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Research Article

Improvements of some Berezin radius inequalities

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ABSTRACT. The Berezin transform $\widetilde{A}$ and the Berezin radius of an operator A on the reproducing kernel Hilbert space over some set Q with normalized reproducing kernel $k_\eta := \frac{K_\eta}{\|K_\eta\|}$ are defined, respectively, by $\widetilde{A}(\eta) = \langle Ak_\eta, k_\eta \rangle$, $\eta \in Q$ and $\operatorname{ber}(A) := \sup_{\eta \in Q} \left|\widetilde{A}(\eta)\right|$. A simple comparison of these properties produces the inequalities $\frac{1}{4}\|A^*A + AA^*\| \leq \operatorname{ber}^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|$. In this research, we investigate other inequalities that are related to them. In particular, for $A \in \mathcal{L}(\mathcal{H}(Q))$ we prove that

$$\operatorname{ber}^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|_{\operatorname{ber}} - \frac{1}{4}\inf_{\eta \in Q}\left(\left(\widetilde{|A|}(\eta)\right) - \left(\widetilde{|A^*|}(\eta)\right)\right)^2.$$

Keywords: Mixed Schwarz inequality, Berezin radius, Furuta inequality.

2020 Mathematics Subject Classification: 47A30, 47A63.

1. INTRODUCTION

Many mathematicians and physicists are interested in the Berezin symbol of an operator defined with the help of a reproducing kernel Hilbert space. Several mathematicians have done extensive study on the Berezin radius inequality, which is presented in (1.1) (see [23]). Indeed, researchers are eager to obtain refinements and additions to this inequality given by (1.1) ([20], [32]). We show various inequalities for Berezin transformations of operators on the reproducing kernel Hilbert space $\mathcal{H}(Q)$ over some set Q in this study. By using Berezin transforms, we study several sharp inequalities involving powers of Berezin radius of some operators.

Remember that a reproducing kernel Hilbert space (abbreviated RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(Q)$ of complex-valued functions on some set Q in which:

(a) the evaluation functionals

$$\varphi_\eta(f) = f(\eta), \eta \in Q,$$

are continuous on $\mathcal{H}$;

(b) for every $\eta \in Q$, there exists a function $f_\eta \in \mathcal{H}$ such that $f_\eta(\eta) \neq 0$.

Then, via the classical Riesz representation theorem, we know that if $\mathcal{H}$ is a RKHS on Q, there is a unique element $K_\eta \in \mathcal{H}$ such that $h(\eta) = \langle h, K_\eta \rangle_{\mathcal{H}}$ for every $\eta \in Q$ and all $h \in \mathcal{H}$. The reproducing kernel at η denoted by the element K_η. In addition, we shall refer to the normalized reproducing kernel at η as $k_\eta := \frac{K_\eta}{\|K_\eta\|}$. Let $\mathcal{L}(\mathcal{H})$ be the Banach algebra of all

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bounded linear operators on a complex Hilbert space $\mathcal{H}$ including the identity operator $1_{\mathcal{H}}$ in $\mathcal{L}(\mathcal{H})$. The Berezin transform (symbol) of A is the complex-valued function on Q defined by

$$\widetilde{A}(\eta) := \langle Ak_\eta, k_\eta \rangle$$

for an operator $A \in \mathcal{L}(\mathcal{H})$.

The Berezin transform $\widetilde{A}$ is obviously a bounded function on Q and $\sup_{\eta \in Q} \left|\widetilde{A}(\eta)\right|$, which is known as the Berezin radius (number) of operator A, i.e.,

$$\operatorname{ber}(A) := \sup_{\eta \in Q} \left|\widetilde{A}(\eta)\right|.$$

The Berezin transform $\widetilde{A}$ is a bounded real-analytic function on Ω for any bounded operator A on $\mathcal{H}$. Properties of the operator A are often reflected in properties of the Berezin transform $\widetilde{A}$. F. Berezin proposed the Berezin transform in [7], and it has proven to be a valuable tool in operator theory, since many fundamental features of significant operators are stored in their Berezin transforms. The Berezin set and number, also known as $\operatorname{Ber}(A)$ and $\operatorname{ber}(A)$, were allegedly first publicly proposed by Karaev in [22].

The range of the Berezin transform $\widetilde{A}$, which is stated to be the Berezin set of operator A, is also obvious from the definition of the Berezin transform, i.e.,

$$\operatorname{Ber}(A) := \operatorname{Range}\left(\widetilde{A}\right) = \left\{\widetilde{A}(\eta) : \eta \in Q\right\}.$$

Recall that the numerical radius of operator A is defined by

$$w(A) := \sup_{\|x\|=1} |\langle Ax, x \rangle|.$$

It is well-known that

$$\operatorname{ber}(A) \le w(A) \le \|A\| \tag{1.1}$$

for any $X \in \mathcal{L}(\mathcal{H})$. See [1, 2, 9, 8, 17, 19, 24, 25, 30, 35] for further details.

