TURKISH JOURNAL OF SCIENCE
VOLUME 7, ISSUE 1, 21-30
ISSN: 2587–0971

http://dergipark.gov.tr/tjos

Lightlike Hypersurfaces of Poly-Norden Semi-Riemannian Manifolds

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Abstract. In this article, we initiate the study of lightlike hypersurfaces in a poly-Norden semi-Riemannian manifold. We introduce invariant and screen semi-invariant lightlike hypersurfaces of a poly-Norden semi-Riemannian manifold. Also, we give some examples of such hypersurfaces.

1. Introduction

In differential geometry, submanifolds equipped with different geometric structure have been studied widely. A submanifold of a semi-Riemann manifold is known a lightlike submanifold if the induced metric is degenerate. The general theory of lightlike submanifold has been examined in [1] (see also [4]). On this subject, some applications of the theory mathematical physics is inspired, especially electromagnetisms [1], black hole theory [4] and general relativity [5]. Many studies on lightlike submanifolds have been reported by many geometers (see [2], [3], [6], [7], [8]).

The golden proportion and the golden rectangle have been found in the harmonious proportion of temples, fractals, paintings etc. Golden structure was revealed by the golden proportion which was charecterized by J. Kepler. The number φ, which is the real positive root of

$$x^2 - x - 1 = 0,$$

(hence $\varphi = \frac{1+\sqrt{5}}{2}$) is the golden proportion. In [9], inspired by golden ratio, golden Riemannian manifolds were introduced. Then many geometers have studied golden (semi) Riemannian manifolds on different manifolds ([10], [11], [12], [13]).

As a generalization of the golden mean, metallic mean family was studied by V. W. de Spinadel [14]. The positive solution of the equation

$$x^2 - px - q = 0,$$

is called member of the metallic means family, where p and q are fixed two positive integers. These number denoted by;

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2},$$

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Received: 25 April 2021; Accepted: 25 February 2022; Published: 30 April 2022
Keywords. Almost poly-Norden structure, poly-Norden manifold, Lightlike hypersurface
2010 *Mathematics Subject Classification*. 53C15; 53C40
Cited this article as: Kılıç E, Acet T, Yüksel Perktaş S. Lightlike Hypersurfaces of Poly-Norden Semi-Riemannian Manifolds, Turkish Journal of Science. 2022, 7(1), 21-30.

are also known $(p, q)-$metallic numbers. Recently many paper about metallic mean have been published ([15], [16], [17], [18]).

On the other hand in [19], the authors has defined Bronze mean which is different from Bronze mean given in [20]. Also note that there is no inclusion relation between the Bronze mean defined in [19] and metallic mean.

In [21], B. Şahin introduce as a new type of manifold which is called almost poly-Norden manifolds and study the geometry of such manifolds. Recently S. Yüksel Perktaş defined and studied submanifolds of almost poly-Norden Riemannian manifolds in [22].

In this article, by inspring from [21] and [15], we study lightlike hypersurfaces of almost poly-Norden manifolds.

2. Preliminaries

The bronze mean [19] which is the positive solution of the equation $x^2 - mx + 1 = 0$, is defined by

$$B_m = \frac{m + \sqrt{m^2 - 4}}{2}. \tag{1}$$

The Bronze Fibonacci numbers $(f_{m,n})$ (resp., the Bronze Lucas numbers $(l_{m,n})$) are the family of sequences defined by recurrence

$$f_{m,n+2} = mf_{m,n+1} - f_{m,n}, \quad (\text{resp.}, l_{m,n+2} = ml_{m,n+1} - l_{m,n}),$$

where $f_{m,0} = 0$ and $f_{m,1} = 1$ (resp., $l_{m,0} = 2$ and $l_{m,1} = m$). The Bronze Fibonacci numbers and Bronze Lucas numbers are related by

$$B_m^n = \frac{l_{m,n} + f_{m,n}\sqrt{m^2 - 4}}{2}.$$

Also note that the recurrence relation $B_m^{n+2} = mB_m^{n+1} - B_m^n$ is satisfied and the covergents of B_m^a are $\frac{f_{m,a(n+1)}}{f_{m,an}}$ [19].

By being inspired of the Bronze mean (1) defined by S. Kalia [19], a new structure on a differentiable manifold which is called a poly-Norden structure was introduced by B. Şahin [21].

