sigma\in\Omega$ *and* $\mu\in\Theta$ *such that*

$$
\Sigma\left(\kappa^3\int_0^{\ell_{\frac{\lambda}{c}}(\Gamma\varpi,\Gamma\xi)}\mu(\rho)\,d\rho\right)\leq k\Sigma(\Xi(\varpi,\xi)),
$$

where $\Xi(\varpi,\xi)$ *is defined as in (4.2) and for all distinct* $\varpi,\xi\in\mathbf{S}_\ell^*$, $c,l\in\mathbf{R}^+$, $c>l$ *for all* $\lambda>0$.

(ii) *The conditions* (M_1) *and* (M_2) *are provided.*

Then, we yield that Γ *admits a unique fixed point* $\mathbf{S}_\ell^*$.

5. Conclusion

Consequently, we extended the results of Fulga and Proca [19, 20] and [23, 24] to modular $b-$metric space via $C-$class functions for four mappings and examined that our results can be applied to graph structure and integral type contractions. In the meantime, our consequences are still valid in the case of

$$
\Xi(\varpi,\xi)=\ell_\lambda(\varpi,\xi)+|\ell_\lambda(\varpi,\Gamma\varpi)-\ell_\lambda(\xi,\Gamma\xi)|.
$$

Moreover, by taking $\kappa=1$, the obtained conclusions for MbMS are valid in the setting of modular metric space.

Acknowledgements

The authors express their gratitude to the referees for their valuable comments, critical remarks, and suggestions, which helped them improve this paper.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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