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CHROMATIC SCHULTZ POLYNOMIAL OF CERTAIN GRAPHS

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Abstract. A topological index of a graph G is a real number which is preserved under isomorphism. Extensive studies on certain polynomials related to these topological indices have also been done recently. In a similar way, chromatic versions of certain topological indices and the related polynomials have also been discussed in the recent literature. In this paper, the chromatic versions of the Schultz polynomial and modified chromatic Schultz polynomial are introduced and determined this polynomial for certain fundamental graph classes.

1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [10]. Further, for graph colouring, see [6, 7]. Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected.

A *proper vertex colouring* of a graph G is an assignment $\varphi : V(G) \to \mathcal{C}$ of the vertices of G, where $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\ell\}$ is a set of colours such that adjacent vertices of G have different colours. The cardinality of the minimum set of colours which allows a proper colouring of G is called the *chromatic number* of G and is denoted $\chi(G)$. The set of all vertices of G which have the colour c_i is called the *colour class* of that colour c_i in G. The cardinality of the colour class of a colour c_i is said to be the *strength* of that colour in G and is denoted by $\theta(c_i)$. We can also define a function $\zeta : V(G) \to \{1, 2, 3, \ldots, \ell\}$ such that $\zeta(v_i) = s$ if and only if $\varphi(v_i) = c_s, c_s \in \mathcal{C}$.

A vertex colouring consisting of the colours having minimum subscripts may be called a *minimum parameter colouring* (see [8]). If we colour the vertices of G in such a way that c_1 is assigned to maximum possible number of vertices, then c_2 is assigned to maximum possible number of remaining uncoloured vertices and

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proceed in this manner until all vertices are coloured, then such a colouring is called a χ^--*colouring* of G. In a similar manner, if c_ℓ is assigned to maximum possible number of vertices, then $c_{\ell-1}$ is assigned to maximum possible number of remaining uncoloured vertices and proceed in this manner until all vertices are coloured, then such a colouring is called a χ^+-*colouring* of G.

A *topological index* of a graph G is a real number which is preserved under isomorphism. The chromatic versions of certain topological indices have been introduced in [8]. The Schultz polynomials and modified Schultz polynomials of graphs are some of such widely studied polynomials (see [1, 2, 4]).

Some chromatic version of topological indices were introduced and studied in [8] and later the idea of chromatic topological polynomials was introduced in [9] In this paper, we discuss the chromatic versions of certain polynomials related to the topological indices of a graph G.

2. Chromatic Schultz Polynomial of Graphs

Note that throughout this study, we use the chromatic colourings of the graphs under consideration. Motivated by the studies on Schultz polynomial of graphs (see [1,2,4,5]), we can now introduce the chromatic version of the Schultz polynomial as follows:

Definition 1. Let G be a connected graph with chromatic number $\chi(G)$. Then, the *chromatic Schultz polynomial* of G, denoted by $\mathcal{S}_\chi(G,x)$, is defined as

$$\mathcal{S}_\chi(G,x) = \sum_{u,v\in V(G)} (\zeta(u)+\zeta(v))x^{d(u,v)}.$$

Definition 2. Let G be a connected graph with chromatic number φ^- and φ^+ be the minimal and maximal parameter colouring of G. Then,

(i) the χ^--*chromatic Schultz polynomial* of G, denoted by $\mathcal{S}_{\chi^-}(G,x)$, is defined as

$$\mathcal{S}_{\chi^-}(G,x) = \sum_{u,v\in V(G)} (\zeta_{\varphi^-}(u)+\zeta_{\varphi^-}(v))x^{d(u,v)};$$

and

(ii) the χ^+-*chromatic Schultz polynomial* of G, denoted by $\mathcal{S}_{\chi^+}(G,x)$, is defined as

$$\mathcal{S}_{\chi^+}(G,x) = \sum_{u,v\in V(G)} (\zeta_{\varphi^+}(u)+\zeta_{\varphi^+}(v))x^{d(u,v)}.$$

Now, we can determine the chromatic Schultz polynomials of certain fundamental graph classes.