Berezin set and Berezin radius of operators are new numerical properties of RKHS operators presented by Karaev in [21]. See [5, 12, 23] for the fundamental features and information about these new categories.

In 2021, Huban et al. [20] improved the inequality (1.1) by proving that

$$\operatorname{ber}(A) \le \frac{1}{2}\left(\|A\|_{\operatorname{ber}} + \left\|A^2\right\|_{\operatorname{ber}}^{1/2}\right) \tag{1.2}$$

for any $A \in \mathcal{L}(\mathcal{H})$.

It has been shown in [20] that if $A \in \mathcal{B}(\mathcal{H})$, then

$$\frac{1}{4}\|A^*A + AA^*\| \le \operatorname{ber}^2(A) \le \frac{1}{2}\|A^*A + AA^*\|. \tag{1.3}$$

Inspired by the numerical radius inequalities in [3], this study proves an extension of the inequality (1.3). In particular, for $A \in \mathcal{L}(\mathcal{H}(Q))$ we prove that

$$\operatorname{ber}^2(A) \le \frac{1}{2}\|A^*A + AA^*\|_{\operatorname{ber}} - \frac{1}{4}\inf_{\eta \in Q}\left(\left(\widetilde{|A|}(\eta)\right) - \left(\widetilde{|A^*|}(\eta)\right)\right)^2.$$

Other general-related outcomes have also been established.

2. Auxiliary Theorems

The following chain of corollaries is required to attain our aim.

According to the basic Schwarz inequality for positive operators, if $A \in \mathcal{L}(\mathcal{H})$ is a positive operators, then

$$|\langle Ax_1, x_2\rangle|^2 \le \langle Ax_1, x_1\rangle \langle Ax_2, x_2\rangle \tag{2.4}$$

for any $x_1, x_2 \in \mathcal{H}$.

Reid [28] demonstrated an inequality in 1951 that may be regarded a version of the Schwarz inequality. In fact, he proved that for all $x_1 \in \mathcal{H}$

$$|\langle ABx_1, x_2\rangle| \le \|B\| \langle Ax_1, x_1\rangle \tag{2.5}$$

for any operators $A \in \mathcal{L}(\mathcal{H})$ where A is positive and AB is selfadjoint.

Kato [27] established a companion inequality of (2.4), the mixed Schwarz inequality, in 1952, which claims

$$|\langle Ax_1, x_2\rangle|^2 \le \left\langle |A|^{2\alpha} x_1, x_1\right\rangle \left\langle |A^*|^{2(1-\alpha)} x_2, x_2\right\rangle, \ 0 \le \alpha \le 1 \tag{2.6}$$

for every operators $A \in \mathcal{L}(\mathcal{H})$ and any vectors $x_1, x_2 \in \mathcal{H}$, where $|A| = (A^*A)^{1/2}$.

In 1988, Kittaneh [24] proved a very interesting extension combining both the Halmos-Reid inequality (2.5) and the mixed Schwarz inequality (2.6). His result reads that

$$|\langle ABx_1, x_2\rangle| \le r(B) \|f(|A|) x_1\| \|g(|A^*|) x_2\| \tag{2.7}$$

for any vectors $x_1, x_2 \in \mathcal{H}$, where $A, B \in \mathcal{L}(\mathcal{H})$ such that $|A|B = B^*|A|$ and f, g are nonnegative continuous functions defined on $[0, \infty)$ satisfying that $f(t)g(t) = t$ $(t \ge 0)$. Clearly, when we select $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ with $B = 1_{\mathcal{H}}$, we are referring to the inequality (2.6). Furthermore, several alterations that are chosen $\alpha = \frac{1}{2}$ pertain to the Halmos version of the Reid inequality.

Furuta [11] demonstrated another extension of Kato's inequality (2.6) in 1994, as follows:

$$\left|\left\langle A|A|^{\alpha+\beta-1} x_1, x_2\right\rangle\right|^2 \le \left\langle |A|^{2\alpha} x_1, x_1\right\rangle \left\langle |A|^{2\beta} x_2, x_2\right\rangle \tag{2.8}$$

for any $x_1, x_2 \in \mathcal{H}$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \ge 1$.

The inequality (2.8) was generalized for any $\alpha, \beta \ge 0$ with $\alpha + \beta \ge 1$ by Dragomir in [10]. Indeed, as Dragomir pointed out, Furuta adopted the condition $\alpha, \beta \in [0, 1]$ to match with the Heinz-Kato inequality, which reads:

$$|\langle Ax_1, x_2\rangle| \le \|T^\alpha x_1\| \left\|S^{1-\alpha} x_2\right\|$$

for any $x_1, x_2 \in \mathcal{H}$ and $\alpha \in [0, 1]$ where T and S are positive operators such that $\|Ax_1\| \le \|Tx_1\|$and $\|A^*x_2\| \le \|Sx_2\|$ for any $x_1, x_2 \in \mathcal{H}$.

