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A Note on Invariant Submanifolds of Metallic Riemannian Manifolds

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Abstract. The goal of this paper is to examine invariant submanifolds in metallic Riemannian manifolds with the help of induced structures on them by the metallic Riemannian structure of the ambient manifold. We obtain a useful characterization of invariant submanifolds. We also discuss some necessary conditions for invariant submanifolds to be totally geodesic. Finally, we provide an example of an invariant submanifold.

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Keywords: Metallic structure, metallic Riemannian manifold, invariant submanifold.

1. Introduction

The notion of a metallic structure on differentiable manifolds was first introduced by C.E. Hreţcanu and M.C. Crâşmăreanu in [17] as a natural extension of golden structures [6]. In [19], M. Özkan and F. Yılmaz analyzed metallic structures by means of the corresponding almost product structures. In [8], A. Gezer and Ç. Karaman investigated the integrability conditions and curvature properties of metallic Riemannian structures by using a special operator. In [5], A.M. Blaga and C.E. Hreţcanu defined and examined the conjugate connections determined by a metallic structure, called metallic conjugate connections, which are also a generalization of golden conjugate connections [3].

Using the similar approach as in the setting of golden Riemannian manifolds [15, 16], by C.E. Hreţcanu and M.C. Crâşmăreanu, the study of the differential geometry of submanifolds of a metallic Riemannian manifold was initiated in [17], particularly, invariant submanifolds were characterized here. A.M. Blaga and C.E. Hreţcanu [4] showed that on an invariant submanifold of locally decomposable metallic Riemannian manifolds, the Nijenhuis tensor of the tensor field of the induced structure is identically zero and it preserves the property of the locally decomposability of the ambient manifold. Later, by the same authors, such a type of submanifold was extended to slant, semi-slant, hemi-slant and bi-slant submanifolds, respectively, in golden and metallic Riemannian manifolds [4, 13, 14]. Moreover, the investigation of totally umbilical semi-invariant submanifolds was made in golden and metallic Riemannian manifolds [7, 10].

On the other hand, inspired by the golden structure [6] and S. Kalia's definition of the bronze mean [18], B. Şahin [21] defined and studied a new type of manifold that is not a metallic manifold, called an almost poly-Norden manifold. In addition, submanifolds of an almost poly-Norden Riemannian manifold were explored by S.Y. Perktaş in [20].

In this work, we continue to investigate invariant submanifolds of a metallic Riemannian manifold. The paper is devoted to three sections and organised as follows: In section 2, we give some fundamental concepts, formulas and notations to provide a background for the main results. Section 3 contains some important results on invariant submanifolds in metallic Riemannian manifolds. For a submanifold in metallic Riemannian manifolds, we get a

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condition equivalent to its invariance under the action of the metallic structure of the ambient manifold. We obtain some necessary conditions for an invariant submanifold to be totally geodesic. Lastly, we construct an example of an invariant submanifold in metallic Riemannian manifolds.

2. Preliminaries

An endomorphism $\widetilde{F}$ on a differentiable manifold $\widetilde{M}$ is said to be a metallic structure [17] if it yields the algebraic equation
$$\widetilde{F}^2 = a\widetilde{F} + bI$$
for $a, b \in \mathbb{N}^+$, where I denotes the Kronecker tensor field on $\widetilde{M}$. In this situation, the pair $\left(\widetilde{M}, \widetilde{F}\right)$ is called a metallic manifold. It is known that the metallic structure $\widetilde{F}$ has two real characteristic values $\sigma_{a,b}$ and $a - \sigma_{a,b}$, where $\sigma_{a,b}$ is the (a, b)-metallic number, which is the positive root of the quadratic equation $x^2 - ax - b = 0$ for positive integer values of a and b. If $\left(\widetilde{M}, \widetilde{F}\right)$ is a metallic manifold admitting an $\widetilde{F}$-compatible Riemannian metric $\widetilde{g}$, then the triple $\left(\widetilde{M}, \widetilde{g}, \widetilde{F}\right)$ is named a metallic Riemannian manifold.