2.1. Chromatic Schultz Polynomials of Paths.

In this section, we discuss the two types of chromatic Schultz polynomials of paths.

Theorem 1. *Let P_n be a path on n vertices. Then, we have*

$$\mathcal{S}_{\chi^-}(P_n, x) = \begin{cases} \sum\limits_{i=0}^{\frac{n-1}{2}} \left[(3n-6i-3)x + (3n-6i-1)\right] x^{2i}; & \textit{if } n \textit{ is odd}; \\ 3 \cdot \sum\limits_{i=0}^{n} (n-i)x^i; & \textit{if } n \textit{ is even}. \end{cases}$$

Proof. Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of P_n, where the vertices are labelled consecutively. Note that $\chi(P_n) = 2$. Let c_1, c_2 be the two colours we use for colouring P_n. We also note that te diameter of P_n is $n-1$. Hence, the power of the variable x varies from 0 to $n-1$ in the Schultz polynomial of P_n. Here, we need to consider the following two cases:

Case-1: Let n be odd. Then, with respect to a χ^--colouring, the vertices $v_1, v_3, v_5, \dots v_n$ get the colour c_1 and the vertices $v_2, v_4, v_6, \dots, v_{n-1}$ get the colour c_2. The possible colour pairs and their numbers in G in terms of the distances between them are listed in Table 1.

TABLE 1. A list of color pairs and the distance between them in an odd path.

Distance $d(u,v)$	Colour pairs	Number of pairs	Total number of pairs
0	(c_1, c_1)	$\frac{n+1}{2}$	n
	(c_2, c_2)	$\frac{n-1}{2}$	
1	(c_1, c_2)	$n-1$	$n-1$
2	(c_1, c_1)	$\frac{n-1}{2}$	$n-2$
	(c_2, c_2)	$\frac{n-3}{2}$	
3	(c_1, c_2)	$n-3$	$n-3$
4	(c_1, c_1)	$\frac{n-3}{2}$	$n-4$
	(c_2, c_2)	$\frac{n-5}{2}$	
5	(c_1, c_2)	$n-5$	$n-5$
6	(c_1, c_1)	$\frac{n-5}{2}$	$n-6$
	(c_2, c_2)	$\frac{n-7}{2}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-3$	(c_1, c_1)	2	3
	(c_2, c_2)	1	
$n-2$	(c_1, c_2)	2	2
$n-1$	(c_1, c_1)	1	1
	(c_2, c_2)	0	

In the above table, the possible distances between different pairs of vertices are written in the first column, the different colour pairs with respect to each distance is written in the second column and the number of corresponding colour pairs with respect to each distance is written in the third column. The total number of vertex pairs corresponding to each distance is written in the fourth column.

From Table-1, we note that for $0 \leq r \leq n$, the number of vertex pairs which are at a distance r is $n-r$ and in this case all colour pairs contain two colours, when r is odd. But, when r is even, all colour pairs contain the same colour - either (c_1, c_1) or (c_2, c_2). In this case, note that the number of (c_1, c_1)-colour pairs is $\frac{n-r+1}{2}$ and the number of (c_2, c_2)-colour pairs is $\frac{n-r-1}{2}$ so that total number of colour pairs is $n-r$. Hence,

$$\begin{aligned}
\mathcal{S}_{\chi^-}(P_n, x) &= \sum_{r \text{ odd}} (1+2)(n-r)x^r + \sum_{r \text{ even}} \left[\frac{n-r+1}{2} \cdot 2 + \frac{n-r-1}{2} \cdot 4\right] x^r \\
&= \sum_{r \text{ odd}} (3n-3r)x^r + \sum_{r \text{ even}} (3n-3r-1)x^r \\
&= \sum_{i=0}^{\frac{n-1}{2}} (3n-6i-3)x^{2i+1} + \sum_{i=0}^{\frac{n-1}{2}} (3n-6i-1)x^{2i} \\
&= \sum_{i=0}^{\frac{n-1}{2}} \left[(3n-6i-3)x + (3n-6i-1)\right] x^{2i}. \\
& \quad (\text{since } 3n-6i-3 = 0 \text{ at } i = \frac{n-1}{2}).
\end{aligned}$$