Definition 2.1. *[21] On a manifold $\check{M}$, a poly-Norden structure is defined by a $(1, 1)$-tensor field Φ which satisfies*

$$\Phi^2 = m\Phi - I, \tag{2}$$

where I is the identity operator on $\check{M}$. So, $(\check{M}, \Phi)$ is called an almost poly-Norden manifold.

Example 2.2. *[21] Let Φ be a map defined by*

$$\begin{aligned} \Phi \quad &: \quad \mathbb{R}^4 \to \mathbb{R}^4 \\ (u_1, u_2, u_3, u_4) \quad &\to \quad (B_m u_1, B_m u_2, \bar{B}_m u_3, \bar{B}_m u_4), \end{aligned}$$

where $B_m = \frac{m+\sqrt{m^2-4}}{2}$, $\bar{B} = m - B_m$ and (u_1, u_2, u_3, u_4) is the standard coordinate system on $\mathbb{R}^4$. One can easily see that Φ satisfies (2). Thus $(\mathbb{R}^4, \Phi)$ is a poly-Norden manifold.

A semi-Riemannian metric $\breve{g}$ is called Φ-compatible, if it satisfies

$$\breve{g}(\Phi X, \Phi Y) = m\breve{g}(\Phi X, Y) - \breve{g}(X, Y), \tag{3}$$

which yields

$$\breve{g}(\Phi X, Y) = \breve{g}(X, \Phi Y). \tag{4}$$

Definition 2.3. *[21] Let $(\check{M}, \breve{g})$ be a semi-Riemannian manifold endowed with a poly-Norden structure Φ. If the semi-Riemannian metric $\breve{g}$ is Φ-compatible, then the manifold is named an almost poly-Norden semi-Riemannain manifold and $(\breve{g}, \Phi)$ is called an almost poly-Norden semi-Riemannian structure on $\check{M}$.*

From now on, we shall consider that $m \neq 0$.

We note that the eigenvalues of Φ are $\frac{m+\sqrt{m^2-4}}{2}$ and $\frac{m-\sqrt{m^2-4}}{2}$. The inverse of Φ is not an almost poly-Norden structure. Additionally, each complex structure on a semi-Riemannian manifold allows defining two poly-Norden structures in the forms [21]

$$\Phi_1 = \frac{m}{2}I + \left(\frac{\sqrt{4-m^2}}{2}\right)J, \quad \Phi_1 = \frac{m}{2}I - \left(\frac{\sqrt{4-m^2}}{2}\right)J, \quad -2 < m < 2.$$

Conversely, each poly-Norden structure Φ on the manifold gives rise to following two almost complex structures on this manifold,

$$J_1 = \frac{-m}{\sqrt{4-m^2}}I + \frac{2}{\sqrt{4-m^2}}\Phi, \quad J_2 = \frac{m}{\sqrt{4-m^2}}I - \frac{2}{\sqrt{4-m^2}}\Phi, \quad -2 < m < 2.$$

Definition 2.4. *[21] Let $(\breve{M}, \breve{g}, \Phi)$ be an almost poly-Norden semi-Riemannian manifold. If the almost poly-Norden structure is parallel with respect to the Levi-Civita connection $\breve{\nabla}$ then $(\breve{M}, \breve{g}, \Phi)$ is called a poly-Norden semi-Riemannian manifold.*

Let $\breve{M}$ be a semi-Riemannian manifold equipped with a semi-Riemannian metric $\breve{g}$ of index q, $0 < q < 2n+1$, and M is a hypersurface of $\breve{M}$ with the induced metric $g = \breve{g}\mid_M$. If the induced metric g is degenerate and the orthogonal complement $TM^{\perp}$ of tangent space TM, given as

$$TM^{\perp} =_{p\in M} \{V_p \in T_p\breve{M} : g_p(U_p, V_p) = 0, \forall U \in \Gamma(T_pM)\}$$

is a distribution of rank 1 on M, then M is called a lightlike hypersurface of $\breve{M}$ [1]. In this case, $TM^{\perp} \subset TM$ and then it coincides with the radical distribution $Rad\, TM = TM \cap TM^{\perp}$.

The complementary bundle of $TM^{\perp}$ in TM, namely screen distribution, is a non-degenerate distribution of constant rank $2n-1$ over M and denoted by $S(TM)$.

