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# Time Path of Capital-Labor Ratio and Steady-State Conditions of the Solow Long-Run Growth Model

*Sermaye Birikiminin Fonksiyonel İfadesi ve Solow Uzun-Dönem Büyüme Modelinde Kararlı-Denge Koşulları*

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**Abstract**

This paper proposes a general solution to Solow's original differential equation explaining the rate of change of capital-labor ratio. Determining the time path of capital-labor ratio, we obtain novel general conditions under which the capital-labor ratio can reach a stable steady-state value. Identifying steady-state conditions, this study obtains exact time period when the economy can reach its stable steady-state capital per labor magnitude. The findings of the study stay at the crossroads of new generation endogenous growth models, propose implications for factor-eliminating technical change and consider the effect of automation on economic growth. Our results state some policy implications that can be outlined as follows. First, in economies where elasticity of substitution between capital and labor is lower than unity, the economic policies addressing productivity growth should be different than those implemented in economies where elasticity is greater than unity. Second, the main source of uncertainty for policy makers is inability to determine exactly when the economic activity would reach a stable steady-state path. Our findings aim to shed light on this uncertainty.

**Jel Codes:** *O40, O41, O42*
**Keywords:** *Economic Growth, Constant Elasticity of Substitution, Production Function, Dynamical Equilibrium, Capital Accumulation, Steady-State Growth Rate*

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**Öz**

Bu çalışma Robert Solow tarafından geliştirilen ve ekonomik büyüme literatürünün temeli olarak kabul edilen sermaye büyüme modelinin geliştirilmesini amaçlanmaktadır. Çalışmamızda, Solow'un sermaye-işgücü oranındaki zamana bağlı değişimi açıklayan diferansiyel denklemine bir çözüm önerdik. Böylece, sermaye-işgücü oranının zamana bağlı fonksiyonel ifadesini elde ederek, hangi gerek koşullar altında ekonominin kararlı bir denge değerine ulaşabileceğini araştırdık. Ayrıca, sonuçlarımız bir bulguya daha ulaşmamıza olanak sağlamıştır. Bu bulguyu şu şekilde özetleyebiliriz: Belirli bir başlangıç değerinden başlamak üzere, zaman içinde evrilen sermaye ve işgücü miktarlarının, kararlı bir dengenin varlığında, ne zaman bu dengeye ulaşabileceklerini elde ettik. Çalışmamızın sonuçları yeni nesil içsel büyüme modellerinin kesişiminde yer almaktadır ve faktör-eleyici teknolojik değişim hakkında bulgular sunarken, otomasyonun ekonomik büyüme oranına katkısını ikame esnekliği seviyesi temelinde değerlendirmektedir. Bu bulgularımızın bir getirisi olarak: Ekonomi politikalarının zorlu salgın süreçleri içerisinde sınandığı günümüz üretim koşullarında, politika yapıcıların uyguladıkları ekonomik reçeteler sonucunda ekonomik bileşenlerin dengeye gelip gelmemesine ek olarak, ne zaman dengenin sağlanacağını bilmelerinin hiç olmadığı kadar önem arz ettiğini değerlendiriyoruz.

**Jel Kodları:** *O40, O41, O42*
**Anahtar Kelimeler:** *Ekonomik Büyüme, Sermaye-İşgücü İkame Esnekliği, Üretim Fonksiyonu, Dinamik Ekonomik Denge*

## 1. Introduction

Solow (1956) first defines the rate of change of capital-labor ratio and second, proposes a general definition of the steady-state through which capital-labor ratio evaluates and that the output of the economy converges. In the steady-state the capital-labor ratio is a constant, and the capital stock must be expanding at the same rate as the labor force. However, Solow (1956) does not give the conditions under which the economy reaches the domain of stable steady-state. Instead, Solow (1956) exemplifies certain production functions having different values of elasticity of substitutions. In those examples, even though Solow (1956) obtains steady-state conditions or thresholds, the general solution is missing especially for the third example on constant elasticity of substitution function (CES function). It is well-known fact that the elasticity of substitution between capital and labor is a parameter which defines the dynamical properties of production function. The purpose of this paper is first, to determine the time evolution of capital-labor ratio; second, to obtain general conditions for the existence of the domain of steady-state and to introduce the economic implications of these conditions for the rate of economic growth and factor productivity from the perspective of neoclassical growth theory. The present study proposes certain contributions to the recent studies in the literature and reveals different findings from the previous ones. These are briefly tabulated below.

Even though our study investigates Solow model capital accumulation dynamics, the findings of the study stays at the crossroads of new generation endogenous growth models such as Aghion et al. (2001), Peretto & Seater (2013), Aghion, Jones & Jones (2019). In addition, our results contribute to studies examining asymptotic behavior of neoclassical production process (Hakanes & Irmen 2008; Ozkaya 2021). More specifically, in perspective of policy making to evaluate the effect of initial conditions of production factors gives more room to addressing sustainable growth path in economies. Moreover, our study also makes a potential contribution to understanding the relationship between elasticity of substitution and growth rate of capital per labor (Klump & Preissler, 2000). Therefore, revisiting the pioneering studies and reconsidering common beliefs in growth theory will be crucial to develop more accurate models in future studies.