Case-2: Let n be even. Then, with respect to a χ^--colouring, the vertices $v_1, v_3, v_5, \ldots v_{n-1}$ get the colour c_1 and the vertices $v_2, v_4, v_6, \ldots v_n$ get the colour c_2. The possible colour pairs and their numbers in G in terms of the distances between them are listed in the following table.
From Table-2, we have

$$\begin{aligned}
\mathcal{S}_{\chi^-}(P_n, x) &= \sum_{r \text{ odd}} (1+2)(n-r)x^r + \sum_{r \text{ even}} \left[\frac{n-r}{2} \cdot 2 + \frac{n-r}{2} \cdot 4\right] x^r \\
&= \sum_{r \text{ odd}} (3n-3r)x^r + \sum_{r \text{ even}} (3n-3r)x^r \\
&= 3 \cdot \sum_{i=0}^{n-1} (n-i)x^i.
\end{aligned}$$

This completes the proof. □

Note that the χ^+-colouring of P_n can be obtained by interchanging the colours c_1 and c_2 in the χ^--colouring. Hence, as explained in the proof of above theorem, we have

TABLE 2. A list of color pairs and the distance between them in an even path.

Distance $d(u,v)$	Colour pairs	Number of pairs	Total number of pairs
0	(c_1,c_1)	$\frac{n}{2}$	n
	(c_2,c_2)	$\frac{n}{2}$	
1	(c_1,c_2)	$n-1$	$n-1$
2	(c_1,c_1)	$\frac{n-2}{2}$	$n-2$
	(c_2,c_2)	$\frac{n-2}{2}$	
3	(c_1,c_2)	$n-3$	$n-3$
4	(c_1,c_1)	$\frac{n-4}{2}$	$n-4$
	(c_2,c_2)	$\frac{n-4}{2}$	
5	(c_1,c_2)	$n-5$	$n-5$
6	(c_1,c_1)	$\frac{n-6}{2}$	$n-6$
	(c_2,c_2)	$\frac{n-6}{2}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-3$	(c_1,c_2)	3	3
$n-2$	(c_1,c_1)	1	2
	(c_2,c_2)	1	
$n-1$	(c_1,c_2)	1	1

Theorem 2. *Let P_n be a path on n vertices. Then, we have*

$$\mathcal{S}_{\chi^+}(P_n,x)=\begin{cases}\sum\limits_{i=0}^{\frac{n-1}{2}}\left[(3n-6i-3)x+(3n-6i+1)\right]x^{2i}; & \textit{if } n \textit{ is odd};\\ 3\cdot\sum\limits_{i=0}^{n-1}(n-i)x^i; & \textit{if } n \textit{ is even}.\end{cases}$$

2.2. **Chromatic Schultz Polynomial of Cycles.** In this section, we discuss the two types of chromatic Schultz polynomials of cycles.

Theorem 3. *Let C_n be a cycle on n vertices. Then, we have*

$$\mathcal{S}_{\chi^-}(C_n,x)=\begin{cases}\frac{3n(1-x^{\frac{n+2}{2}})}{1-x}; & \textit{if } n \textit{ is even};\\ \frac{3(n+1)(1-x^{\frac{n+1}{2}})}{1-x}; & \textit{if } n \textit{ is odd}.\end{cases}$$

Proof. Let $V=\{v_1,v_2,\ldots,v_n\}$ be the vertex set of C_n, where the vertices are labelled consecutively from one end vertex to the other in a clockwise manner.