Let M be an m-dimensional isometrically immersed submanifold of an $(m + s)$-dimensional metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{F}\right)$. If we denote by T_pM and $T_pM^\perp$ its tangent and normal spaces at a point $p \in M$, respectively, then the ambient tangent space $T_p\widetilde{M}$ splits as an orthogonal direct sum given by
$$T_p\widetilde{M} = T_pM \oplus T_pM^\perp$$
for each point $p \in M$. The induced Riemannian metric g on M is a 2-tensor field defined by
$$g = i_*\widetilde{g},$$
where i_* stands for the differential of the isometric immersion $i : M \longrightarrow \widetilde{M}$. We consider a local orthonormal frame $\{N_1, \ldots, N_s\}$ of the normal bundle $TM^\perp$. For every vector field $X \in \Gamma(TM)$, the decompositions of the vector fields $\widetilde{F}(i_*X)$ and $\widetilde{F}(N_\lambda)$ on $\widetilde{M}$ into tangential and normal components are given by
$$\widetilde{F}(i_*X) = i_*(FX) + \sum_{\lambda=1}^{s} u_\lambda(X) N_\lambda \tag{2.1}$$
and
$$\widetilde{F}N_\lambda = i_*(\xi_\lambda) + \sum_{\mu=1}^{s} \mathcal{A}_{\lambda\mu} N_\mu, \tag{2.2}$$
respectively, where F is a linear transformation on M, ξ_λ are tangent vector fields on M, u_λ are 1-forms on M and $\left[\mathcal{A}_{\lambda\mu}\right]$ is a matrice of type $s \times s$ of real functions on M for any $\lambda, \mu \in \{1, \ldots, s\}$. Thus, the quintet $\sum = \left(F, g, u_\lambda, \xi_\lambda, \left[\mathcal{A}_{\lambda\mu}\right]_{s\times s}\right)$ determines a structure induced on M by the metallic Riemannian structure $\left(\widetilde{g}, \widetilde{F}\right)$. Such a structure is said to be a $\sum$-metallic Riemannian structure on (M, g) (see, e.g., [4, 12, 17], for more details).

We denote by $\widetilde{\nabla}$ and ∇ the Riemannian connections on $\widetilde{M}$ and M, respectively. Then, Gauss and Weingarten formulas of M in $\widetilde{M}$ are given, respectively, by
$$\widetilde{\nabla}_{i_*X} i_*Y = i_*\nabla_X Y + \sum_{\lambda=1}^{s} h_\lambda(X, Y) N_\lambda$$
and
$$\widetilde{\nabla}_{i_*X} N_\lambda = -i_*A_\lambda X + \sum_{\mu=1}^{s} l_{\lambda\mu}(X) N_\mu$$
for all vector fields $X, Y \in \Gamma(TM)$, where h_λ are the second fundamental tensors corresponding to N_λ, i.e., $h(X, Y) = \sum\limits_{\lambda=1}^{s} h_\lambda(X, Y) N_\lambda$, A_λ are the Weingarten endomorphisms associated with the normal vector fields N_λ and $l_{\lambda\mu}$ are the 1-forms on M corresponding to the normal connection $\nabla^\perp$ for any $\lambda, \mu \in \{1, \ldots, s\}$, i.e., $\nabla^\perp_X N_\lambda = \sum\limits_{\mu=1}^{s} l_{\lambda\mu}(X) N_\mu$ [12].

Now, we recall the following well known definitions for a submanifold M [2, 22]: If $h = 0$ (or equivalently $h_\lambda = 0$ for any $\lambda \in \{1, \ldots, s\}$), M is called a totally geodesic submanifold; if $H = 0$, M is named a minimal submanifold; if

$h(X,Y) = g(X,Y)H$ for all vector fields $X, Y \in \Gamma(TM)$, M is said to be a totally umbilical submanifold, where H stands for the mean curvature vector of M.

3. Main Results

Any isometrically immersed submanifold M of a metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{F}\right)$ is said to be invariant if

$$\widetilde{F}\left(T_pM\right) \subseteq T_pM$$

for each point $p \in M$. In this situation, it follows that $\widetilde{F}\left(T_pM^{\perp}\right) \subseteq T_pM^{\perp}$ for each point $p \in M$.