In their model Peretto & Seater (2013) introduce factor-eliminating process, in which elimination of the non-reproducible factor enables increase in the growth rate. This is done by devoting resources to R&D process and changing factor elasticities of output. However, in that setting the production function is Cobb-Douglas, and Inada conditions are imposed. The present study contributes to the literature by adopting authors' framework to constant elasticity of substitution production function. To do this, we make use of the findings reported in Ozkaya (2021). We consider two types of CES functions, differing from each other regarding the elasticity of substitution levels with respect to unity. For an economy where elasticity level is higher-than-unity, the R&D investment should address minimum marginal productivity. On the other hand, for the economies where elasticity level is lower-than-unity R&D investment should address maximum marginal productivity. In this vein, our results support the findings of Peretto & Seater (2013), which imply that R&D investment should be employed to increase the upper frontier of technology. From overall perspective, our findings contribute to the result of Peretto & Seater (2013). Because, in either type of economy, R&D investment causes an increase in average marginal productivity, which potentially generates an increase in real growth rate for output per capita. This contribution is crucial to shed light on productivity slowdown in G-7 countries (Moss, Nunn & Shambaugh, 2020; Sprague, 2021).

In a related study, Aghion, Jones & Jones (2019) analyzed the effect of automation on capital share and output of the economy. The authors focused on constant elasticity of substitution production function, particularly for the case where elasticity is lower-than-one. The authors propose a simulation which assumes that a constant fraction of the tasks that have not yet been automated become automated each year. The outcome of the simulation produces steady exponential growth. The authors state that "However, when tasks are complements, the depletion effect dominates, and automation is capital depleting." However, Aghion, Jones & Jones (2019) do not analytically prove their result. The present study contributes to the literature by giving an analytical proof which explains the overall effect of automation on rate of growth for output per capita. Our finding contributes the analysis in Aghion, Jones & Jones (2019).

The third contribution of this study is as follows. Our result challenges the view of Solow (1956: 70). Solow states that "The time path of capital and output will not be exactly exponential except asymptotically (There is an exception to this. If $\mathrm{K}=0, \mathrm{r}=0$ and the system can't get started; with no capital there is no output and hence no accumulation. But this equilibrium is unstable: the slightest windfall capital accumulation will start the system off toward steady state)." Our result shows that the time path of capital will be exactly exponential. In this perspective, our study contributes to the literature and gives support to findings in Hakenes

& Irmen (2008), which showed that a neoclassical economy may take-off even though the initial capital stock is zero and capital is essential.

Another contribution of the present study is to reinvestigate and to revise the role of elasticity of substitution on capital growth. Beginning from the influential studies in last two decades (de La Grandville 1989, 1997; Yuhn 1991; Klump & de La Grandville 2000; Saam 2008; Klump et al. 2007), the concept of elasticity of substitution between capital and labor has been widely analyzed from theoretical perspective. Our study shows that the effect of elasticity of substitution on growth rate of capital per labor is not positive. This result refines the results in Klump & Preissler (2000: 48). The authors state that for the case of elasticity of substitution lower-than-one, an increase in elasticity of substitution increases the probability that a permanent decline is prevented. Moreover, Klump & Preisler (2000) state that "It is possible for the economy to reach or leave the domain of steady states as a consequence of suitable changes in the elasticity of substitution." However, we demonstrate that their argument on the "threshold" is misleading.

Finally, in perspective of policy making our results contribute to the literature. The findings of the present study on the time path of capital per labor accumulation enables policy makers to determine the timing of the policy implications. The prediction of the exact time period when the economy would reach a stable equilibrium is crucial in various respects. Amid the Covid-19 pandemics, macroprudential policies addressed the financial system mainly to efficiently sustain the production activity. The main source of uncertainty in perspective of policy makers is not knowing exactly when the economic activity would reach a stable steady-state path. If this will be known, the efficiency of the policies and measures can be observed and revised. In terms of capital per labor expansion, our results can shed light on this question.

## 2. Time Path of Capital-Labor Ratio

Solow (1956) obtains the differential equation involving the capital-labor ratio alone. This differential equation determines the rate of change of the capital-labor ratio as the difference between the increment of capital and the increment of labor. We conserve Solow's notation in (1).

$$\dot{k} = sf(k) - nk \tag{1}$$

where $k = k(t)$ and $t$ denotes the time and $f(k)$ is the production function. Mean Value Theorem (MVT) enables us to define

$\frac{f(k)-f(k(0))}{k-k(0)} = f'(c)$ where $0 < c < k$ , and enables us to state that there exists such a $c \in R$, (Strang, 1991). We already know by the definition of per capita production function that $0 \leq f\big(k(0)\big) < \infty$. Then we rearrange the per capita production function and give in (2). Now assume that an initial condition for capital-labor ratio is denoted by $f(0)$, denoting the limiting case where the labor force as the sole production factor.

$$f(k) = f(0) + k.f'(c) \tag{2}$$

Let (2) be imposed to (1). The rate of change of capital-labor ratio is determined in (3) as the sum of two terms, the source, and the accumulation rate of capital-labor ratio.

$$\dot{k} = sf(0) + k(sf'(c) - n) \qquad (3)$$

For notational easiness let the accumulation rate of capital-labor ratio be denoted as

$H = sf'(c) - n$ and the source be denoted as $M = sf(0)$.

Since the initial and terminal conditions of the production function under lower-than-unity substitution level ($\sigma < 1$) is different from that of under higher-than-unity substitution level ($\sigma > 1$), we propose to continue our analysis by distinguishing the case $\sigma < 1$ from the case $\sigma > 1$, respectively. The terms "minimum marginal product of an input factor" and "maximum marginal product of an input factor" refer to initial and terminal conditions of production function for level of elasticity of substitution higher-than-unity and for lower-than-unity, respectively.