Theorem 2.5. *[1] Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $\breve{M}$. Then there exists a unique rank 1 vector subbundle $ltr(TM)$ of $T\breve{M}$, with base space M, such that for any non-zero section E of $Rad\, TM$ on a coordinate neighbourhood $\mathfrak{I} \subset M$, there exists a unique section N of $ltr(TM)$ on $\mathfrak{I}$ satisfying:*

$$\breve{g}(N,N) = 0, \quad \breve{g}(N,W) = 0, \quad \breve{g}(N,E) = 1, \quad \text{for } W \in \Gamma(S(TM)\mid_{\mathfrak{I}}\,.$$

$ltr(TM)$ is called the lightlike transversal vector bundle of M with respect to $S(TM)$.

Therefore we get

$$TM = S(TM) \perp Rad\, TM, \tag{5}$$

$$\begin{aligned} T\breve{M} &= TM \oplus ltr(TM) \\ &= S(TM) \perp \{Rad\, TM \oplus ltr(TM)\}. \end{aligned} \tag{6}$$

Let $\omega : \Gamma(TM) \to \Gamma(S(TM))$ be the projection morphism of TM. So we have

$$\breve{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{7}$$

$$\breve{\nabla}_X N = -A_N X + \tau(X)N, \tag{8}$$

$$\nabla_X \omega Y = \nabla^*_X \omega Y + C(X, \omega Y)E, \tag{9}$$

$$\nabla_X E = -A^*_E X - \tau(X)E, \tag{10}$$

where ∇ (resp., ∇^*) is a linear connection on M (resp., $S(TM)$) and B, A_N and τ are called the local second fundamental form, the local shape operator, the transversal differential 1−form, respectively.

The induced linear connection ∇ is not a metric connection in general and we have

$$(\nabla_X g)(Y,Z) = B(X,Z)\theta(Y) + B(X,Y)\theta(Z), \tag{11}$$

where θ is a differential 1-form such that

$$\theta(X) = \breve{g}(N,X). \tag{12}$$

3. LIGHTLIKE HYPERSURFACES OF ALMOST POLY-NORDEN SEMI-RIEMANNIAN MANIFOLDS

Let M be a lightlike hypersurface of an almost poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then, for every $X \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$, we write

$$\Phi X = \phi X + u(X)N, \tag{13}$$

$$\Phi N = \zeta + v(E)N, \tag{14}$$

where ϕX, $\zeta \in \Gamma(TM)$ and u, v are 1-forms given by

$$u(X) = g(X, \Phi E), \qquad v(X) = g(X, \Phi N). \tag{15}$$

Lemma 3.1. *Let M be a lightlike hypersurface of an almost poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then we have*

$$\phi^2 X = m\phi X - X - u(X)\zeta, \tag{16}$$

$$u(\phi X) = -mu(X) - u(X)v(E), \tag{17}$$

$$\phi\zeta = m\zeta - v(E)\zeta, \tag{18}$$

$$v(E)^2 = mv(E) - 1 - u(\zeta), \tag{19}$$

$$g(\phi X, Y) = g(X, \phi Y) - u(X)\theta(Y) + u(Y)\theta(X), \tag{20}$$

$$\begin{aligned} g(\phi X, \phi Y) &= mg(X, \phi Y) - g(X, Y) + mu(Y)\theta(X) \\ &\quad - u(Y)g(\phi X, N) - u(X)g(\phi Y, N). \end{aligned} \tag{21}$$

In case of $\breve{M}$ is being a poly-Norden semi-Riemannian manifold, we give the following:

Lemma 3.2. *Let M be a lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then we have*

$$(\nabla_X \phi)Y = u(Y)(A_N X) + B(X, Y)\zeta, \tag{22}$$

$$(\nabla_X u)Y = v(E)(B(X, Y)) - B(X, \phi Y) - u(Y)\tau(X), \tag{23}$$

$$\nabla_X \zeta = -\phi A_N X + \tau(X)\zeta + v(E)(A_N X), \tag{24}$$

$$X(v(E)) = -B(X, \zeta) - u(A_N X). \tag{25}$$

4. INVARIANT LIGHTLIKE HYPERSURFACES OF A POLY-NORDEN SEMI-RIEMANNIAN MANIFOLD

Definition 4.1. *Let M be a lightlike hypersurface of an almost poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then M is called an invariant lightlike hypersurface of $\breve{M}$ if*

$$\begin{aligned} \Phi(Rad\,TM) &= Rad\,TM, \\ \Phi(ltr(TM)) &= ltr(TM). \end{aligned} \tag{26}$$