Lemma 2.1. *If $B \in \mathcal{L}(\mathcal{H})$, $B \ge 0$ and $x_1 \in H$ is any unit vector, then there's*

$$\langle Bx, x\rangle^r \le (\ge) \langle B^r x, x\rangle, \ r \ge 1 \ (0 \le r \le 1). \tag{2.9}$$

Kittaneh and Manasrah [26] discovered this conclusion, which is a refinement of the scalar Young inequality.

Lemma 2.2. *If $a, b \ge 0$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$ab + \min\left\{\frac{1}{p}, \frac{1}{q}\right\} \left(a^{p/2} - b^{q/2}\right)^2 \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.10}$$

Sheikhhosseini et al. [31] recently found the following generalization of (2.10).

Lemma 2.3. *If $a, b > 0$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then for $m = 1, 2, 3, ...$,*

$$\left(a^{1/p}b^{1/q}\right)^m + r_0^m \left(a^{m/2} - b^{m/2}\right)^2 \leq \left(\frac{a^r}{p} + \frac{b^r}{q}\right)^{m/r}, \ r \geq 1, \tag{2.11}$$

where $r_0 = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$. In particular, if $p = q = 2$, then

$$\left(a^{1/2}b^{1/2}\right)^m + 2^{-m}\left(a^{m/2} - b^{m/2}\right)^2 \leq 2^{-m/r}\left(a^r + b^r\right)^{m/r}.$$

For $m = 1$,

$$\left(a^{1/2}b^{1/2}\right) + 2^{-1}\left(a^{1/2} - b^{1/2}\right)^2 \leq 2^{-1/r}\left(a^r + b^r\right)^{1/r}.$$

3. Main result

We are now prepared to provide this section's primary results. The section's next finding is a revised Berezin radius inequality.

Theorem 3.1. *If $A \in \mathcal{L}(\mathcal{H}(Q))$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 1$, then we get*

$$\begin{aligned} \mathrm{ber}^m\left(A\left|A\right|^{\alpha+\beta-1}\right) \leq{}& \frac{1}{2^{m/r}}\left\|\left|A\right|^{2r\alpha} + \left|A^*\right|^{2r\beta}\right\|_{\mathrm{ber}}^{m/r} \\ &- \frac{1}{2^m}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2\alpha}}(\eta)\right)^{m/2} - \left(\widetilde{\left|A^*\right|^{2\beta}}(\eta)\right)^{m/2}\right)^2 \end{aligned} \tag{3.12}$$

for all $r \geq 1$.

Proof. For all $m \geq 1$, on choosing $x_1 = k_\eta$ and $x_2 = k_\tau$ in the inequality (2.8), we get

$$\left|\left\langle A\left|A\right|^{\alpha+\beta-1}k_\eta, k_\tau\right\rangle\right|^m \leq \left\langle\left|A\right|^{2\alpha}k_\eta, k_\eta\right\rangle^{\frac{m}{2}}\left\langle\left|A^*\right|^{2\beta}k_\tau, k_\tau\right\rangle^{\frac{m}{2}}.$$

By the inequalities (2.9) and (2.11), for $\eta \in Q$ with $\eta = \tau$ we have

$$\begin{aligned} \left|\left\langle A\left|A\right|^{\alpha+\beta-1}k_\eta, k_\eta\right\rangle\right|^m &\leq \left\langle\left|A\right|^{2\alpha}k_\eta, k_\eta\right\rangle^{\frac{m}{2}}\left\langle\left|A^*\right|^{2\beta}k_\eta, k_\eta\right\rangle^{m/2}. \\ &\leq \left(\frac{\left\langle\left|A\right|^{2\alpha}k_\eta, k_\eta\right\rangle^r + \left\langle\left|A^*\right|^{2\beta}k_\eta, k_\eta\right\rangle^r}{2}\right)^{m/r} \\ &\quad - \frac{1}{2^m}\left(\left\langle\left|A\right|^{2\alpha}k_\eta, k_\eta\right\rangle^{m/2} - \left\langle\left|A^*\right|^{2\beta}k_\eta, k_\eta\right\rangle^{m/r}\right)^2 \\ &\leq \left(\frac{\left\langle\left|A\right|^{2r\alpha}k_\eta, k_\eta\right\rangle + \left\langle\left|A^*\right|^{2r\beta}k_\eta, k_\eta\right\rangle}{2}\right)^{m/r} \\ &\quad - \frac{1}{2^m}\left(\left\langle\left|A\right|^{2\alpha}k_\eta, k_\eta\right\rangle^{m/2} - \left\langle\left|A^*\right|^{2\beta}k_\eta, k_\eta\right\rangle^{m/2}\right)^2, \end{aligned}$$

and

$$\begin{aligned} \sup_{\eta\in Q}\left|\left\langle A\left|A\right|^{\alpha+\beta-1}k_\eta, k_\eta\right\rangle\right|^m \leq{}& \frac{1}{2^{m/r}}\sup_{\eta\in Q}\left(\left\langle\left|A\right|^{2r\alpha}k_\eta, k_\eta\right\rangle + \left\langle\left|A^*\right|^{2r\beta}k_\eta, k_\eta\right\rangle\right)^{m/r} \\ &- \frac{1}{2^m}\inf_{\eta\in Q}\left(\left\langle\left|A\right|^{2\alpha}k_\eta, k_\eta\right\rangle^{m/2} - \left\langle\left|A^*\right|^{2\beta}k_\eta, k_\eta\right\rangle^{m/2}\right)^2 \end{aligned}$$