1. Let us assume that $\sigma < 1$: the elasticity of substitution is lower-than-unity. The source is equal to zero, namely $M = sf(0) = 0$. Then the fundamental differential equation (3) is rearranged:

$\dot{k} = k(sf'(c) - n)$. Solving this differential equation enables us to obtain the time path of the capital-labor ratio in (4).

$$k(t) = e^{H.t}k(0) \qquad (4)$$

Solution (4) depicts that there are now two main possibilities:

1.1. If the maximum marginal product of capital-labor ratio is smaller or equal to $\frac{n}{s}$, namely $\frac{n}{s} \geq f'(0)$ then real output declines permanently. Hence, the steady-state capital-labor ratio is zero, namely $k(\infty) = 0$, and the steady-state output is zero as well, $f\big(k(\infty)\big) = 0$.

1.2. Conversely, if the maximum marginal product of capital is greater than $\frac{n}{s}$, then there are three subsequent cases to be considered:

i) The case $\frac{n}{s} < f'(c) < f'(0)$ denotes that the average value of marginal product of capital per labor is greater than $\frac{n}{s}$, but smaller than maximum marginal product of capital per labor $f'(0)$. This case leads to a growth of capital-labor ratio, $\dot{k} > 0$ until average marginal product attains the magnitude $\frac{n}{s}$.

ii) $f'(c) < \frac{n}{s} < f'(0)$ shows the case that average marginal product of capital-labor ratio is smaller than $\frac{n}{s}$. This case should lead to a decline, $\dot{k} < 0$ until average marginal product reaches $\frac{n}{s}$. In this case there is a stable steady-state capital-labor ratio.

As can be seen, the stable steady-state capital-labor ratio depends on the maximum marginal product of each input factor and on the elasticity of substitution level as well.

iii) $H = sf'(c) - n = 0$ indicates that the accumulation rate is zero and that the average value of marginal product of capital per labor equals $\frac{n}{s}$. In this case the capital-labor ratio does not move, $\dot{k} = 0$. This can be occur at initial period as well: $k(t) = k(0)$.

2. Let us assume that $\sigma > 1$: the elasticity of substitution is higher-than-unity. In this case the source is greater than zero: $M = sf(0) > 0$. Then the fundamental differential equation equals to equation (3). By resolving (3), we obtain the solution given in (5).

$$k(t) = e^{H.t}\left(k(0) + \frac{M}{H}\right) - \frac{M}{H} \quad (5)$$

In equation (5), $H = sf'(c) - n$ and $M = sf(0)$. There are now two main possibilities according to solution (5).

2.1. The case $\frac{n}{s} \leq f'(\infty) < f'(c)$ shows that if the minimum marginal product of capital-labor ratio is greater or equal to $\frac{n}{s}$, then the accumulation rate will be positive $H > 0$, so does the rate of growth of capital-labor ratio $\dot{k} > 0$. This case yields a permanent growth. That is to say, the steady-state capital-labor ratio is $k(\infty) = \infty$, and the steady-state output is $f\big(k(\infty)\big) = \infty$.

2.2. Conversely, assume that the minimum marginal product of capital per labor is smaller than $\frac{n}{s}$, namely $f'(\infty) < \frac{n}{s}$. Then three subsequent cases originate from the relationship between average value of marginal product of capital-labor ratio and $\frac{n}{s}$. These cases are tabulated below.

i) $f'(\infty) < f'(c) < \frac{n}{s}$ shows that the average value of marginal product of capital per labor is smaller, rendering a negative accumulation rate $H < 0$ and therefore as $t \to \infty$, a stable steady-state capital-labor ratio value exists and is finite $k(\infty) < \infty$. This stable steady-state capital-labor ratio $k(\infty)$ can be computed by the following expression:

$k(\infty) = \frac{sf(0)}{n - sf'(c)}$ and the steady-state output per capita is finite, $f\big(k(\infty)\big) < \infty$. As it can be easily seen in the steady-state $sf(0) > 0$ balances the decumulation $k(sf'(c) - n) < 0$.

ii) The case $f'(\infty) < \frac{n}{s} < f'(c)$ denotes that if the average value of marginal product of capital-labor ratio is greater, then an accumulation occurs $H > 0$ which leads to a temporary growth of capital-labor ratio, $\dot{k} > 0$ until $k(\infty)$, stable steady-state capital-labor ratio is reached.

iii) $H = 0$, implies that even though the accumulation rate is zero, the source renders the rate of change of capital-labor ratio, which is greater than zero, $\dot{k} > 0$. The time path of capital-labor ratio then becomes a linear expression:

$$k(t) = k(0) + sf(0).t$$

This is a function of initial conditions. In this case, the capital-labor ratio accumulates with a constant rate, namely with the savings from the minimum marginal product of labor, $f(0)$. In this perspective, labor force (labor productivity) is the sole determinant for the capital growth.

$$\dot{k} = sf(0).$$

## 3. Factor-Eliminating and Automation Process

Based on the definition of CES production function in Arrow et al. (1961), we are able to determine the distribution parameter and the efficiency parameter in terms of initial and the terminal conditions. To do this, we use appropriate differential equations and compute initial and terminal conditions of the function. The elasticity of substitution is denoted by $\sigma$.

