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A hybrid algorithm for solving fractional Fokker-Planck equations arising in physics and engineering

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Abstract — In this work, we proposed a hybrid algorithm to approximate the solution of Conformable Fractional Fokker-Planck Equation (CFFPE). This algorithm comprises of unification of two methods named Fractional Wave Transformation Method (FWTM) and Differential Transform Method (DTM). The method is based on two steps. The first step is to reduce the given CFPDEs to corresponding Partial Differential Equations (PDEs). Then, the second step is to solve obtained PDEs iteratively by using DTM. Moreover, the algorithm's efficiency is shown by employing the method successfully to conformable time-fractional Fokker-Planck equation arising in surface physics, plasma physics, polymer physics, laser physics, biophysics, engineering, neurosciences, nonlinear hydrodynamics, population dynamics, pattern formation and marketing. As a result, the obtained data demonstrate that the algorithm is reliable and applicable.

Subject Classification (2020): 35R11, 35A20.

Nomenclature

DTM differential transform method
FWTM fractional wave transformation method
CFFPE Conformable Fractional Fokker-Planck Equations
PDE Partial Differential Equations
u unknown function
A drift coefficient
B diffusion coefficient
x coordinate
t time
Γ gamma function
U differential transform of u
T wave transform of t
X wave transform of x
k, h order of differential transform
α, β fractional orders

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1. Introduction

In the last decade, arbitrary order calculus has seen rapid growth and is widely used in various applications of diverse fields in science and technology. Differential equations involving fractional-order derivatives are progressively implemented to model problems in fluid mechanics, chemistry, biology, electromagnetism, fluid mechanics, signal processing, material science and many other physical processes [1,2]. Solving fractional partial differential equations has attracted considerable attention from scientists in the last decades. As is known that the analytical results for most fractional differential equations cannot be acquired. However, it is possible to get an approximate solution for these problems using numerical methods such as linearization or perturbation. In the '90s, some mathematical tools, such as the Adomian decomposition method [3], variational iteration method [4], homotopy perturbation method [5], and homotopy analysis method [6] have been found to solve various fractional problems analytically. The differential transform method (DTM) was expressed by Zhou in 1986 [7]. With the aid of this method, the considered differential equation can be changed into a recurrence equation and then we can construct the approximate analytical results in a polynomial form.

Scientists have been applying various numerical and analytical methods to obtain solutions to mathematical models arising in nature. The transform methods are used more often due to their simplicity and efficiency. There are many transform methods to solve those problems, and the most popular ones are the Laplace transform [8], the Fourier transform [9], the integral transform (IT) [10], and the fractional IT [11].

This study regards the conformable fractional Fokker-Planck equation [12,13]. Some critical applications of the Fokker-Planck equation can be seen in various fields, such as surface physics, plasma physics, polymer physics, laser physics, biophysics, engineering, neurosciences, nonlinear hydrodynamics, population dynamics, pattern formation and marketing [12-14]. The Conformable Fractional Fokker-Planck Equations (CFFPEs) can be written in the following form. Assume that u(x, t) that depends on the space and time variables is the solution of the CFFP1

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\left[-\frac{\partial^\beta}{\partial x^\beta}A(t,x,u)+\frac{\partial^{2\beta}}{\partial x^{2\beta}}B(t,x,u)\right]u(t,x), t>0 \text{ and } 0<\alpha,\beta\leq 1 \tag{1.1}$$

where the parameter α denotes the order of the conformable fractional time derivative, while β denotes the order of the conformable fractional space derivative and $\frac{\partial^{2\beta}}{\partial x^{2\beta}}$ denotes two times successive conformable fractional derivative also, $A(t,x,u)$ and $B(t,x,u)$ are conformable differentiable functions. One can obtain the classical Fokker-Planck equation [14] by letting $\alpha=1$ and $\beta=1$.

The key idea of the method expressed in this paper is to convert CFPDEs into integer order PDEs by using FWTM. After this transformation, we can find the solutions to the considered equation using DTM. The rest of the paper is organized as follows: Section 2 presents the basics of conformable fractional calculus. Section 3 expresses the basic idea of DTM. Section 4 states the Hybrid Wave Transform method. Section 5 presents two case studies and a discussion of the results. Finally, Section 6 provides a brief conclusion.