Note that if n is odd, then the diameter of C_n is $\frac{n-1}{2}$ and if n is even, the diameter of C_n is $\frac{n}{2}$. Hence, we have to consider the following two cases:

Case-1: Let n be even. Then, C_n is 2-colourable and we can label the vertices $v_1, v_3, v_5 \ldots, v_{n-1}$ by colour c_1 and the vertices $v_2, v_4, v_6 \ldots, v_n$ by colour c_2. Then, for $0 \leq i \leq \frac{n}{2}$, the possible colour pairs and their numbers can be obtained from the following table.

Table 3. A list of color pairs and the distance between them in an even cycle.

Distance $d(u,v)$	Colour pairs	Number of pairs	Total number of pairs
i, even	(c_1, c_1)	$\frac{n}{2}$	n
	(c_2, c_2)	$\frac{n}{2}$	
i, odd	(c_1, c_2)	n	n

Then, from Table 3, we have

$$\begin{aligned}
\mathcal{S}_{\chi^-}(C_n, x) &= \sum_{i \text{ odd}} 3nx^i + \sum_{i \text{ even}} \left[\frac{n}{2}\cdot 2 + \frac{n}{2}\cdot 4\right] x^i \\
&= \sum_{i \text{ odd}} 3nx^i + \sum_{i \text{ even}} (n+2n)x^i \\
&= \sum_{i=0}^{\frac{n}{2}} 3nx^i \\
&= \frac{3n(1-x^{\frac{n+2}{2}})}{1-x}.
\end{aligned}$$

Case-2: Let n be odd. Then, $\chi(C_n) = 3$ and the vertices $v_1, v_3, v_5 \ldots, v_{n-1}$ by colour c_1 and the vertices $v_2, v_4, v_6 \ldots, v_{n-2}$ by colour c_2 and the vertex v_n gets colour c_3. Then, for $0 \leq i \leq \frac{n-1}{2}$, the possible colour pairs and their numbers can be obtained from Table 4.

When $i = 0$, we have

$$\sum_{v \in V} (\zeta(v) + \zeta(v))x^{d(v,v)} = \left[(2+4)\cdot\frac{n-1}{2} + 6\cdot 1\right] x^0 = 3(n+1)x^0$$

When $i > 0$ and is even, we have

$$\begin{aligned}
\sum_{d(u,v)=i} (\zeta(u) + \zeta(v))x^{d(u,v)} &= \left[(2+4)\cdot\frac{n-r-1}{2} + 3(r-1) + (4+5)\cdot 1\right] x^i \\
&= 3(n+1)x^i
\end{aligned}$$

TABLE 4. A list of color pairs and the distance between them in an odd cycle.

Distance $d(u,v)$	Colour pairs	Number of pairs	Total number of pairs
$i=0$	(c_1,c_1)	$\frac{n-1}{2}$	n
	(c_2,c_2)	$\frac{n-1}{2}$	
	(c_3,c_3)	1	
$i>0$ and even	(c_1,c_1)	$\frac{n-r-1}{2}$	n
	(c_1,c_2)	$r-1$	
	(c_2,c_2)	$\frac{n-r-1}{2}$	
	(c_1,c_3)	1	
	(c_2,c_3)	1	
$i>$, odd	(c_1,c_1)	$\frac{r-1}{2}$	n
	(c_1,c_2)	$n-r-1$	
	(c_2,c_2)	$\frac{r-1}{2}$	
	(c_1,c_3)	1	
	(c_2,c_3)	1	

Similarly, when $i>0$ and is odd, we have

$$\begin{aligned}\sum_{d(u,v)=i}(\zeta(u)+\zeta(v))x^{d(u,v)} &= \left[(2+4)\cdot\frac{r-1}{2}+3(n-r-1)+(4+5)\cdot 1\right]x^i\\ &= 3(n+1)x^i\end{aligned}$$

Therefore, $\mathcal{S}_{\chi^-}(C_n,x)=\sum\limits_{i=0}^{\frac{n+1}{2}}3(n+1)x^i=\frac{3(n+1)(1-x^{\frac{n+1}{2}})}{1-x}$, completing the proof. $\square$

Note that in the χ^--colouring of an even cycle C_n if we the colours c_1 and c_2, we get its χ^+-colouring. It can be observed that this change makes no change in the corresponding Schultz polynomial. But, for an odd cycle C_n, we have to interchange the colours c_1 and c_3 in its χ^--colouring and keep c_2 as it is to get a χ^+-colouring.