which is equivalent to

$$\mathrm{ber}^m\left(A\left|A\right|^{\alpha+\beta-1}\right) \leq \frac{1}{2^{m/r}}\left\|\left|A\right|^{2r\alpha}+\left|A^*\right|^{2r\beta}\right\|_{\mathrm{ber}}^{m/r} - \frac{1}{2^m}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2\alpha}}\left(\eta\right)\right)^{m/2}-\left(\widetilde{\left|A^*\right|^{2\beta}}\left(\eta\right)\right)^{m/2}\right)^2,$$

and completes the theorem's proof. □

We get the following result by putting $m=2$ in (3.12).

Corollary 3.1. *If $A\in\mathcal{L}(\mathcal{H}\left(Q\right))$ and $\alpha,\beta\geq 0$ such that $\alpha+\beta\geq 1$, then we have*

$$\mathrm{ber}^2\left(A\left|A\right|^{\alpha+\beta-1}\right) \leq \frac{1}{2^{2/r}}\left\|\left|A\right|^{2r\alpha}+\left|A^*\right|^{2r\beta}\right\|_{\mathrm{ber}}^{2/r} - \frac{1}{4}\inf_{\eta\in Q}\left(\widetilde{\left|A\right|^{2\alpha}}\left(\eta\right)-\widetilde{\left|A^*\right|^{2\beta}}\left(\eta\right)\right)^2 \tag{3.13}$$

for all $r\geq 1$.

By choosing $r=1$ in (3.13), we get

$$\mathrm{ber}^2\left(A\left|A\right|^{\alpha+\beta-1}\right) \leq \frac{1}{4}\left\|\left|A\right|^{2\alpha}+\left|A^*\right|^{2\beta}\right\|_{\mathrm{ber}}^{2} - \frac{1}{4}\inf_{\eta\in Q}\left(\widetilde{\left|A\right|^{2\alpha}}\left(\eta\right)-\widetilde{\left|A^*\right|^{2\beta}}\left(\eta\right)\right)^2 \tag{3.14}$$

for all $\alpha,\beta\geq 0$ such that $\alpha+\beta\geq 1$.

Also for $\alpha=\beta=\frac{1}{2}$ in (3.14), we get

$$\mathrm{ber}^2\left(A\right) \leq \frac{1}{4}\left\|\left|A\right|+\left|A^*\right|\right\|_{\mathrm{ber}}^{2} - \frac{1}{4}\inf_{\eta\in Q}\left(\widetilde{\left|A\right|}\left(\eta\right)-\widetilde{\left|A^*\right|}\left(\eta\right)\right)^2.$$

In particular, take $\alpha=\beta=1$, we have

$$\mathrm{ber}^2\left(A\left|A\right|\right) \leq \frac{1}{4}\left\|\left|A\right|^{2}+\left|A^*\right|^{2}\right\|_{\mathrm{ber}}^{2} - \frac{1}{4}\inf_{\eta\in Q}\left(\widetilde{\left|A\right|^{2}}\left(\eta\right)-\widetilde{\left|A^*\right|^{2}}\left(\eta\right)\right)^2$$

and

$$\mathrm{ber}^2\left(A\left|A\right|\right) \leq \frac{1}{4}\left\|A^*A+AA^*\right\|_{\mathrm{ber}}^{2} - \frac{1}{4}\inf_{\eta\in Q}\left(\widetilde{\left[A^*A-AA^*\right]}\left(\eta\right)\right)^2.$$

A generalization of the above findings may be expressed as follows:

Theorem 3.2. *If $A\in\mathcal{L}(\mathcal{H}\left(Q\right))$ and $\alpha,\beta\geq 0$ such that $\alpha+\beta\geq 1$, then we have*

$$\mathrm{ber}^{2s}\left(A\left|A\right|^{\alpha+\beta-1}\right) \leq \frac{1}{2^{2/r}}\left\|\left|A\right|^{2rs\alpha}+\left|A^*\right|^{2rs\beta}\right\|_{\mathrm{ber}}^{2/r} - \frac{1}{4}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2sr\alpha}}\left(\eta\right)\right)-\left(\widetilde{\left|A^*\right|^{2sr\beta}}\left(\eta\right)\right)\right) \tag{3.15}$$

for all $r,s\geq 1$.