Let $\sum = \left(F, g, u_\lambda, \xi_\lambda, \left[\mathcal{A}_{\lambda\mu}\right]_{s\times s}\right)$ be the $\sum$-metallic Riemannian structure on an m-dimensional invariant submanifold M of an $(m+s)$-dimensional metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{F}\right)$. In this case, $\xi_\lambda = 0$ (or equivalenty $u_\lambda = 0$) for any $\lambda \in \{1, \ldots, s\}$. Therefore, (2.1) and (2.2) are written as follows:

$$\widetilde{F}(i_* X) = i_*(F(X))$$

and

$$\widetilde{F}(N_\lambda) = \sum_{\lambda=1}^{s} \mathcal{A}_{\lambda\mu} N_\mu, \tag{3.1}$$

respectively. For the $\sum$-metallic Riemannian structure $\sum = \left(F, g, u_\lambda, \xi_\lambda, \left[\mathcal{A}_{\lambda\mu}\right]_{s\times s}\right)$ on (M, g), the following relations hold [17]:

$$\begin{aligned} F^2 X &= aFX + bX, \\ \mathcal{A}_{\lambda\mu} &= \mathcal{A}_{\mu\lambda}, \\ \sum_{\nu=1}^{s} \mathcal{A}_{\lambda\nu}\mathcal{A}_{\mu\nu} &= a\mathcal{A}_{\lambda\mu} + b\delta_{\lambda\mu}, \\ g(FX, Y) &= g(X, FY) \end{aligned} \tag{3.2}$$

and

$$g(FX, FY) = ag(FX, Y) + bg(X, Y)$$

for all vector fields $X, Y \in \Gamma(TM)$, where $\delta_{\lambda\mu}$ is the Kronecker delta. In addition to these, if $\widetilde{\nabla}\widetilde{F} = 0$, i.e., $\left(\widetilde{M}, \widetilde{g}, \widetilde{F}\right)$ is a locally decomposable metallic Riemannian manifold, then we have from [12]:

$$\begin{aligned} \nabla F &= 0, \\ h_\lambda(X, FY) - \sum_{\mu=1}^{s} h_\mu(X,Y)\mathcal{A}_{\lambda\mu} &= 0, \\ F(A_\lambda X) - \sum_{\mu=1}^{s} \mathcal{A}_{\lambda\mu} A_\mu X &= 0 \end{aligned} \tag{3.3}$$

and

$$X\left(\mathcal{A}_{\lambda\mu}\right) + \sum_{\nu=1}^{s} \mathcal{A}_{\lambda\nu} l_{\nu\mu}(X) + \sum_{\nu=1}^{s} \mathcal{A}_{\mu\nu} l_{\nu\lambda}(X) = 0$$

for all vector fields $X, Y \in \Gamma(TM)$.

Theorem 3.1. *Let M be an m-dimensional isometrically immersed submanifold of an $(m+s)$-dimensional metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{F}\right)$. Then, M is an invariant submanifold if and only if the normal bundle $TM^{\perp}$ admits a local orthonormal frame consisting of characteristic vectors of the metallic structure $\widetilde{F}$.*

Proof. By a suitable transformation [1] of a local orthonormal frame of the normal bundle $TM^{\perp}$ to another one, $\mathcal{A}_{\lambda\mu}$ can be reduced to $\mathcal{A}'_{\lambda\mu} = \sigma_{\lambda}\delta_{\lambda\mu}$, where σ_{λ} are the characteristic values of the matrice $\left[\mathcal{A}_{\lambda\mu}\right]_{s\times s}$ for any $\lambda, \mu \in \{1, \ldots, s\}$. If M is an invariant submanifold, i.e., each of the tangent vector fields ξ'_{λ} is zero, then it follows from (3.1) that

$$\widetilde{F}\left(N'_{\lambda}\right) = \sigma_{\lambda}N'_{\lambda},\ \lambda = 1, \ldots, s,$$

in other words, the normal vector fields N'_{λ} are the characteristic vectors of the metallic structure $\widetilde{F}$.