Example 4.2. *Let $\breve{M} = \mathbb{R}^7_3$ be a semi-Euclidean space with coordinate system $(x_1, x_2, ..., x_7)$ and signature $(-, +, -, +, -, +, +)$. Taking*

$$\Phi(x_1, x_2, ..., x_7) = (B_m x_1, B_m x_2, B_m x_3, B_m x_4, B_m x_5, B_m x_6, B_m x_7),$$

then Φ is an almost poly-Norden structure on $\breve{M}$.

Now, we consider a hypersurface M of $\breve{M}$ with

$$x_5 = x_7.$$

Then TM of M is spanned by

$$\Pi_1 = \frac{\partial}{\partial x_1}, \quad \Pi_2 = \frac{\partial}{\partial x_2},$$
$$\Pi_3 = \frac{\partial}{\partial x_3}, \quad \Pi_4 = \frac{\partial}{\partial x_4}, \quad \Pi_5 = \frac{\partial}{\partial x_6},$$
$$\Pi_6 = \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7}.$$

In this case, Rad TM and ltr(TM) are given by

$$Rad\,TM = Sp\{E = \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7}\},$$

and

$$ltr(TM) = Sp\{N = -\frac{1}{2}(\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7})\},$$

respectively. Thus, we find

$$\Phi E = B_m E \quad \text{and} \quad \Phi N = B_m N,$$

which implies that M is an invariant lightlike hypersurface of $\breve{M}$.

Theorem 4.3. *Let M be a lightlike hypersurface of an almost poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then ϕ is an almost poly-Norden structure on M.*

Proof. It is well known that, M is an invariant lightlike hypersurface if and only if

$$\Phi X = \phi X,$$

that is

$$u(X) = 0.$$

Then, from (16) and (20), we get

$$\phi^2 X = m\phi X - X,$$

and

$$g(\phi X, Y) = g(X, \phi Y).$$

So, we get our assertion. □

Theorem 4.4. *Let M be an invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then we have*

$$\begin{aligned} B(X, \Phi Y) &= B(\Phi X, Y) = \Phi B(X, Y), \\ B(\Phi X, \Phi Y) &= mB(X, \Phi Y) + B(X, Y). \end{aligned}$$

Proof. It is obvious from (7). □

5. SCREEN SEMI-INVARIANT LIGHTLIKE HYPERSURFACES OF A POLY-NORDEN SEMI-RIEMANNIAN MANIFOLD

Definition 5.1. *Let M be a lightlike hypersurface of an almost poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. If*

$$\begin{aligned} \Phi(Rad\,TM) &\subset S(TM), \\ \Phi(ltr(TM)) &\subset S(TM), \end{aligned} \tag{27}$$

then M is called a screen semi-invariant lightlike hypersurface of $\breve{M}$.

Example 5.2. *Let $\breve{M} = \mathbb{R}^5_2$ be semi-Euclidean space with coordinate system $(x^1, x^2, x^3, x^4, x^5)$ and signature $(-,+,-,+,+)$. Taking*

$$\Phi(x^1, x^2, x^3, x^4, x^5) = ((m - B_m)x^1, (m - B_m)x^2, B_m x^3, B_m x^4, B_m x^5),$$

then we can say that Φ is a poly-Norden structure on $\breve{M}$.

Consider a hypersurface M of $\breve{M}$ with

$$x^5 = B_m x^1 + B_m x^2 + x^3.$$

Then TM of M is spanned by

$$\Omega_1 = \frac{\partial}{\partial x^1} + B_m \frac{\partial}{\partial x^5}, \quad \Omega_2 = \frac{\partial}{\partial x^2} + B_m \frac{\partial}{\partial x^5}$$

$$\Omega_3 = \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^5}, \quad \Omega_4 = \frac{\partial}{\partial x^4}$$