Whenever $\sigma > 1$, the CES production function is:

$$F(K,L) = \left((F_K(K,0).K)^{\frac{\sigma-1}{\sigma}} + (F_L(0,L).L)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{6}$$

For the case where $\sigma < 1$, the production function can be obtained by (7).

$$F(K,L) = \left((F_K(K,\infty).K)^{\frac{\sigma-1}{\sigma}} + (F_L(\infty,L).L)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{7}$$

For $\sigma > 1$, the coefficients $F_K(K,0)$ and $F_L(0,L)$ constitute the initial condition of the production function and denote the minimum marginal product of capital and minimum marginal product of labor, respectively. Each is constant with respect to input factor. For $\sigma < 1$, the coefficients $F_K(K,\infty)$ and $F_L(\infty,L)$ are terminal conditions of the production function. The terminal conditions are the maximum marginal product of capital and maximum marginal product of labor, respectively. $F_K(K,\infty)$ and $F_L(\infty,L)$ are constants. If Inada conditions are applied to (6) and (7), the following results can be reached, respectively.

i.) Whenever $\sigma > 1$, and L is essential product, then capital elasticity $\lim_{K\to 0}\frac{KF_k}{F} = a_i = 0$ and $1 - a_i = 1$. This transforms CES function (6) into the function $Y = A.L$ . On the other hand, if K is essential product $\lim_{L\to 0}\frac{KF_k}{F} = a_i = 1$ , this transforms CES function (6) into the function $Y = A.K$. In this transformation, $A$ is a constant.

ii.) Whenever $\sigma < 1$, and L is essential product, then capital elasticity is $\lim_{K\to\infty}\frac{KF_k}{F} = a_i = 0$ and hence $\lim_{K\to\infty} 1 - \frac{KF_k}{F} = 1 - a_i = 1$ . This transforms CES function given in (7) into the function $Y = A_2.L$ . On the other hand, if $K$ is essential product, then capital elasticity will equal $\lim_{L\to\infty}\frac{KF_k}{F} = a_i = 1$, generating the function $Y = A_2.K$. At the asymptotes, Inada conditions make CES function transform into Cobb-Douglas function. In this transformation $A_2$ is a constant which may not be equal to $A$. These findings contribute the results in Peretto &Seater (2013). Our findings clearly show how the essentiality of factors occurs, with respect to the value of elasticity of substitution. Therefore, factor eliminating can be implemented when L is essential product, and this can be done for both cases $\sigma < 1$ and $\sigma > 1$. The other extreme case endogenously yields perpetual growth.

When adopted to our findings, R&D investment can be considered as an effect which increases maximum marginal product of capital $f'(0) = F_K(K,\infty)$ . This is consistent with expression (6) in Peretto & Seater (2013), which implies that R&D investment should be made to increase the upper frontier of technology. We determined that $\frac{n}{s} \geq f'(0)$ is a necessary condition for permanent decline for growth rate. Therefore, if R&D investment is assumed to increase $f'(0)$, then this increment will augment the probability of $f'(0)$ exceeding $\frac{n}{s}$, as well.

Secondly, (4) indicates that to enhance growth rate of capital-labor ratio average marginal product of capital should exceed $f'(c) > \frac{n}{s}$ .Therefore, an increase in minimum marginal productivity causes an increase in average marginal productivity, which makes the probability of positive rate of growth higher.

For the case of elasticity of substitution is higher-than-one $f'(\infty) < f'(c) < \frac{n}{s}$ , the output is higher than steady-state growth rate and rate of growth declines until equilibrium value reached. Since $f'(\infty) = F_K(K,0)$ the R&D investment addresses an increase in $f'(\infty)$, which augments average value of marginal product of capital, $f'(c)$ . This makes more probable that $f'(c)$ exceeds $\frac{n}{s}$, which is necessary and sufficient condition for growth $\frac{n}{s} \leq f'(c)$.

Above arguments indicate that agents can learn to change the limiting values of marginal products, and hence range of average marginal productivity, $f'(c)$. Indeed, an increase in minimum marginal productivity results with an increase in average productivity. Similarly, an increase in maximum marginal productivity causes an increase in average marginal productivity as well. These results give support to the findings in Peretto & Seater (2013) which state that agents can learn to change exponents of a Cobb–Douglas production function.

Let us shed light on another point. In above arguments, we consider the "increase in probability of limiting values of marginal products". This has partially been studied in Aghion et al. (2001). In their model, competition motivates R&D investment which increases the probability of innovation within a step-by-step innovation process. The authors conclude that imitation promotes growth rate. In a related study Ozkaya (2010) showed that within step-by-step innovation process, if firms engage in R&D investment to advance knowledge frontier (rather than technology import or imitation) the growth rate will be greater. Therefore, factor-eliminating can also be adopted these models. R&D teams can be considered to be composed of labor with novel knowledge creativity capacity and labor equipped with knowledge-imitation capacity. On the other hand, the maximum product of labor is equal to maximum price of labor under competitive markets. Therefore, according to Aghion et al. (2001), investment in R&D can be considered to decrease maximum marginal product of unskilled labor and decreases the maximum price of unskilled labor. Hence, the growth rate augments.