2. Conformable Fractional Derivative

The definition of conformable derivative and integral are firstly presented by Khalil et al. [15] as follows:

Definition 2.1. [15] Consider the function $f:(0,\infty)\to\mathbb{R}$. The $\alpha-th$ order *"conformable fractional derivative"* of f is stated as,

$$D_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

for all $t>0,\alpha\in(0,1)$.

Definition 2.2. Let f be α -differentiable in some $(0,a), a>0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t)$. The *"conformable fractional integral"* of a function f starting from $a\geq 0$ is defined as:

$$I_\alpha^a(f)(t)=\int_a^t f\,(x)d_\alpha x=\int_a^t\frac{f(x)}{x^{1-\alpha}}dx$$

where the integral is the Riemann improper integral, and $\alpha\in(0,1]$.

In fact, the conformable fractional derivative has great, decisive advantages over other well-known derivatives such as Grunwald-Letnikov, Caputo, and Riemann-Liouville-type fractional derivatives [2,16,17]. For instance

- The Riemann-Liouville derivative of a constant is not zero. For instance, $D_a^\alpha c\neq 0$ (Caputo derivative satisfies) if α is not a natural number.
- The derivative of the known formula of the derivative of the product of two functions is not satisfied by all fractional derivatives.

$$D_a^\alpha(fg)=gD_a^\alpha(f)+fD_a^\alpha(g)$$

- The derivative of the quotient of two functions is not satisfied by all fractional derivatives.

$$D_a^\alpha\left(\frac{f}{g}\right)=\frac{fD_a^\alpha(f)-gD_a^\alpha(g)}{g^2}$$

- The chain rule does not satisfy by all fractional derivatives.

$$D_a^\alpha(fog)(t)=f^\alpha(g(t))g^\alpha(t)$$

- All fractional derivatives do not satisfy $D^\alpha D^\beta=D^{\alpha+\beta}$ in general.
- In the Caputo definition, it is assumed that the function f is differentiable.

After these advantages came out, it attracted many researchers, and many studies have been done. For instance, Rezazadeh et al. [18] used two methods for solving the Conformable Fractional Diffusion-Reaction Equation (CFDRE), which is commonly applied in mathematical biology. Korkmaz et al. [19] investigated "investigated a method for the solution of the Conformable Fractional Zakharov-Kuznetsov Equation (CFZKE). Hashemi [20] obtained the exact solutions of integrable nonlinear Schrödinger type equation with conformable time-fractional derivative. Chen et al. [21] used the simplest equation method for acquiring the exact solutions of some FPDEs with conformable derivatives. Özkan et al. [22] introduced the conformable double Laplace transform and investigated the solutions of the fractional heat equation and fractional telegraph equation using this transform. See the references [15,23].

3. Differential Transform Method

A brief description of the two-dimensional DTM [13,14] is expressed as follows:

Definition 3.1. [14,24,25] Let $u(t,x)$ be a continuously differentiable function and an analytic function in the defined domain. Then we can express the two-dimensional DTM of the function $u(t,x)$ as

$$U(k,h)=\frac{1}{k!\,h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial x^h}u(t,x)\right]_{\substack{t=0\\x=0}} \tag{3.1}$$

where $u(t,x)$ is the original function, and $U(k,h)$ is the transformed function.

Definition 3.2. [14,24,25] The inverse DTM of $U(k,h)$ can be stated as

$$u(t,x)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)t^k x^h \tag{3.2}$$

Unifying Equations (3.1) and (3.2), we get

$$u(t,x)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!\,h!}\left[\frac{\partial^{k+h}}{\partial t^k \partial x^h}u(t,x)\right]_{\substack{t=0\\x=0}}t^k x^h \tag{3.3}$$

Equation (3.3) indicates that the concept of the two-dimensional DTM is obtained from to two-dimensional Taylor series expansion. In this work, the lower-case letters denote the original functions, and the upper-case letters show the transformed version of the considered functions (T-functions). With the help of the above definitions, the two-dimensional DTM of some basic mathematical operations and functions can be obtained as [14,24,25] and are listed in Table 1.