Proof. Setting $x_1 = x_2 = k_\eta$ in (2.8) and then using Lemma 2.3 with $p = q = 2$ and $m = 2$, we get

$$\begin{aligned}
\left|\left\langle A\left|A\right|^{\alpha+\beta-1}k_\eta,k_\eta\right\rangle\right|^{2s} &\le \left\langle\left|A\right|^{2\alpha}k_\eta,k_\eta\right\rangle^{s}\left\langle\left|A^*\right|^{2\beta}k_\eta,k_\eta\right\rangle^{s}\ (t^s\text{ increasing})\\
&\le \left\langle\left|A\right|^{2s\alpha}k_\eta,k_\eta\right\rangle\left\langle\left|A^*\right|^{2s\beta}k_\eta,k_\eta\right\rangle\ (\text{by convexity of }t^s)\\
&\le \frac{1}{2^{2/r}}\left(\left\langle\left|A\right|^{2s\alpha}k_\eta,k_\eta\right\rangle^{r}+\left\langle\left|A^*\right|^{2s\beta}k_\eta,k_\eta\right\rangle^{r}\right)^{2/r}\\
&\quad(\text{by the inequality (2.11)})\\
&\quad-\frac{1}{4}\left[\left\langle\left|A\right|^{2sr\alpha}k_\eta,k_\eta\right\rangle-\left\langle\left|A^*\right|^{2rs\beta}k_\eta,k_\eta\right\rangle\right]\\
&\le \frac{1}{2^{2/r}}\left(\left\langle\left|A\right|^{2rs\alpha}k_\eta,k_\eta\right\rangle+\left\langle\left|A^*\right|^{2rs\beta}k_\eta,k_\eta\right\rangle\right)^{2/r}\\
&\quad(\text{by the inequality (2.9)})\\
&\quad-\frac{1}{4}\left[\left\langle\left|A\right|^{2sr\alpha}k_\eta,k_\eta\right\rangle-\left\langle\left|A^*\right|^{2rs\beta}k_\eta,k_\eta\right\rangle\right].
\end{aligned}$$

Equivalenty, we may write

$$\begin{aligned}
\left|A\widetilde{\left|A\right|^{\alpha+\beta-1}}(\eta)\right|^{2s} &\le \frac{1}{2^{2/r}}\left(\widetilde{\left|A\right|^{2rs\alpha}}(\eta)+\widetilde{\left|A^*\right|^{2rs\beta}}(\eta)\right)^{2/r}\\
&\quad-\frac{1}{4}\inf_{\eta\in Q}\left[\left(\widetilde{\left|A\right|^{2sr\alpha}}(\eta)\right)-\left(\widetilde{\left|A^*\right|^{2sr\beta}}(\eta)\right)\right].
\end{aligned}$$

By taking the supremum over $\eta \in Q$, we obtain

$$\begin{aligned}
\operatorname{ber}^{2s}\left(A\left|A\right|^{\alpha+\beta-1}\right) &\le \frac{1}{2^{2/r}}\left\|\left|A\right|^{2rs\alpha}+\left|A^*\right|^{2rs\beta}\right\|_{\text{ber}}^{2/r}\\
&\quad-\frac{1}{4}\inf_{\eta\in Q}\left[\left(\widetilde{\left|A\right|^{2sr\alpha}}(\eta)\right)-\left(\widetilde{\left|A^*\right|^{2sr\beta}}(\eta)\right)\right],
\end{aligned}$$

which completes the proof. □

We get the following result by setting $r = 1$ in (3.15).

Corollary 3.2. *If $A \in \mathcal{L}(\mathcal{H}(Q))$ and $\alpha, \beta \ge 0$ such that $\alpha + \beta \ge 1$, then we have*

$$\begin{aligned}
\operatorname{ber}^{2s}\left(A\left|A\right|^{\alpha+\beta-1}\right) &\le \frac{1}{4}\left\|\left|A\right|^{2s\alpha}+\left|A^*\right|^{2s\beta}\right\|_{\text{ber}}^{2}\\
&\quad-\frac{1}{4}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2s\alpha}}(\eta)\right)-\left(\widetilde{\left|A^*\right|^{2s\beta}}(\eta)\right)\right)
\end{aligned}\tag{3.16}$$

for all $s \ge 1$.

In (3.16), let $\alpha = \beta = \frac{1}{2}$ we get

$$\operatorname{ber}^{2s}(A) \le \frac{1}{4}\left\|\left|A\right|^{s}+\left|A^*\right|^{s}\right\|_{\text{ber}}^{2}-\frac{1}{4}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{s}}(\eta)\right)-\left(\widetilde{\left|A^*\right|^{s}}(\eta)\right)\right)$$

for every $s \ge 1$. We have, in particular, for $s = 1$

$$\operatorname{ber}^{2}(A) \le \frac{1}{4}\left\|\left|A\right|+\left|A^*\right|\right\|_{\text{ber}}^{2}-\frac{1}{4}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|}(\eta)\right)-\left(\widetilde{\left|A^*\right|}(\eta)\right)\right).$$