In view of this fact, the χ^+-chromatic Schultz polynomial of C_n is obtained in the following theorem.

Theorem 4. *Let C_n be a cycle on n vertices. Then, we have*

$$\mathcal{S}_{\chi^+}(C_n,x)=\begin{cases}\frac{3n(1-x^{\frac{n+2}{2}})}{1-x}; & \text{if } n \text{ is even;}\\ \frac{(5n-3)(1-x^{\frac{n+3}{2}})}{1-x}; & \text{if } n \text{ is odd.}\end{cases}$$

2.3. **Chromatic Schultz Polynomial of Complete Graphs.** Next, we consider the complete graph K_n. In K_n, we have $d(u,v) = 1$ for any two $u, v \in V(G)$. Therefore, $\mathcal{S}_{\chi^-}(K_n, x)$ and $\mathcal{S}_{\chi^+}(K_n, x)$ are the same and are first degree polynomials. The following result provides the Schultz polynomial of a complete graph K_n.

Proposition 1. *For $n \geq 2$, $\mathcal{S}_{\chi^-}(K_n, x) = \mathcal{S}_{\chi^+}(K_n, x) = (n^2 + n) + (2n^2 - n - 3)x$.*

Proof. In any proper vertex colouring, distinct vertices in K_n get distinct colours. Now, $\sum\limits_{v \in V} 2\zeta(v)x^0 = (2 + 4 + 6 + \ldots + 2n)x^0 = n(n+1)$. Also, we have

$$\begin{aligned} \sum_{d(u,v)=1} (\zeta(u) + \zeta(v))x^1 &= (3 + 4 + 5 + \ldots + (2n-1))x = \left(\frac{2n-3}{2}(2n+2)\right)x \\ &= (2n-3)(n+1)x. \end{aligned}$$

Therefore, $\mathcal{S}_{\chi^-}(K_n, x) = (n^2 + n) + (2n^2 - n - 3)x = \mathcal{S}_{\chi^+}(K_n, x)$. □

2.4. **Chromatic Schultz Polynomial of Complete Bipartite Graphs.** Next, let us consider the complete bipartite graphs $K_{a,b}$, where $a \geq b$.

Theorem 5. *For a complete bipartite $K_{a,b}, a \geq b, a+b = n$, we have $\mathcal{S}_{\chi^-}(K_n, x) = (2a + 4b) + 3abx + (a(a-1) + 2b(b-1))x^2$ and $\mathcal{S}_{\chi^+}(K_n, x) = (4a + 2b) + 3abx + (2a(a-1) + b(b-1))x^2$.*

Proof. Note that $K_{a,b}$ is 2-colourable and its diameter is 2. Since $a \geq b$, with respect to all a vertices in the first partition get the colour c_1 and all b vertices in the second partition get colour c_2. Then, we have the following table. Then,

Table 5. A list of color pairs and the distance between them in a complete bipartite graph.

Distance $d(u,v)$	Colour pairs	Number of pairs	Total number of pairs
$i = 0$	(c_1, c_1)	a	$a + b$
	(c_2, c_2)	b	
$i = 1$	(c_1, c_2)	ab	ab
$i = 2$	(c_1, c_1)	$\binom{a}{2}$	$\binom{a}{2} + \binom{b}{2}$
	(c_2, c_2)	$\binom{b}{2}$	

$$\begin{aligned} \mathcal{S}_{\chi^-}(K_{m,n}, x) &= (2a + 4b) + 3abx + (2 \cdot \binom{a}{2} + 4 \cdot \binom{b}{2})x^2 \\ &= (2a + 4b) + 3abx + (a(a-1) + 2b(b-1))x^2. \end{aligned}$$