Table 1. Some operations of the two-dimensional differential transformation

Original Function	Transformed Function
$u(t,x)=w(x,y)\pm v(x,y)$	$U(k,h)=W(k,h)\pm V(k,h)$
$u(t,x)=\alpha w(t,x)$	$U(k,h)=\alpha W(k,h)$
$u(t,x)=e^{\lambda(t+x)}$	$U(k,h)=\frac{\lambda^{k+h}}{k!\,h!}$
$u(t,x)=w(t,x)v(t,x)$	$U(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}W(r,h-s)V(k-r,s)$
$u(t,x)=\frac{\partial^{r+s}}{\partial x^r \partial y^s}w(t,x)$	$U(k,h)=\frac{(k+r)(h+s)}{r!\,s!}W(k+r,h+s)$
$u(t,x)=t^m x^n$	$U(k,h)=\delta(k-m,h-n)=\begin{cases}1, for\ k=m,h=n\\0, otherwise\end{cases}$

4. A Hybrid Method for CFPDEs: Wave Transform Method with DTM

To show the solution procedure of the method for obtaining the solution of CFPDEs by using the Hybrid Wave Transform Method (HWTM), which is a combination of the FWTM and DTM, we will consider the following nonlinear time CFFPE:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\frac{\partial}{\partial x}A(t,x,u) + \frac{\partial^2}{\partial x^2}B(t,x,u)\right]u(t,x),\ \ t \in \mathbb{R}_0^+, x \in \mathbb{R}, \text{and } 0 < \alpha \leq 1 \tag{4.1}$$

together with the initial condition

$$u(0,x) = f(x) = \sum_{n=0}^{\infty} a_n\, x^n \tag{4.2}$$

Where $D^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ indicates the conformable derivative. The method is based on transforming conformable fractional partial differential Equation (4.1) into a partial differential equation via FWTM with the aid of $T = \frac{pt^\alpha}{\alpha}$ wave transform, setting $p = 1$ and applying the chain rule [23] result in a PDE as follows:

$$\frac{\partial u}{\partial T} = \left[-\frac{\partial}{\partial x}A(T,x,u) + \frac{\partial^2}{\partial x^2}B(T,x,u)\right]u(T,x) \tag{4.3}$$

$$u(0,x) = f(x) = \sum_{n=0}^{\infty} a_n\, x^n \tag{4.4}$$

which can be solved via DTM. Employing DTM to both sides of the Equation (4.3) yields the following recurrence formula

$$(k+1)U(k+1,h) = A(h)V(k,h) \tag{4.5}$$

where $A(h)$ is the coefficient of $V(k,h)$, which is the differential transform of the right-hand side of Equation (4.3). Similarly, we transform the initial condition (4.4) to $U(0,h) = a_h$, $h = 0,1,2,3,\cdots$ by using DTM. By using $U(0,h)$ and Equation (4.5), we can iteratively obtain $U(k,h), k = 1,2,3,\cdots, h = 0,1,2,3,\cdots$. Here, it should not be forgotten that $U(k,h)$ values are the components of the spectrum of $u(T,x)$. Finally, we acquire the solution of Equation (4.3) by

$$u(T,x) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U\,(k,h) T^k x^h \tag{4.6}$$

By turning back to the original variables, we get

$$u(T,x) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U\,(k,h)\left(\frac{t^\alpha}{\alpha}\right)^k x^h \tag{4.7}$$

which is the solution of problems (4.1)-(4.2).

5. Applications

In this part, the practicality of the algorithm shall be exemplified by two examples.