Conversely, if $\widetilde{F}\left(N'_{\lambda}\right) = \sigma_{\lambda}N'_{\lambda}$ for any $\lambda \in \{1, \ldots, s\}$, then we get from (2.2) that

$$\xi'_{\lambda} = 0,\ \lambda = 1, \ldots, s,$$

which implies that the submanifold M is invariant. □

Remark 3.2. For an isometrically immersed submanifold of codimension 1, namely a hypersurface, in metallic Riemannian manifolds, Theorem 3.1 was proven by C.E. Hreţcanu and M.C. Crâşmăreanu in [17, Proposition 5.2].

Theorem 3.3. *Let M be an m-dimensional isometrically immersed submanifold of an $(m+s)$-dimensional metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{F}\right)$. If*

$$\widetilde{F}i_* = \sigma_{a,b}i_* \tag{3.4}$$

and

$$\widetilde{F}N_{\lambda} = \left(a - \sigma_{a,b}\right)N_{\lambda},\ \lambda = 1, \ldots, s, \tag{3.5}$$

then M is a totally geodesic invariant submanifold.

Proof. What (3.4) and (3.5) say is that M is an invariant submanifold. By the use, again, of (3.4) and (3.5) in (2.1) and (2.2), respectively, we have

$$F = \sigma_{a,b}I$$

and

$$\mathcal{A}_{\lambda\mu} = \left(a - \sigma_{a,b}\right)\delta_{\lambda\mu}$$

for any $\lambda, \mu \in \{1, \ldots, s\}$. Thus, we derive from (3.3) that M is a totally geodesic submanifold. □

Theorem 3.4. *Let M be an m-dimensional isometrically immersed submanifold of an $(m+s)$-dimensional metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{F}\right)$. If*

$$\widetilde{F}i_* = \left(a - \sigma_{a,b}\right)i_*$$

and

$$\widetilde{F}N_{\lambda} = \sigma_{a,b}N_{\lambda},\ \lambda = 1, \ldots, s,$$

then M is a totally geodesic invariant submanifold.

Proof. The proof can be done by a similar technique used in that of Theorem 3.3. □

Theorem 3.5. *Let M be an m-dimensional totally umbilical invariant submanifold of an $(m+s)$-dimensional metallic Riemannian manifold $\left(\widetilde{M}, \widetilde{g}, \widetilde{F}\right)$. If*

$$\{trace(F)\}^2 \neq m\{mb + trace(F)a\},$$

or equivalently

$$trace(F) \neq \sigma m,$$

then M is a totally geodesic submanifold, where $\sigma = \sigma_{a,b}$ or $\sigma = a - \sigma_{a,b}$.

Proof. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of the tangent space T_pM at a point $p \in M$. Because M is a totally umbilical submanifold, it is known that the existence of constants ρ_{λ} such that $h_{\lambda} = \rho_{\lambda}g$ for any $\lambda \in \{1, \ldots, s\}$. In this case, from (3.3), we get

$$\rho_{\lambda}g(X, FY) = \sum_{\mu=1}^{s}\mathcal{A}_{\mu\lambda}\rho_{\mu}g(X, Y) \tag{3.6}$$

for all vector fields $X, Y \in \Gamma(TM)$. Substituting $X_p = Y_p = e_i$ for any $i \in \{1, \ldots, m\}$ at the point $p \in M$ in (3.6), then we obtain

$$\rho_\lambda g(e_i, Fe_i) = g(e_i, e_i) \sum_{\mu=1}^{s} \mathcal{A}_{\mu\lambda}\rho_\mu. \tag{3.7}$$

It follows by summing over i in (3.7) that

$$\sum_{i=1}^{m} \rho_\lambda g(e_i, Fe_i) = m \sum_{\mu=1}^{s} \mathcal{A}_{\mu\lambda}\rho_\mu,$$

from which we have

$$trace(F)\rho_\lambda = m \sum_{\mu=1}^{s} \mathcal{A}_{\mu\lambda}\rho_\mu. \tag{3.8}$$