In a similar way, by interchanging c_1 and c_2, we can prove that $\mathcal{S}_{\chi^+}(K_{m,n}, x) = (4a + 2b) + 3abx + (2a(a-1) + b(b-1))x^2$. □

3. Modified Chromatic Schultz Polynomials

Definition 3. Let G be a connected graph with chromatic number $\chi(G)$. Then, the *modified chromatic Schultz polynomial* of G, denoted by $\mathcal{S}^*_\chi(G,x)$, is defined as

$$\mathcal{S}^*_\chi(G,x) = \sum_{u,v\in V(G)} (\zeta(u)\zeta(v))x^{d(u,v)}.$$

Definition 4. Let G be a connected graph with chromatic number φ^- and $varphi^+$ be the minimal and maximal parameter colouring of G. Then,

(i) the *modified χ^--chromatic Schultz polynomial* of G, denoted by $\mathcal{S}^*_{\chi^-}(G,x)$, is defined as

$$\mathcal{S}^*_{\chi^-}(G,x) = \sum_{u,v\in V(G)} (\zeta_{\varphi^-}(u)\cdot\zeta_{\varphi^-}(v))x^{d(u,v)};$$

and

(ii) the *χ^+-chromatic Schultz polynomial* of G, denoted by $\mathcal{S}^*_{\chi^+}(G,x)$, is defined as

$$\mathcal{S}^*_{\chi^+}(G,x) = \sum_{u,v\in V(G)} (\zeta_{\varphi^+}(u)\cdot\zeta_{\varphi^+}(v))x^{d(u,v)}.$$

The following theorems discuss the modified chromatic Schultz polynomials of paths.

Theorem 6. *Let P_n be a path on n vertices. Then, we have*

$$\mathcal{S}^*_{\chi^-}(P_n,x) = \begin{cases} \sum\limits_{i=0}^{\frac{n-1}{2}} \left[(2n-4i-2)x + (\frac{5n-10i-3}{2})\right] x^{2i}; & \text{if } n \text{ is odd;}\\ \sum\limits_{i=0}^{\frac{n-1}{2}} \left[(2n-4i-2)x + (\frac{5n-10i}{2})\right] x^{2i}; & \text{if } n \text{ is even.}\end{cases}$$

Proof. If n is odd, then from Table 1, we have

$$\begin{aligned}\mathcal{S}^*_{\chi^-}(P_n,x) &= \sum_{r \text{ odd}} 2(n-r)x^r + \sum_{r \text{ even}}\left[\frac{n-r+1}{2}\cdot 1 + \frac{n-r-1}{2}\cdot 4\right]x^r\\ &= \sum_{r \text{ odd}} (2n-2r)x^r + \sum_{r \text{ even}}\left(\frac{5n-5r-3}{2}\right)x^r\\ &= \sum_{i=0}^{\frac{n-1}{2}} (2n-4i-2)x^{2i+1} + \sum_{i=0}^{\frac{n-1}{2}}\left(\frac{5n-10i-3}{2}\right)x^{2i}\\ &= \sum_{i=0}^{\frac{n-1}{2}}\left[(2n-4i-2)x + \left(\frac{5n-10i-3}{2}\right)\right]x^{2i}.\end{aligned}$$

If n is even, then from Table 2, we have

$$\begin{aligned}
\mathcal{S}^*_{\chi^-}(P_n,x) &= \sum_{r \text{ odd}} 2(n-r)x^r + \sum_{r \text{ even}} \left[\frac{n-r}{2}\cdot 1 + \frac{n-r}{2}\cdot 4\right] x^r \\
&= \sum_{r \text{ odd}} (2n-2r)x^r + \sum_{r \text{ even}} \left(\frac{5n-5r}{2}\right) x^r \\
&= \sum_{i=0}^{\frac{n-1}{2}} (2n-4i-2)x^{2i+1} + \sum_{i=0}^{\frac{n-1}{2}} \left(\frac{5n-10i}{2}\right) x^{2i} \\
&= \sum_{i=0}^{\frac{n-1}{2}} \left[(2n-4i-2)x + \left(\frac{5n-10i}{2}\right)\right] x^{2i}.
\end{aligned}$$

This completes the proof. □

Similarly, by interchanging the colours c_1 and c_2, we have the following result.