So, $Rad\,TM$ and $ltr(TM)$ are given by

$$Rad\,TM = Sp\{E = B_m\Omega_1 - B_m\Omega_2 + \Omega_3\},$$

$$ltr(TM) = Sp\left\{N = \frac{1}{2}\left(-B_m \frac{\partial}{\partial x^1} + B_m \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3} + \frac{\partial}{\partial x^5}\right)\right\}.$$

Also $S(TM)$ is spanned by $\{\Pi_1, \Pi_2, \Pi_3\}$, where

$$\Pi_1 = -\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + B_m \frac{\partial}{\partial x^3} + B_m \frac{\partial}{\partial x^5},$$

$$\Pi_2 = \frac{1}{2}\left\{ \begin{array}{c} -\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} \\ -B_m \frac{\partial}{\partial x^3} + B_m \frac{\partial}{\partial x^5} \end{array} \right\},$$

$$\Pi_3 = \frac{\partial}{\partial x_4}.$$

Thus we arrive at

$$\Pi_1 = \Phi E \quad \text{and} \quad \Pi_2 = \Phi N,$$

which imply that M is a screen semi-invariant lightlike hypersurface of $\breve{M}$.

We know that $S(TM)$ is non-degenerate, so we can define a distribution ϑ such that

$$S(TM) = \{\Phi(Rad\,TM) \oplus \Phi(ltr(TM))\} \perp \vartheta, \tag{28}$$

from which

$$TM = \{\Phi(Rad\,TM) \oplus \Phi(ltr(TM))\} \perp \vartheta \perp Rad\,TM, \tag{29}$$

$$T\breve{M} = \{\Phi(Rad\,TM) \oplus \Phi(ltr(TM))\} \perp \vartheta \perp \{Rad\,TM \oplus ltr(TM)\}. \tag{30}$$

Taking $\hat{D} = Rad\,TM \perp \Phi(Rad\,TM) \perp \vartheta$ and $\mathring{D} = \Phi(ltr(TM))$ on M. So, we get

$$TM = \hat{D} \oplus \mathring{D}. \tag{31}$$

Let $\xi = \Phi N$ and $\Psi = \Phi E$ be local lightlike vector fields. For $X \in \Gamma(TM)$, we can write

$$X = RX + QX, \tag{32}$$

where R and Q are projections of TM into $\hat{D}$ and $\mathring{D}$, respectively.

Also, for $X, Y \in \Gamma(TM)$, $\xi \in \mathring{D}$ and $\Psi \in \hat{D}$,

$$\phi^2 X = m\phi X - X - u(X)\xi, \tag{33}$$

$$u(\phi X) = mu(X), \quad u(\xi) = -1, \tag{34}$$

$$g(X, \phi Y) = g(\phi X, Y) + u(X)\theta(Y) - u(Y)\theta(X), \tag{35}$$

$$\begin{aligned} g(\phi X, \phi Y) &= mg(X, \phi Y) - g(X, Y) - mu(Y)\theta(X) \\ &\quad -u(Y)g(\phi X, N) - u(X)g(\phi Y, N), \end{aligned} \tag{36}$$

$$(\nabla_X \phi)Y = g(A_E^* X, Y)\xi + u(Y)A_N X, \tag{37}$$

$$\nabla_X \xi = -\phi A_N X + \tau(X)\xi, \tag{38}$$

$$\nabla_X \Psi = -\phi A_E^* X - \tau(X)\Psi \tag{39}$$

$$B(X, \xi) = -C(X, \Psi). \tag{40}$$

Theorem 5.3. *Assume that M is a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $\breve{M}$. Then the lightlike vector field Ψ is parallel on M if and only if*
i) M is totally geodesic on $\breve{M}$,
ii) $\tau = 0$.

Proof. Assume that Ψ is a parallel vector fields. From (13) and (39), for any $X \in \Gamma(TM)$, we have

$$\begin{aligned} 0 &= -\phi A_E^* X - \tau(X)\Psi \\ &= -\Phi A_E^* X + u(A_E^* X)N - \tau(X)\Psi. \end{aligned} \tag{41}$$

Applying Φ to (41) and in view of (13) with (2), we get

$$-m\phi(A_E^* X) - mu(A_E^* X)N + A_E^* X - m\tau(X)\Psi + \tau(X)E + u(A_E^* X)\xi = 0. \tag{42}$$