Multiplying (3.8) by the matrice entry $\mathcal{A}_{\mu\lambda}$ and then summing over λ, we find

$$trace(F) \sum_{\lambda=1}^{s} \mathcal{A}_{\mu\lambda}\rho_\lambda = m \sum_{\nu=1}^{s} \sum_{\lambda=1}^{s} \mathcal{A}_{\mu\lambda}\mathcal{A}_{\lambda\nu}\rho_\nu. \tag{3.9}$$

From (3.2), (3.9) can be written in the form

$$trace(F) \sum_{\lambda=1}^{s} \mathcal{A}_{\mu\lambda}\rho_\lambda = mb\rho_\mu + ma \sum_{\nu=1}^{s} \mathcal{A}_{\mu\nu}\rho_\nu,$$

which implies that

$$\rho_\mu = \frac{1}{mb}\left(trace(F) - ma\right) \sum_{\lambda=1}^{s} \mathcal{A}_{\mu\lambda}\rho_\lambda. \tag{3.10}$$

Substituting (3.10) into (3.8), we derive

$$\left\{trace(F)\left(trace(F) - ma\right) - m^2 b\right\} \sum_{\mu=1}^{s} \mathcal{A}_{\lambda\mu}\rho_\mu = 0. \tag{3.11}$$

Using the assumption $\{trace(F)\}^2 \neq m\{mb + trace(F)a\}$, it can be easily seen from (3.11) that

$$\sum_{\mu=1}^{s} \mathcal{A}_{\lambda\mu}\rho_\mu = 0.$$

Therefore, we conclude from (3.10) that

$$\rho_\mu = 0, \mu = 1, \ldots, s,$$

which completes the proof. □

Remark 3.6. On golden Riemannian manifolds, Theorems 3.1, 3.3, 3.4 and 3.5 were demonstrated by M. Gök, S. Keleş and E. Kılıç in [9].

Now, let us give an example.

Example 3.7. We consider the $(2k + l)$-dimensional Euclidean space E^{2k+l} with the usual scalar product $\langle , \rangle$, where k and l are positive integers. Let us define a tensor field $\widetilde{F}$ of type $(1, 1)$ by

$$\widetilde{F}\left(X^\alpha, Y^\alpha, Z^\beta\right) = \left((a - \sigma_{a,b})X^\alpha, \sigma_{a,b}Y^\alpha, (a - \sigma_{a,b})Z^\beta\right)$$

for every tangent vector $\left(X^\alpha, Y^\alpha, Z^\beta\right) \in T_{\left(x^\alpha, y^\alpha, z^\beta\right)}E^{2k+l}$ at each point $\left(x^\alpha, y^\alpha, z^\beta\right)$, where

$$\left(x^\alpha, y^\alpha, z^\beta\right) = \left(x^1, \ldots, x^k, y^1, \ldots, y^k, z^1, \ldots, z^l\right)$$

and

$$\left(X^\alpha, Y^\alpha, Z^\beta\right) = \left(X^1, \ldots, X^k, Y^1, \ldots, Y^k, Z^1, \ldots, Z^l\right).$$

In this case, it is not difficult to verify that $\left(\langle , \rangle, \widetilde{F}\right)$ is a metallic Riemannian structure and $\left(E^{2k+l}, \langle , \rangle, \widetilde{F}\right)$ is a locally decomposable metallic Riemannian manifold.

Due to the fact that $E^{2k+l} = E^k \times E^k \times E^l$, we can mention the following three hyperspheres:

$$S^{k-1}(r_1) = \left\{ \left(x^1, \ldots, x^k\right) : \sum_{\alpha=1}^{k} (x^\alpha)^2 = r_1^2 \right\} \text{ in } E^k,$$

$$S^{k-1}(r_2) = \left\{ \left(y^1, \ldots, y^k\right) : \sum_{\alpha=1}^{k} (y^\alpha)^2 = r_2^2 \right\} \text{ in } E^k$$

and

$$S^{l-1}(r_3) = \left\{ \left(z^1, \ldots, z^l\right) : \sum_{\beta=1}^{l} \left(z^\beta\right)^2 = r_3^2 \right\} \text{ in } E^l.$$