Finally, factor eliminating model can be adopted to CES function for the range of inputs other than asymptote values. However, Peretto & Seater (2013) do not explain what happens when Inada conditions do not apply, or in other words in case of a CES function. Aghion, Jones & Jones (2019) can answer this question. In Aghion et al. (2019) in expression (12) the production function given is $Y_t = A_t\left(\beta_t^{1-\rho}K_t^{\rho} + (1-\beta_t)^{1-\rho}L_t^{\rho}\right)^{\frac{1}{\rho}}$ for elasticity of substitution lower-than one. Then the substitution parameter is $\rho < 0$ . This production function is a special case of Arrow et al. (1961). To align this production function with Arrow's equation, we refer to equation (7) given above. In expression (12) in Aghion, Jones & Jones (2019), the ratio of automated to nonautomated output—or the ratio of the capital share to the labor share—equals

$$\frac{\alpha_K}{\alpha_L} = \left(\frac{\beta_t}{1-\beta_t}\right)^{1-\rho}\left(\frac{K_t}{L_t}\right)^{\rho}$$

From (1) and (2), the parameters in expression (12) in Aghion et al. (2019) correspond to,

$A_t^{\frac{\sigma-1}{\sigma}}\beta_t^{\frac{1}{\sigma}} = F_K(K,\infty)^{\frac{\sigma-1}{\sigma}}$ and thus $F_K(K,\infty) = A_t.\beta_t^{\frac{1}{\sigma-1}}$

$A_t^{\frac{\sigma-1}{\sigma}}(1-\beta_t)^{\frac{1}{\sigma}} = F_L(\infty,L)^{\frac{\sigma-1}{\sigma}}$ and thus $F_L(\infty,L) = A_t.(1-\beta_t)^{\frac{1}{\sigma-1}}$

The ratio of automated to unautomated output is then

$\frac{\alpha^*{}_K}{\alpha^*{}_L} = \left(\frac{\beta_t}{1-\beta_t}\right)^{\frac{1}{\sigma}}\left(\frac{K_t}{L_t}\right)^{\frac{\sigma-1}{\sigma}}$ where elasticity of substitution is lower-than-one, $\sigma < 1$ .

$$\frac{\alpha^*{}_K}{\alpha^*{}_L} = \left(\frac{F_K(K,\infty)}{F_L(\infty,L)}\right)^{\frac{1}{\sigma}}\left(\frac{K_t}{L_t}\right)^{\frac{\sigma-1}{\sigma}} \text{ ; where } \frac{\beta_t}{1-\beta_t} = \left(\frac{F_K(K,\infty)}{F_L(\infty,L)}\right)^{\sigma-1} \quad (8)$$

We suggest that an increase in the effect of automation is observed on $F_K(K,\infty)$ and $F_L(\infty,L)$, which are maximum product of capital and maximum product of labor, respectively as $\frac{\partial F_K(K,\infty)}{\partial\beta_t} < 0$ and $\frac{\partial F_L(\infty,L)}{\partial\beta_t} > 0$. Up to this point, this result is consistent with the result obtained by expression (16) in Aghion, Jones & Jones (2019). However, our finding on capital-accumulation differs.

In our setting, automation directly decreases maximum marginal product of capital and increases maximum marginal product of labor. Finally, this increases the average marginal product of capital $f'(c)$ , which in turn augments capital-accumulation and can easily be seen in equation (7). However, Aghion, Jones & Jones (2019) state that "That is, automation is equivalent to a combination of labor- augmenting technical change and capital-depleting technical change. This is surprising. One might have thought of automation as somehow capital augmenting. Instead, it is very different: it is labor augmenting and simultaneously dilutes the stock of capital. "The authors do not give analytical proof but propose a simulation where growth rate of output is positive. Our setup enables to give the analytical proof.

**Proposition 1.** Let $\dot{f}/f$ denote growth rate of output per capita, $\frac{\dot{k}}{k}$ denote growth rate of capital per labor. For notational easiness, let us denote $b = F_K(K,\infty)$ and $a = F_L(\infty,L)$ in (7).

The sensitivity of rate of growth rate of output per capita to automation is computed by the chain $\frac{\partial(\dot{f}/f)}{\partial(\beta_t)} = \frac{\partial(\dot{f}/f)}{\partial\left(\frac{a}{b}\right)}.\frac{\partial\left(\frac{a}{b}\right)}{\partial(\beta_t)}$. For $\sigma < 1$, the effect of $\frac{a}{b}$ on the growth rate of per capita income depends on the sign of $\left(\frac{\dot{k}}{k} + n\sigma\right)$.

**Proof 1.** Rewrite (2) in intensive form and take derivative with respect to $\frac{a}{b}$. Then we obtain $\frac{\partial(\dot{f}/f)}{\partial\left(\frac{a}{b}\right)}$ as: $\frac{\partial(\dot{f}/f)}{\partial\left(\frac{a}{b}\right)} = \left(\frac{\partial}{\partial\left(\frac{a}{b}\right)}\left(\frac{df}{dk}\frac{k}{f}\right)\right).\frac{\dot{k}}{k} + \left(\frac{\partial}{\partial\left(\frac{a}{b}\right)}\left(\frac{\dot{k}}{k}\right)\right)\frac{df}{dk}\frac{k}{f}$

$\frac{\partial(\dot{f}/f)}{\partial\left(\frac{a}{b}\right)} = \frac{sb(1+v)^{\frac{\sigma}{\sigma-1}}\left(\frac{1}{\sigma-1}\right)+n}{(1+v)^2}.\left(\frac{\sigma-1}{\sigma}\right)v.\frac{b}{a}$ , where $v = \left(\frac{a}{bk}\right)^{\frac{\sigma-1}{\sigma}}$ is the ratio of labor share to capital share. According to this equality, following expression holds.