Taking tangential and transversal part of equation (42), we arrive at

$$A_E^* X = -\tau(X)E - u(A_E^* X)\xi, \quad mu(A_E^* X) = 0.$$

So, we get the proof of our assertion. □

Theorem 5.4. *Assume that M is a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $\breve{M}$. Then the lightlike vector field ξ is parallel on M if and only if M and $S(TM)$ is totally geodesic on $\breve{M}$.*

Proof. Since ξ is parallel vector fields on M, in view of (13) and (38), for any $X \in \Gamma(TM)$, we have

$$\begin{aligned} 0 &= -\phi A_N X + \tau(X)\xi \\ &= -\Phi A_N X + u(A_N X)N + \tau(X)\xi. \end{aligned} \tag{43}$$

Applying Φ to (43) and by use of (13) with (2), we get

$$-m\phi(A_N X) - mu(A_N X)N + A_N X + m\tau(X)\xi - \tau(X)N + u(A_N X)\xi = 0. \tag{44}$$

Taking tangential and transversal part of equation (44), we find

$$A_N X = -u(A_N X)\xi, \quad mu(A_N X) = \tau(X).$$

This completes the proof. □

Definition 5.5. *Let M be a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. If the second fundamental form*

$$B(X, Z) = 0,$$

for any $X \in \Gamma(\hat{D})$ and $Z \in \Gamma(\mathring{D})$, then M is called a mixed geodesic lightlike hypersurface.

Theorem 5.6. *Let M be a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then M is a mixed geodesic lightlike hypersurface if and only if*
i) There is no component of A_N, $\hat{D}$−valuable.
ii) There is no component of A^*_E, $\mathring{D}$−valuable.

Proof. Assume that M is mixed geodesic, i.e.

$$B(X, \xi) = 0. \tag{45}$$

In view of (4) and (8) in (45), we have

$$\begin{aligned} 0 &= B(X, \xi) = B(X, \Phi N) \\ &= \breve{g}(\breve{\nabla}_X \Phi N, E) \\ &= \breve{g}((\breve{\nabla}_X \Phi)N + \Phi\breve{\nabla}_X N, E) \\ &= \breve{g}(\breve{\nabla}_X N, \Phi E) \\ &= -\breve{g}(A_N X, \Phi E). \end{aligned}$$

Therefore we arrive at (i).

On the other hand, since

$$-\breve{g}(A_N X, \Phi E) = \breve{g}(A^*_E X, \Phi N),$$

we obtain (ii). □

Now, we consider the distribution ϑ. From (29) and taking

$$\beta = \{\Phi(Rad(TM)) \oplus \Phi(ltr(TM))\} \perp Rad(TM),$$

for any $X \in \Gamma(TM)$, $Y \in \Gamma(\vartheta)$ and $Z \in \Gamma(\beta)$, we can write

$$\nabla_X Y = \overset{\vartheta}{\nabla}_X Y + \overset{\vartheta}{h}(X, Y), \tag{46}$$

$$\nabla_X Z = -\overset{\vartheta}{A}_Z X + \nabla^\perp_X Z, \tag{47}$$

where $\overset{\vartheta}{h} : \Gamma(TM) \times \Gamma(\vartheta) \to \Gamma(\beta)$ is an $\Im(M)$ bilinear, $\overset{\vartheta}{A}$ is an $\Im(M)$ linear operator on $\Gamma(\vartheta)$, $\overset{\vartheta}{\nabla}$ and $\nabla^\perp$ is a linear connection on ϑ and β, respectively.

Let $\Im \subset M$ be a coordinate neighborhood. If we consider the decomposition (29), we take

$$\begin{aligned} \rho_1(U, Y) &= -g(\overset{\vartheta}{h}(U, Y), \Phi N), \\ \rho_2(U, Y) &= -g(\overset{\vartheta}{h}(U, Y), \Phi E), \\ \rho_3(U, Y) &= g(\overset{\vartheta}{h}(U, Y), N), \end{aligned} \tag{48}$$

for any $U, Y \in \Gamma(\vartheta\,|_{\Im})$.