Thus, as in [11, Example 3], we can build the product manifold $S^{k-1}(r_1) \times S^{k-1}(r_2) \times S^{l-1}(r_3)$ such that its each point has the coordinates $\left(x^\alpha, y^\alpha, z^\beta\right)$ satisfying the equation

$$\sum_{\alpha=1}^{k} (x^\alpha)^2 + \sum_{\alpha=1}^{k} (y^\alpha)^2 + \sum_{\beta=1}^{l} \left(z^\beta\right)^2 = R^2,$$

where $R^2 = r_1^2 + r_2^2 + r_3^2$. Also, if we denote this product manifold by M for short, then M is a submanifold of codimension 3 in the Euclidean space E^{2k+l} and M is a submanifold of codimension 2 in the sphere $S^{2k+l-2}(R)$. Hence, there are successive embeddings such that

$$M \hookrightarrow S^{2k+l-2}(R) \hookrightarrow E^{2k+l}.$$

Moreover, its tangent space $T_{\left(x^\alpha, y^\alpha, z^\beta\right)}M$ at each point $\left(x^\alpha, y^\alpha, z^\beta\right)$ is given by

$$T_{\left(x^\alpha, y^\alpha, z^\beta\right)}M = T_{\left(x^\alpha, 0^\alpha, 0^\beta\right)}S^{k-1}(r_1) \oplus T_{\left(0^\alpha, y^\alpha, 0^\beta\right)}S^{k-1}(r_2) \oplus T_{\left(0^\alpha, 0^\alpha, z^\beta\right)}S^{l-1}(r_3),$$

which means that any tangent vector $\left(X^\alpha, Y^\alpha, Z^\beta\right) \in T_{\left(x^\alpha, y^\alpha, z^\beta\right)}E^{2k+l}$ lies in $T_{\left(x^\alpha, y^\alpha, z^\beta\right)}M$ for each point $\left(x^\alpha, y^\alpha, z^\beta\right) \in M$ if and only if

$$\sum_{\alpha=1}^{k} x^\alpha X^\alpha = \sum_{\alpha=1}^{k} y^\alpha Y^\alpha = \sum_{\beta=1}^{l} z^\beta Z^\beta = 0.$$

Also, since $\left(X^\alpha, Y^\alpha, Z^\beta\right)$ is a tangent vector on the sphere $S^{2k+l-2}(R)$, we obtain

$$T_{\left(x^\alpha, y^\alpha, z^\beta\right)}M \subset T_{\left(x^\alpha, y^\alpha, z^\beta\right)}S^{2k+l-2}(R)$$

for each point $\left(x^\alpha, y^\alpha, z^\beta\right) \in M$.

If $\{N_1, N_2, N_3\}$ is a local orthonormal basis of the normal space $T_{\left(x^\alpha, y^\alpha, z^\beta\right)}M^\perp$ at any point $\left(x^\alpha, y^\alpha, z^\beta\right)$, then the normal vectors N_1, N_2 and N_3 can be chosen as follows:

$$N_1 = \frac{1}{R}\left(x^\alpha, y^\alpha, z^\beta\right),$$

$$N_2 = \frac{1}{R}\left(\frac{r_3}{r}x^\alpha, \frac{r_3}{r}y^\alpha, -\frac{r}{r_3}z^\beta\right)$$

and

$$N_3 = \frac{1}{r}\left(\frac{r_2}{r_1}x^\alpha, -\frac{r_1}{r_2}y^\alpha, 0^\beta\right),$$

where $r^2 = r_1^2 + r_2^2$. Identifying $i_* X$ with X for any tangent vector $X \in T_{\left(x^\alpha, y^\alpha, z^\beta\right)}M$, from (2.2), we have

$$\widetilde{F}N_\lambda = \xi_\lambda + \sum_{\mu=1}^{3} \mathcal{A}_{\lambda\mu} N_\mu \tag{3.12}$$