Example 5.1. Consider the nonlinear time-fractional Fokker-Planck equation:

$$D_t^{\alpha} u = \left[-\frac{\partial}{\partial x}\left(3u - \frac{x}{2}\right) + \frac{\partial^2}{\partial x^2}(xu) \right] u(t,x) \tag{5.1}$$

where $t > 0$, $x > 0$, and $0 < \alpha \leq 1$. D_t^{α} indicates the CFD of function $u(x,t)$ due to the initial condition:

$$u(0,x) = x \tag{5.2}$$

Applying the transformation $T = \frac{pt^{\alpha}}{\alpha}$ and for convenience, we set $p = 1$. Thus, we acquire the following PDE

$$\frac{\partial u}{\partial T} = \left[-\frac{\partial}{\partial x}\left(3u - \frac{x}{2}\right) + \frac{\partial^2}{\partial x^2}(xu) \right] u(T,x) \tag{5.3}$$

which can be solved with the aid of DTM. Implementing the DTM to the Equations (5.2) and (5.3), we have the following relations

$$\begin{aligned} U(k+1,h) = \frac{1}{(k+1)} \Bigg[-(h+1) \Bigg(3 \sum_{r=0}^{k} \sum_{s=0}^{h+1} U(r,h+1-s) U(k-r,s) - \frac{1}{2} U(k,h) \\ +(h+1)(h+2) \sum_{r=0}^{k} \sum_{s=0}^{h+1} U(r,h+1-s) U(k-r,s) \Bigg) \Bigg] \end{aligned} \tag{5.4}$$

By using the initial condition, we get

$$U(0,h) = \delta(h-1) \tag{5.5}$$

Utilizing relation (5.4) and transformed condition (5.5), for $k = 1,2,\cdots$ we obtain

$$U(k,h) = \begin{cases} \frac{1}{k!} = \frac{1}{\Gamma(k+1)}, & h=1 \\ 0, & \text{otherwise} \end{cases} \tag{5.6}$$

By using the inverse transform of DTM given in Equation (4.6), we have

$$u(T,x) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k,h) \; T^k x^h = x + xT + \frac{x}{2!}T^2 + \frac{x}{3!}T^3 + \frac{x}{4!}T^4 + \cdots$$

The inverse transformation of FCTM will yield the solution, which is expressed by,

$$u(t,x) = x + x\frac{t^{\alpha}}{\alpha} + \frac{x}{2!}\frac{t^{2\alpha}}{\alpha^2} + \frac{x}{3!}\frac{t^{3\alpha}}{\alpha^3} + \frac{x}{4!}\frac{t^{4\alpha}}{\alpha^4} + \cdots$$

According to [23, Theorem 4.1.], the above series corresponds to the conformable fractional power series expansion of

$$u(x,t) = xe^{\frac{t^{\alpha}}{\alpha}}$$

which is the analytical solution of problems (5.1)-(5.2).

Example 5.2. Regarding the nonlinear time-fractional Fokker-Planck equation:

$$D_t^\alpha u = \left[-\frac{\partial}{\partial x}\left(\frac{4u}{x} - \frac{x}{3}\right) + \frac{\partial^2}{\partial x^2}(u)\right] u(t,x) \tag{5.7}$$

Where $t > 0$, $x > 0$, and $0 < \alpha \leq 1$ are subject to the initial condition:

$$u(0,x) = x^2$$

Utilizing the considered method, as we have employed in Example 5.1, we obtain the following solution:

$$u(t,x) = x^2 \left(\frac{t^\alpha}{\alpha} + \frac{1}{2!}\frac{t^{2\alpha}}{\alpha^2} + \frac{1}{3!}\frac{t^{3\alpha}}{\alpha^3} + \frac{1}{4!}\frac{t^{4\alpha}}{\alpha^4} + \cdots\right)$$

Now, for the fractional power series expansion [23], this series has the closed form of the solution

$$u(x,t) = x^2 e^{\frac{t^\alpha}{\alpha}}$$

which is also an exact solution of the given diffusion equation.

6. Conclusion

In this study, we have successfully expressed FWTM with the help of DTM to get the approximate solution of the conformable fractional Fokker-Planck equation. Conformable fractional PDE can easily be changed into integer order PDE by fractional wave transform; thus, one can easily use the differential transform algorithm to solve this equation. The fractional wave transform is straightforward and applicable. The method is accessible to all with basic knowledge of advanced calculus and little fraction calculus. It is understood that FWTM-DTM is a very powerful and efficient technique for finding analytical and numerical solutions for broad classes of conformable fractional differential equations.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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