$$\frac{\partial\left(\dot{f}/_f\right)}{\partial\left(\frac{a}{b}\right)}\ is \begin{cases} < 0 & if\ \frac{sf}{k} < (1-\sigma)n \\ = 0 & if\ \frac{sf}{k} = (1-\sigma)n \\ > 0 & if\ \frac{sf}{k} > (1-\sigma)n \end{cases}$$

Imposing $\frac{\partial\left(\dot{f}/_f\right)}{\partial\left(\frac{a}{b}\right)}$ into $\frac{\partial\left(\dot{f}/_f\right)}{\partial(\beta_t)} = \frac{\partial\left(\dot{f}/_f\right)}{\partial\left(\frac{a}{b}\right)}.\frac{\partial\left(\frac{a}{b}\right)}{\partial(\beta_t)}$ and rearranging yields the result given in (9).

$$\frac{\partial\left(\dot{f}/_f\right)}{\partial\left(\frac{a}{b}\right)} = \frac{\left(\frac{\dot{k}}{k}+n\right)}{\sigma(1+v)^2}.v.\frac{b}{a}$$

$$\frac{\partial\left(\dot{f}/_f\right)}{\partial\left(\frac{a}{b}\right)}.\frac{\partial\left(\frac{a}{b}\right)}{\partial(\beta_t)} = \frac{sb(1+v)^{\frac{\sigma}{\sigma-1}}\left(\frac{1}{\sigma-1}\right)+n}{(1+v)^2}.\overbrace{\left(\frac{\sigma-1}{\sigma}\right)v.\left(\frac{-1}{\beta_t(1-\beta_t)}\right)}^{>0}$$

$$\frac{\partial\left(\dot{f}/_f\right)}{\partial\left(\frac{a}{b}\right)}.\frac{\partial\left(\frac{a}{b}\right)}{\partial(\beta_t)} = \frac{sb(1+v)^{\frac{\sigma}{\sigma-1}}\left(\frac{1}{\sigma-1}\right)+n}{(1+v)^2}.\overbrace{\frac{\partial\left(\frac{a}{b}\right)}{\partial(\beta_t)}}^{>0} \quad (9)$$

From (9) we can conclude that: if $\frac{sf}{k} > (1-\sigma)n$ then an increase in automation will augment growth rate of output per capita, $\frac{\partial\left(\dot{f}/_f\right)}{\partial(\beta_t)} > 0$. Note that this case does not require $\frac{sf}{k} > n$, which is necessary condition for capital-accumulation. In addition, at the steady state, the automation increases growth rate of output per capita. Aghion, Jones & Jones (2019) states that "However, when tasks are complements ($\rho < 0$), the depletion effect dominates, and automation is capital depleting." However, the authors do not compute the overall effect of automation on growth rate of output per capita. Our finding completes their argument. Even though growth rate of capital per labor decreases, the growth rate of output per capita may increase in automation.

As we have indicated in the introduction, Klump & Preissler (2000) states that for the case of elasticity of substitution lower-than-one, an increase in elasticity of substitution increases the probability that a permanent decline is prevented. On the other hand, we investigate the relationship between elasticity of substitution and growth rate of capital labor ratio. By using (1), (4) and (7) we rewrite growth rate of capital labor ratio, $\frac{\dot{k}}{k} = s\frac{f(k)}{k} - n$. The derivative with respect to the elasticity of substiution is as follow.

$\frac{\partial}{\partial\sigma}\left(\frac{\dot{k}}{k}\right) = \frac{s.b.(1+v)^{\frac{\sigma}{\sigma-1}}}{(\sigma-1)^2}.\left(ln\frac{v^{\frac{v}{1+v}}}{1+v}\right)$ where $v = \left(\frac{a}{bk}\right)^{\frac{\sigma-1}{\sigma}}$.This result is in contrast with the finding of the Klump & Preissler (2000). We leave a more detailed analysis of the relationship between elasticity of substitution and growth rate of neoclassical economy for the future studies.

These findings make us to extend the study in another direction and to propose certain relevance for the debate on the trivial steady-state $k = 0$. In this vein, our study gives support to the findings of Hakenes & Irmen (2008). The authors show that a neoclassical economy may take-off even though the initial capital stock is zero and capital is essential, i.e., the system can start, even without a slight windfall capital. When this happens, the ignition of the process of capital accumulation has no cause, $k = 0$ is not steady state. Solow (1956: 68) defined capital $K(t) = k(t).L(0).e^{nt}$ .First, make use of (1) and (4) in our study, we obtain time evolution of capital as an exponential function: $K(t) = e^{sf'(c)t}.K(0)$. Therefore, output will be exponential too. Our result challenges the view of Solow (1956: 70). Solow states that "The time path of capital and output will not be exactly exponential except asymptotically (There is an exception to this. If $K = 0$,$r = 0$ and the system can't get started; with no capital there is no output and hence no accumulation. But this equilibrium is unstable: the slightest windfall capital accumulation will start the system off toward steady state)."

Make use of (3) and (5) in our study, we obtain time evolution of capital as an exponential function:

$$K(t) = e^{sf'(c).t}\left(K(0) + \frac{sF(0,L(0))}{sf'(c)-n}\right) - \frac{sF(0,L(0))}{sf'(c)-n}.e^{nt}$$

$$K(t) = \frac{sF(0,1)L(t)}{sf'(c)-n}\left(e^{(sf'(c)-n).t}\right) \quad \text{as } K(0) = 0.$$

Since our focus is on the trivial solution, now we restrict attention to the initial value problem with $K = 0$ at date $t = 0$. We focus on above-defined CES functions. According to (4) there are two paths for capital-labor ratio. However, with $K = 0$ at date $t = 0$ we have one solution for $\dot{k} = k(sf'(c) - n)$.