for any $\lambda \in \{1, 2, 3\}$, which shows that

$$\mathcal{A}_{\lambda\mu} = \left\langle \widetilde{F}N_\lambda, N_\mu \right\rangle$$

for any $\lambda, \mu \in \{1, 2, 3\}$. Hence, by a straightforward computation, the entries of the matrice $\mathcal{A} = \left[\mathcal{A}_{\lambda\mu}\right]_{3\times 3}$ are calculated as follows:

$$\mathcal{A}_{11} = \frac{1}{R^2}\left(\left(a - \sigma_{a,b}\right) R^2 + r_2^2 \sqrt{\Delta}\right),$$

$$\mathcal{A}_{12} = \mathcal{A}_{21} = \frac{r_3 r_2^2 \sqrt{\Delta}}{rR^2},$$

$$\mathcal{A}_{13} = \mathcal{A}_{31} = -\frac{r_1 r_2 \sqrt{\Delta}}{rR},$$

$$\mathcal{A}_{22} = \frac{1}{r^2 R^2}\left(\left(a - \sigma_{a,b}\right) r^2 R^2 + r_2^2 r_3^2 \sqrt{\Delta}\right),$$

$$\mathcal{A}_{23} = \mathcal{A}_{32} = -\frac{r_1 r_2 r_3 \sqrt{\Delta}}{r^2 R}$$

and

$$\mathcal{A}_{33} = \frac{1}{r^2}\left(\left(a - \sigma_{a,b}\right) r^2 + r_1^2 \sqrt{\Delta}\right),$$

where $\Delta = a^2 + 4b$. Also, by virtue of $\mathcal{A}_{\lambda\mu}$ for any $\lambda, \mu \in \{1, 2, 3\}$ given above, it follows from (3.12) that

$$\xi_1 = \xi_2 = \xi_3 = 0_{2k+l}, \tag{3.13}$$

in other words,

$$\widetilde{F}\left(T_{\left(x^\alpha, y^\alpha, z^\beta\right)} M^\perp\right) \subseteq T_{\left(x^\alpha, y^\alpha, z^\beta\right)} M^\perp.$$

Taking into consideration the fact $u_\lambda\left(X^\alpha, Y^\alpha, Z^\beta\right) = \left\langle\left(X^\alpha, Y^\alpha, Z^\beta\right), \xi_\lambda\right\rangle$ for any $\lambda \in \{1, 2, 3\}$, we see from (3.13) that

$$u_1 = u_2 = u_3 = 0 \tag{3.14}$$

for any $\lambda \in \{1, 2, 3\}$. On the other hand, from (2.1), we have

$$\widetilde{F}\left(X^\alpha, Y^\alpha, Z^\beta\right) = F\left(X^\alpha, Y^\alpha, Z^\beta\right) + \sum_{\lambda=1}^{3} u_\lambda\left(X^\alpha, Y^\alpha, Z^\beta\right) N_\lambda. \tag{3.15}$$

Thus, by the help of (3.14) and (3.15), we get

$$\widetilde{F}\left(X^\alpha, Y^\alpha, Z^\beta\right) = F\left(X^\alpha, Y^\alpha, Z^\beta\right),$$

which means that

$$\widetilde{F}\left(T_{\left(x^\alpha, y^\alpha, z^\beta\right)} M\right) \subseteq T_{\left(x^\alpha, y^\alpha, z^\beta\right)} M$$

and

$$F^2 = aF + bI.$$

Consequently, we derive an induced structure $(F, \langle,\rangle, \xi_\lambda = 0_{2k+l}, u_\lambda = 0, \mathcal{A})$ on the product of spheres M by the metallic Riemannian structure $\left(\langle,\rangle, \widetilde{F}\right)$ on the Euclidean space E^{2k+l} and $(F, \langle,\rangle)$ is a metallic Riemannian structure. Thus, by [17, Proposition 4.3], M is an invariant submanifold of codimension 3 in the Euclidean space E^{2k+l}. Furthermore, [4, Proposition 3.10] implies that M is a locally decomposable metallic Riemannian manifold.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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