Theorem 7. *Let P_n be a path on n vertices. Then,*

$$\mathcal{S}^*_{\chi^+}(P_n,x) = \begin{cases} \sum\limits_{i=0}^{\frac{n-1}{2}} \left[(2n-4i-2)x + (\frac{5n-10i+3}{2})\right] x^{2i}; & \text{if } n \text{ is odd;} \\ \sum\limits_{i=0}^{\frac{n-1}{2}} \left[(2n-4i-2)x + (\frac{5n-10i}{2})\right] x^{2i}; & \text{if } n \text{ is even.} \end{cases}$$

The following theorems discuss the modified chromatic Schultz polynomials of cycles.

Theorem 8. *Let C_n be a cycle on n vertices. Then, we have*

$$\mathcal{S}^*_{\chi^-}(C_n,x) = \begin{cases} \sum\limits_{i=0}^{\frac{n}{2}} \left(2nx + \frac{5n}{2}\right) x^{2i}; & \text{if } n \text{ is even;} \\ \frac{5n+17}{2} + \sum\limits_{i=1}^{\frac{n-1}{2}} \left[(2n+9i+10)x + \frac{5n-18i+13}{2}\right] x^{2i}; & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is even and $r = d(u,v), u, v \in V(C_n)$, then from Table 3, we have

$$\begin{aligned}
\mathcal{S}^*_{\chi^-}(C_n,x) &= \sum_{r \text{ odd}} 2nx^r + \sum_{r \text{ even}} \left(\frac{n}{2}\cdot 1 + \frac{n}{2}\cdot 4\right) x^r \\
&= \sum_{r \text{ odd}} 2nx^r + \sum_{r \text{ even}} \frac{5n}{2} x^r \\
&= \sum_{i=0}^{\frac{n}{2}} 2nx^{2i+1} + \sum_{i=0}^{\frac{n}{2}} \frac{5n}{2} x^{2i} \\
&= \sum_{i=0}^{\frac{n}{2}} \left[2nx + \frac{5n}{2}\right] x^{2i}.
\end{aligned}$$

Let n be odd. Then, from Table 4,

$$\begin{aligned}
\mathcal{S}^*_{\chi^-}(C_n,x) &= \sum_{r=0}(\frac{n-1}{2}\cdot 1+\frac{n-1}{2}\cdot 4+9\cdot 1)+\\
&\quad \sum_{r>0 \text{ and odd}}\left(\frac{r-1}{2}(1+4)+2(n-r-1)+(3+6)\cdot 1\right)x^r+\\
&\quad \sum_{r>0 \text{ and even}}\left((1+4)\frac{n-r-1}{2}+2(r-1)+(3+6)\cdot 1\right)x^r\\
&= \frac{5n+17}{2}+\sum_{r \text{ odd}}\frac{4n+9r+11}{2}x^r+\sum_{r \text{ even}}\frac{5n-9r+13}{2}x^r\\
&= \frac{5n+17}{2}+\sum_{i=1}^{\frac{n-1}{2}}\frac{4n+18i+20}{2}x^{2i+1}+\sum_{i=1}^{\frac{n-1}{2}}\frac{5n-18i+13}{2}x^{2i}\\
&= \frac{5n+17}{2}+\sum_{i=1}^{\frac{n-1}{2}}\left[(2n+9i+10)x+\frac{5n-18i+13}{2}\right]x^{2i}.
\end{aligned}$$

This completes the proof. □

Similarly, interchanging c_1 and c_2 in even cycles and interchanging c_1 and c_3 in even cycles, we get