Therefore, from (3), we get

$$\breve{\nabla}_U Y = \overset{\vartheta}{\breve{\nabla}}_U Y - \rho_1(U, Y)\Phi E - \rho_2(U, Y)\Phi N + \rho_3(U, Y)E. \tag{49}$$

If we compute ρ_1, ρ_2 and ρ_3 in terms of B and C we arrive at

$$\begin{aligned} g(\breve{\nabla}_U Y, \Phi N) = \rho_1(U, Y) &= -C(U, \Phi Y), \\ g(\breve{\nabla}_U Y, \Phi E) = \rho_2(U, Y) &= -B(U, \Phi Y), \\ g(\breve{\nabla}_U Y, N) = \rho_3(U, Y) &= -C(U, Y). \end{aligned} \tag{50}$$

Thus, we can rewrite equation (49) with

$$\nabla_U Y = \overset{\vartheta}{\breve{\nabla}}_U Y - C(U, \Phi Y)\Phi E - B(U, \Phi Y)\Phi N - C(U, Y)E. \tag{51}$$

Theorem 5.7. *Let M be a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then the distribution ϑ is integrable if and only if*

$$C(U, \Phi Y) = C(\Phi U, Y), \quad B(U, \Phi Y) = B(\Phi U, Y), \quad C(U, Y) = C(Y, U), \tag{52}$$

for every $U, Y \in \Gamma(\vartheta)$.

Proof. We know that $\breve{\nabla}$ is a linear connection. Therefore, in view of (51), we have

$$\begin{aligned} [U, Y] &= \overset{\vartheta}{\nabla}_U Y - \overset{\vartheta}{\nabla}_Y U \\ &\quad + (C(U, \Phi Y) - C(\Phi U, Y))\Phi E \\ &\quad + (B(U, \Phi Y) - B(\Phi U, Y))\Phi N \\ &\quad + (C(U, Y) - C(Y, U))E. \end{aligned}$$

If ϑ is integrable then the components of $[U, Y]$ with respect to $\Phi E, \Phi N$ and E vanish. Thus, we get (52).
Contrary to, if (52) is satisfied we arrive at

$$[U, Y] \in \Gamma(\vartheta).$$

This completes the proof. □

Theorem 5.8. *Let M be a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then the distribution $\hat{D}$ is integrable if and only if*

$$B(\Phi U, \Phi Y) = mB(\Phi U, Y) - B(U, Y), \tag{53}$$

for every $U, Y \in \Gamma(\hat{D})$.

Proof. If we take $Y \in \Gamma(\hat{D})$, we get $\Phi Y \in \Gamma(\hat{D})$. Then $\hat{D}$ is integrable if and only if

$$\begin{aligned} \breve{g}([\Phi U, Y], \Phi E) &= \breve{g}(\breve{\nabla}_{\Phi U} Y, \Phi E) - \breve{g}(\breve{\nabla}_Y \Phi U, \Phi E) \\ &= \breve{g}(\Phi \breve{\nabla}_{\Phi U} Y, E) - \breve{g}(\Phi \breve{\nabla}_Y U, \Phi E) \\ &= \breve{g}(\breve{\nabla}_{\Phi U} \Phi Y, E) - m\breve{g}(\breve{\nabla}_Y U, \Phi E) + \breve{g}(\breve{\nabla}_Y U, E) \\ &= B(\Phi U, \Phi Y) - mB(\Phi U, Y) + B(U, Y), \end{aligned}$$

which yields (53). □

Theorem 5.9. *Let M be a screen semi-invariant lightlike hypersurface of a poly-Norden semi-Riemannian manifold $(\breve{M}, \breve{g}, \Phi)$. Then the distribution $\hat{D}$ is parallel if and only if $\hat{D}$ is totally geodesic on M.*

Proof. From the definition of the distribution $\mathring{D}$ we know that $\mathring{D}$ is parallel if and only if

$$g(\nabla_U Y, \Psi) = 0.$$

From this equation, we get

$$\begin{aligned} 0 &= g(\breve{\nabla}_U Y, \Psi) \\ &= \breve{g}(\breve{\nabla}_U Y, \Psi) \\ &= \breve{g}(\breve{\nabla}_U Y, \Phi E) \\ &= \breve{g}(\Phi \nabla_U Y, E) \\ &= \breve{g}(-((\breve{\nabla}_U \Phi) Y + \breve{\nabla}_U \Phi Y, E) \\ &= \breve{g}(\breve{\nabla}_U \Phi Y, E) = B(U, \tilde{J} Y), \end{aligned}$$

which gives the proof of our assertion. □

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