1. $K(t) = e^{sf'(c)t}.K(0)$ if $K(0) = 0$. $\dot{K}(t) = 0$
2. $k(t) = 0 \leftrightarrow K(t) = 0$ and $\dot{k} = 0 \leftrightarrow \dot{K}(t) = 0$.

Then capital remains at zero.This result support the second case in Theorem 1 in Hakenes & Irmen (2008). Specifically, this result corresponds to case where elasticity of substitution is lower-than-one. According to (3), there are two paths for capital-labor ratio

$\dot{k} = sf(0) + k(sf'(c) - n)$ we have two solutions at $k(0) = 0$

1. $k(t) = e^{H.t}\left(\frac{M}{H}\right) - \frac{M}{H}$
2. $k(t) = sf(0).t$ and $\dot{k}(0) > 0$ .

Since $\mathrm{F_L}(0, \mathrm{L}) > 0$ the capital takes-off immediately. This result support the third case in Theorem 1 in Hakenes & Irmen (2008). Specifically, this result corresponds to case where elasticity of substitution is higher-than-one. On the other hand, the first case in Theorem 1 concerns with different functional from, this is scope of our study. Finally, to examine the third case in Theorem 1, we must consider $f(0) = 0$ and $f'(c) \to \infty$ in above-given equation, $\dot{k} = sf(0) + k(sf'(c) - n)$.

If $f'(c) > \frac{n}{s}$ is satisfied, the economy takes-off. Therefore, the condition given in 2.2 in explains this case.

## 4. Demonstration Example

In this section we demonstrate our findings on Solow's example 3 (see Solow, 1956).The per capita intensive production function is given as $f(k) = \left(ak^{\frac{1}{2}} + 1\right)^2$.We know from (2) that an intermediate value $c$ always exists, $0 < c < k$, which satisfies the average value of the marginal product of capital-labor ratio, namely $f'(c)$. We obtain $f'(c)$ such that

$f'(c) = a^2 + ac^{\frac{-1}{2}}$ . The task is now to compute whether $f'(c) - \frac{n}{s} > 0$ or not.

The roots of the polynomial $a^2 + ac^{\frac{-1}{2}} - \frac{n}{s} = 0$ are denoted by $r_1$ and $r_2$, respectively. The first of the roots is smaller than zero and by the definition, we do not need to consider it. On the other hand, the second root is greater than zero.

$$r_1 = \frac{-c^{\frac{-1}{2}} - \sqrt{\left(c^{-1} + \frac{4.n}{s}\right)}}{2} < 0 \quad \text{and} \quad 0 < r_2 = \frac{-c^{\frac{-1}{2}} + \sqrt{\left(c^{-1} + \frac{4.n}{s}\right)}}{2} \tag{10}$$

3.1. If the inequality $0 < r_2 < a$ hold, then $f'(c) - \frac{n}{s} > 0$ , leading to a positive accumulation rate, $H > 0$ and a positive rate of change of capital-labor ratio, $\dot{k} > 0$ respectively. In this case there are two possibilities regarding the steady-state domain:

i) If $0 < r_2 < a < \sqrt{\frac{n}{s}}$, then the inequality $s.a^2 < n$ is satisfied. This condition shows that although the inequality $s.a^2 < n$ hold, there is no balanced growth according to expression (5): the capital-labor ratio increases through a steady-state value.

ii) On the other hand, the condition $0 < r < \sqrt{\frac{n}{s}} < a$ implies the following inequality: $s.a^2 > n$. There is no balanced growth according to (5), and hence $\dot{k} > 0$. This result is in accordance with Solow's result.

3.2. If $0 < a < r_2 < \sqrt{\frac{n}{s}}$ then $f'(c) - \frac{n}{s} < 0$, resulting with $H < 0$ and $k(\infty) < \infty$ .Therefore, a stable balanced growth exists. This condition corresponds to $s.a^2 < n$ and is in accordance with Solow's interpretation. By using MVT, let us determine the relation between *k* and *c*. The MVT $\frac{f(k)-f(0)}{k} = f'(c)$ yields $\frac{\left(ak^{\frac{1}{2}}+1\right)^2 - 1}{k} = a^2 + ac^{\frac{-1}{2}}$. This leads to $k = 4c$. Inserting this result into equation (5) capital-labor ratio is rewritten as

$k(t) = e^{H.t}\left(k(0) + \frac{M}{H}\right) - \frac{M}{H}$, where $H$ becomes $H = sf'\left(\frac{k}{4}\right) - n$ and $M$ does not change $M = sf(0) = s$. Rearranging this, we obtain equation (11).

$$\frac{k + \frac{1}{\left(sf'\left(\frac{k}{4}\right) - n\right)}}{k(0) + \frac{1}{\left(sf'\left(\frac{k}{4}\right) - n\right)}} = e^{\left(sf'\left(\frac{k}{4}\right) - n\right).t} \tag{11}$$

Now assume that $0 < a < r_2 < \sqrt{\frac{n}{s}}$ and that there exists a stable steady-state equilibrium. Then, in equation (11), replacing $k$ with $k(\infty)$ reveals a strict period, a *t* value, ready to be determined. Beginning from an initial capital per labor, $k(0)$, the $t$ value depicts the exact time period showing when the economy can reach a stable steady-state rate of growth for capital per labor. One other implication of this finding is the determination of dynamic efficiency level of the economy. This level can be defined as $\frac{\partial f(k(\infty))}{\partial k(\infty)} = n$ (Acemoglu, 2009: 60).