By choosing $\alpha=\beta=\frac{1}{s}$, $(s\geq 1)$, in (3.16) we get

$$\operatorname{ber}^{2s}\left(A\left|A\right|^{\frac{2}{s}-1}\right)\leq\frac{1}{4}\left\|\left|A\right|^{2}+\left|A^{*}\right|^{2}\right\|_{\operatorname{ber}}^{2}-\frac{1}{4}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2}}\left(\eta\right)\right)-\left(\widetilde{\left|A^{*}\right|^{2}}\left(\eta\right)\right)\right).\tag{3.17}$$

Also for $s=1$ in (3.17), we get

$$\operatorname{ber}^{2}\left(A\left|A\right|\right)\leq\frac{1}{4}\left\|\left|A\right|^{2}+\left|A^{*}\right|^{2}\right\|_{\operatorname{ber}}^{2}-\frac{1}{4}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2}}\left(\eta\right)\right)-\left(\widetilde{\left|A^{*}\right|^{2}}\left(\eta\right)\right)\right),\tag{3.18}$$

and

$$\operatorname{ber}^{2}\left(A\left|A\right|\right)\leq\frac{1}{4}\left\|A^{*}A+AA^{*}\right\|_{\operatorname{ber}}^{2}-\frac{1}{4}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2}}\left(\eta\right)\right)-\left(\widetilde{\left|A^{*}\right|^{2}}\left(\eta\right)\right)\right).$$

Remark 3.1. *By choosing $\alpha=\beta=\frac{1}{2}$, $s=1$, $r=2$ in (3.16), we have*

$$\operatorname{ber}^{2}\left(A\right)\leq\frac{1}{2}\left\|\left|A\right|+\left|A^{*}\right|\right\|_{\operatorname{ber}}^{2}-\frac{1}{4}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2}}\left(\eta\right)\right)-\left(\widetilde{\left|A^{*}\right|^{2}}\left(\eta\right)\right)\right)$$

or

$$\operatorname{ber}^{2}\left(A\right)\leq\frac{1}{2}\left\|A^{*}A+AA^{*}\right\|_{\operatorname{ber}}^{2}-\frac{1}{4}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2}}\left(\eta\right)\right)-\left(\widetilde{\left|A^{*}\right|^{2}}\left(\eta\right)\right)\right).\tag{3.19}$$

This improves the upper bound of the inequality (1.2).

Theorem 3.3. *If $A\in\mathcal{L}(\mathcal{H}\left(Q\right))$ and $\alpha,\beta\geq 0$ such that $\alpha+\beta\geq 1$, then we have*

(i)

$$\operatorname{ber}^{2s}\left(A\left|A\right|^{\alpha+\beta-1}\right)\leq\left\|\frac{1}{p}\left|A\right|^{2sp\alpha}+\frac{1}{q}\left|A^{*}\right|^{2sq\beta}\right\|_{\operatorname{ber}}-r_{0}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2s\alpha}}\left(\eta\right)\right)^{p/2}-\left(\widetilde{\left|A^{*}\right|^{2s\beta}}\left(\eta\right)\right)^{q/2}\right)^{2}\tag{3.20}$$

for every $s\geq 1$ and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, where $r_{0}:=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$.

(ii)

$$\operatorname{ber}^{2s}\left(A\left|A\right|^{\alpha+\beta-1}\right)\leq\frac{1}{2}\left\|\left|A\right|^{4s\alpha}+\left|A^{*}\right|^{4s\beta}\right\|_{\operatorname{ber}}-\frac{1}{2}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2s\alpha}}\left(\eta\right)\right)-\left(\widetilde{\left|A^{*}\right|^{2s\beta}}\left(\eta\right)\right)\right)^{2}.\tag{3.21}$$

Proof. Now, as in (2.8) but with $x_1 = x_2 = k_\eta$, we have by convexity of t^s

$$\begin{aligned}
\left|\left\langle A\left|A\right|^{\alpha+\beta-1}k_\eta,k_\eta\right\rangle\right|^{2s} &\le \left\langle \left|A\right|^{2\alpha}k_\eta,k_\eta\right\rangle^{s}\left\langle \left|A^*\right|^{2\beta}k_\eta,k_\eta\right\rangle^{s} \\
&\text{(by the inequaltiy (2.8))} \\
&\le \left\langle \left|A\right|^{2s\alpha}k_\eta,k_\eta\right\rangle \;\left\langle \left|T^*\right|^{2s\beta}k_\eta,k_\eta\right\rangle \\
&\le \frac{1}{p}\left\langle \left|A\right|^{2s\alpha}k_\eta,k_\eta\right\rangle^{p}+\frac{1}{q}\left\langle \left|A^*\right|^{2s\beta}k_\eta,k_\eta\right\rangle^{q} \\
&\text{(by the inequality (2.10))} \\
&-r_0\left(\left\langle \left|A\right|^{2s\alpha}k_\eta,k_\eta\right\rangle^{\frac{p}{2}}-\left\langle \left|A^*\right|^{2s\beta}k_\eta,k_\eta\right\rangle^{\frac{q}{2}}\right)^2 \\
&\le \frac{1}{p}\left\langle \left|A\right|^{2sp\alpha}k_\eta,k_\eta\right\rangle+\frac{1}{q}\left\langle \left|A^*\right|^{2sq\beta}k_\eta,k_\eta\right\rangle \\
&\text{(by the inequality (2.9))} \\
&-r_0\left(\left\langle \left|A\right|^{2s\alpha}k_\eta,k_\eta\right\rangle^{\frac{p}{2}}-\left\langle \left|A^*\right|^{2s\beta}k_\eta,k_\eta\right\rangle^{\frac{q}{2}}\right)^2
\end{aligned}$$