Theorem 9. *Let C_n be a cycle on n vertices. Then, we have*

$$\mathcal{S}^*_{\chi^+}(C_n,x)=\begin{cases}\frac{13n-11}{2}+\sum\limits_{i=1}^{\frac{n-1}{2}}\left[(6n+i-7)x+\frac{13n-2i-15}{2}\right]x^{2i}; & \text{if } n \text{ is odd;}\\ \sum\limits_{i=0}^{\frac{n}{2}}\left(2nx+\frac{5n}{2}\right)x^{2i}; & \text{if } n \text{ is even.}\end{cases}$$

The following result provides the modified Schultz polynomial of a complete bipartite graph $K_{a,b}$.

Theorem 10. *For a complete bipartite $K_{a,b}, a\geq b, a+b=n$, we have $\mathcal{S}^*_{\chi^-}(K_n,x)=(a+4b)+2abx+\left(\frac{a(a-1)}{2}+2b(b-1)\right)x^2$ and $\mathcal{S}_{\chi^+}(K_n,x)=(4a+b)+2abx+\left(2a(a-1)+\frac{b(b-1)}{2}\right)x^2$.*

Proof. The proof similar to that of Theorem 5. □

4. Conclusion

In this article, we have introduced a particular type of polynomial, called chromatic Schultz polynomial of graphs, as an analogue of the Schultz polynomial of graphs and determined this polynomial for certain fundamental graphs.

The study seems to be promising for further studies as the polynomial can be computed for many graph classes and classes of derived graphs. The chromatic Schultz polynomial can be determined for graph operations, graph products and graph powers. The study on Schultz polynomials with respect to different types of graph colourings also seem to be much promising. The concept can be extended to edge colourings and map colourings also.

These polynomials have so many applications in various fields like Mathematical Chemistry, Distribution Theory, Optimisation Techniques etc. In Chemistry, some interesting studies using the above-mentioned concepts are possible if $c(v_i)$ (or $\zeta(v_i)$) assumes the values such as energy, valency, bond strength etc. Similar studies are possible in various other fields. All these facts highlight the wide scope for further research in this area.

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References

[1] Abdullah, M. M., Ali, A. M., Schultz and modified Schultz polynomials for edge-identification chain and ring-for square graphs, *Baghdad Sci. J.*, 19(3) (2022), 560–568.

[2] Abdullah, M. M., Ali, A. M., Schultz and modified Schultz polynomials for edge-identification chain and ring for pentagon and hexagon graphs, *J. Phys: Conf. Ser.* , 1818(1) (2021), 012063.

[3] Brandstädt, A., Le, V. B., Spinrad, J. P., Graph Classes: A Survey, SIAM, Monographs on Discrete Mathematics and Applications, Philadelphia, 1999. http://dx.doi.org/10.1137/1.9780898719796

[4] Eliasi, M., Taeri, B., Schultz polynomials of composite graphs, *Appl. Anal. Discrete Math.*, 2 (2008), 285–296. doi:10.2298/AADM0802285E

[5] Eu, S. P., Yang, B. Y., Yeh, Y. N., Theoretical and computational developments generalized Wiener indices in hexagonal chains, *Int. J. Quantum Chem.*, 106(2) (2006), 426–435.

[6] Jensen, T. R., Toft, B., Graph Colouring Problems, John Wiley & Sons, New York, 1995.

[7] Kubale, M., Graph Colourings, American Math. Soc., Rhode Island, 2004.

[8] Kok, J., Sudev, N. K., Mary, U., On chromatic Zagreb indices of certain graphs, *Discrete Math. Algorithm. Appl.*, 9(1) (2017), 1–11. DOI:10.1142/S1793830917500148.

[9] Rose S., David, I., Naduvath, S., On chromatic *D*-polynomial of graphs, *Contemp. Stud. Discrete Math.*, 2(1) (2018), 31–43.

[10] West, D. B., Introduction to Graph Theory, Pearson Education, Delhi, 2001.