The special value for steady-state $k(\infty) = \left(\frac{a}{n-a^2}\right)^2$ together with $n > f'(\infty)$ satisfies the condition. If economy is above this level, then dynamic inefficiency will occur.

The equation (11) also enables to compute when the economy can reach this special consumption-saving equilibrium level.

## 5. Conclusion and discussion

The findings of the study stay at the crossroads of new generation endogenous growth models such as Aghion et al. (2001), Peretto & Seater (2013), Aghion, Jones & Jones (2019). In addition, our results contribute to studies examining asymptotic behavior of neoclassical production process (Hakanes & Irmen 2008; Ozkaya 2021). More specifically, in perspective of policy making to evaluate the effect of initial conditions of production factors gives more room to addressing sustainable growth path in economies. Moreover, our study also makes a potential contribution to the understanding of relationship between elasticity of substitution and growth rate of capital per labor (Klump & Preissler, 2000). Our results suggest some policy implications that can be outlined as follows. First, in economies where elasticity of substitution between capital and labor is lower than unity, the economic policies should be different than those implemented in economies where elasticity is greater than unity. This result is supported by empirical literature. As demonstrated by Herrendorf et al. (2015), Chirinko & Mallick (2017), Knoblach et al. (2020) estimates of the elasticity of substitution in U.S. economy at aggregate level have fluctuated between 0.4-0.6 interval. The U.S. economy historically has elasticity of substitution level lower-than-unity. Recent empirical surveys suggest that most evidence favors elasticities ranging between 0.4–0.6 for the United States (Chirinko, 2008). More recently, Knoblach et al. (2020) estimates a long-run meta-elasticity for the aggregate economy in the range of 0.45–0.87 and conclude that "*Estimates of σ at the aggregate or manufacturing level of the U.S. economy are characterized by large heterogeneity. Although the range of estimates is wide, most of the empirical evidence suggests that σ is below the Cobb–Douglas value of unity*". From the perspective of the results of present study, this empirical evidence suggest that U.S. growth policy should focus on maximum marginal productivities of input factors. Otherwise, the deceleration of factor productivities in U.S. economy will continue in the long-term. The empirical literature has already examined the time evolution of factor productivities. U.S. Bureau of Labor Statistics reports show that percent change in annual rate of real output growth in private business sector[2] decelerated over the period from 2007 to 2018. The contribution of capital productivity, labor productivity and multifactor productivity are all decreasing. Similarly, in

[2] Please refer to https://www.bls.gov/productivity/articles-and-research/source-of-output-growth-private-business-productivity-1987-2018.pdf.

manufacturing sector[3] average productivity of capital services is decreasing. Specifically, Sprague (2021) states that "In the years since 2005, labor productivity has grown at an average annual rate of just 1.3 percent, which is lower than the 2.1-percent long-term average rate from 1947 to 2018. The slow growth observed since 2010 has been even more striking: labor productivity grew just 0.8 percent from 2010 to 2018."

In addition, Oberfield & Raval, (2014), Cantore et al. (2017), de La Grandville & Solow (2017) show evidence that the elasticity of substitution for the U.S. economy is rising over time. These empirical findings show that elasticity of substitution is lower than Cobb Douglas level, but is increasing over time, which is consistent with our result pointing out that an increase in elasticity level makes the growth rate of capital per labor decelerated. This result is in contrast with the result in Klump & Preissler (2000).

As the elasticity level is lower than one, the growth policy should address maximum marginal productivity. Increasing maximum marginal productivity correspond to an increase in cutting-edge of technology, which can be realized by employing R&D investment and can be measured by evolution of patent intensity[4] (Ozkaya, 2010). This can be achieved by factor eliminating (Peretto & Seater, 2013). Therefore, for economies having elasticity of substitution lower than Cobb-Douglas level, adequate policy would be supporting maximum marginal productivities. On the other hand, for economies with elasticity level higher than unity, policy makers should consider increasing average productivities. This can be done by supporting economy-wide automation, step-by-step innovation and hence addressing minimum marginal productivity (Aghion et al., 2001; Aghion, Jones & Jones 2019). Moreover, the findings of the present study can be extended to analyze productivity growth in G-7 economies. Moss, Nunn & Shambaugh (2020) state that labor productivity growth and factor productivity growth both have declined since 1995 in every G-7 economy. Therefore, the results of the present study are important and point out a hot debate in leading literature.

Accordingly, these results can be used to explain structural differences in higher growth rate of East Asia economies and slowing growth rate in other developing economies. We believe that the extensive approach proposed in this study will be a source for different points of view. Our results give more room to theoretical and empirical studies which focus on the economic growth rate across countries with differing elasticity of substitution among production factors, and hence which can explain stable technological progress in those countries. The future studies which will be based on existing literature on modern growth theory should consider our findings.

[3] Please refer to U.S. Bureau of Labor Statistics, Manufacturing Sector: Output per Unit Capital Services [MPU9900072], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/MPU9900072, July 17, 2022.

[4] According to the 2008 Statistics of National Accounts (SNA), "Intellectual property products are the result of research, development, investigation, or innovation leading to knowledge that the developers can market or use to their own benefit in production because use of the knowledge is restricted by means of legal or other protection".

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