for $s \ge 1$. Thus,

$$\begin{aligned}
\left|\left\langle A\left|A\right|^{\alpha+\beta-1}k_\eta,k_\eta\right\rangle\right|^{2s} &\le \frac{1}{p}\left\langle \left|A\right|^{2sp\alpha}k_\eta,k_\eta\right\rangle+\frac{1}{q}\left\langle \left|A^*\right|^{2sq\beta}k_\eta,k_\eta\right\rangle \\
&-r_0\left(\left\langle \left|A\right|^{2s\alpha}k_\eta,k_\eta\right\rangle^{\frac{p}{2}}-\left\langle \left|A^*\right|^{2s\beta}k_\eta,k_\eta\right\rangle^{\frac{q}{2}}\right)^2,
\end{aligned}$$

and by taking supremum over $\eta \in Q$, we then obtain the first inequality

$$\operatorname{ber}^{2s}\left(A\left|A\right|^{\alpha+\beta-1}\right) \le \frac{1}{2}\left\|\left|A\right|^{4s\alpha}+\left|A^*\right|^{4s\beta}\right\|_{\operatorname{ber}}-\frac{1}{2}\inf_{\eta\in Q}\left(\left(\widetilde{\left|A\right|^{2s\alpha}}(\eta)\right)-\left(\widetilde{\left|A^*\right|^{2s\beta}}(\eta)\right)\right)^2$$

as required. Taking $p = q = 2$, we get the particular case (3.21). □

Several intriguing particular situations might be drawn from this (3.12). When we put $\alpha = \beta = \frac{1}{2}$ in (3.13), we get

$$\operatorname{ber}^{2s}(A) \le \frac{1}{2}\left\|\left|A\right|^{2s}+\left|A^*\right|^{2s}\right\|_{\operatorname{ber}}-\frac{1}{2}\inf_{\eta\in Q}\left(\widetilde{\left|A\right|^{s}}(\eta)-\widetilde{\left|A^*\right|^{s}}(\eta)\right)^2$$

for every $s \ge 1$. We have, in particular, for $s = 1$

$$\operatorname{ber}^{2}(A) \le \frac{1}{2}\left\|\left|A\right|^{2}+\left|A^*\right|^{2}\right\|_{\operatorname{ber}}-\frac{1}{2}\inf_{\eta\in Q}\left(\widetilde{\left|A\right|}(\eta)-\widetilde{\left|A^*\right|}(\eta)\right)^2,$$

which can be written as

$$\operatorname{ber}^{2}(A) \le \frac{1}{2}\left\|A^*A+AA^*\right\|_{\operatorname{ber}}-\frac{1}{2}\inf_{\eta\in Q}\left(\widetilde{\left|A\right|}(\eta)-\widetilde{\left|A^*\right|}(\eta)\right)^2. \tag{3.22}$$

and this refines the upper bound of the refinement of the inequality (1.3). Clearly, (3.22) is better than (3.19) which in turn bettern that (1.2).

Remark 3.2. *(i) When we set $\alpha = \beta = 1$ in (3.20), we get*

$$\operatorname{ber}^{2s}(A|A|) \leq \left\| \frac{1}{p}|A|^{2sp} + \frac{1}{q}|A^*|^{2sq} \right\|_{\text{ber}} - r_0 \inf_{\eta\in Q}\left(\left(\widetilde{|A|^{2s}}(\eta)\right)^{p/2} - \left(\widetilde{|A^*|^{2s}}(\eta)\right)^{q/2} \right)^2$$

for every $s \geq 1$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, where $r_0 := \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

(ii) Choose $s = 1$ and $p = q = 2$ in the above inequality, we get

$$\operatorname{ber}^2(A|A|) \leq \left\| \frac{1}{p}|A|^4 + \frac{1}{q}|A^*|^4 \right\|_{\text{ber}} - \frac{1}{2}\inf_{\eta\in Q}\left(\left(\widetilde{|A|^2}(\eta)\right) - \left(\widetilde{|A^*|^2}(\eta)\right) \right)^2.$$

The Berezin radius inequality of Hilbert space operators of a certain kind for commutators may be proven as follows:

Theorem 3.4. *If $A, B \in \mathcal{L}(\mathcal{H}(Q))$ and $\alpha, \beta, \gamma, \delta \geq 0$ such that $\alpha + \beta \geq 1$ and $\gamma + \delta \geq 1$, then we have*

$$\begin{aligned} &\operatorname{ber}\left(A|A|^{\alpha+\beta-1} + B|B|^{\gamma+\delta-1}\right) \\ \leq& \frac{1}{2^{1/r}}\left\| |A|^{2r\alpha} + |A^*|^{2r\beta} \right\|_{\text{ber}}^{1/r} + \frac{1}{2^{1/r}}\left\| |B|^{2r\gamma} + |B^*|^{2r\delta} \right\|_{\text{ber}}^{1/r} \\ &- \frac{1}{2}\inf_{\eta\in Q}\left(\left(\widetilde{|A|^{2\alpha}}(\eta)\right)^{1/2} - \left(\widetilde{|A^*|^{2\beta}}(\eta)\right)^{1/2} \right)^2 \\ &- \frac{1}{2}\inf_{\eta\in Q}\left(\left(\widetilde{|B|^{2\gamma}}(\eta)\right)^{1/2} - \left(\widetilde{|B^*|^{2\delta}}(\eta)\right)^{1/2} \right)^2 \end{aligned} \tag{3.23}$$

for all $r \geq 1$.

Proof. Using the triangle inequality, we get

$$\begin{aligned} &\left|\left\langle \left(A|A|^{\alpha+\beta-1} + B|B|^{\gamma+\delta-1}\right) k_\eta, k_\eta \right\rangle\right| \\ \leq& \left|\left\langle A|A|^{\alpha+\beta-1} k_\eta, k_\eta \right\rangle\right| + \left|\left\langle B|B|^{\gamma+\delta-1} k_\eta, k_\eta \right\rangle\right| \\ \leq& \left\langle |A|^{2\alpha} k_\eta, k_\eta \right\rangle^{\frac{1}{2}} \left\langle |A^*|^{2\beta} k_\eta, k_\eta \right\rangle^{\frac{1}{2}} \left\langle |B|^{2\gamma} k_\eta, k_\eta \right\rangle^{\frac{1}{2}} \left\langle |B^*|^{2\delta} k_\eta, k_\eta \right\rangle^{1/2} \\ &\text{(by the inequality (2.8))} \\ \leq& \frac{1}{2^{1/r}}\left(\left\langle |A|^{2\alpha} k_\eta, k_\eta \right\rangle^r + \left\langle |A^*|^{2\beta} k_\eta, k_\eta \right\rangle^r \right)^{1/r} \\ &\text{(by the inequality (2.11))} \\ &- \frac{1}{2}\left(\left\langle |A|^{2\alpha} k_\eta, k_\eta \right\rangle^{1/2} - \left\langle |A^*|^{2\beta} k_\eta, k_\eta \right\rangle^{1/2} \right)^2 \\ &+ 2^{\frac{-1}{r}}\left(\left\langle |B|^{2\gamma} k_\eta, k_\eta \right\rangle^r + \left\langle |B^*|^{2\delta} k_\eta, k_\eta \right\rangle^r \right)^{1/r} \\ &- \frac{1}{2}\left(\left\langle |B|^{2\gamma} k_\eta, k_\eta \right\rangle^{1/2} - \left\langle |B^*|^{2\delta} k_\eta, k_\eta \right\rangle^{1/2} \right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq 2^{\frac{-1}{r}}\left(\left\langle |A|^{2r\alpha}k_\eta,k_\eta\right\rangle+\left\langle |A^*|^{2r\beta}k_\eta,k_\eta\right\rangle\right)^{1/r} \\
&\quad \text{(by the inequality (2.9))} \\
&-\frac{1}{2}\left(\left\langle |A|^{2\alpha}k_\eta,k_\eta\right\rangle^{1/2}-\left\langle |A^*|^{2\beta}k_\eta,k_\eta\right\rangle^{1/2}\right)^2 \\
&+\frac{1}{2^{1/r}}\left(\left\langle |B|^{2r\gamma}k_\eta,k_\eta\right\rangle+\left\langle |B^*|^{2r\delta}k_\eta,k_\eta\right\rangle\right)^{1/r} \\
&-\frac{1}{2}\left(\left\langle |B|^{2\gamma}k_\eta,k_\eta\right\rangle^{1/2}-\left\langle |B^*|^{2\delta}k_\eta,k_\eta\right\rangle^{1/2}\right)^2,
\end{aligned}$$

and so

$$\begin{aligned}
\left|\left(\widetilde{A|A|^{\alpha+\beta-1}+B|B|^{\gamma+\delta-1}}(\eta)\right)\right| \leq\ & \frac{1}{2^{1/r}}\left(\widetilde{|A|^{2r\alpha}}(\eta)+\widetilde{|A^*|^{2r\beta}}(\eta)\right)^{1/r} \\
&+\frac{1}{2^{1/r}}\left(\widetilde{|B|^{2r\gamma}}(\eta)+\widetilde{|B^*|^{2r\delta}}(\eta)\right)^{1/r} \\
&-\frac{1}{2}\inf_{\eta\in Q}\left(\left(\widetilde{|A|^{2\alpha}}(\eta)\right)^{1/2}-\left(\widetilde{|A^*|^{2\beta}}(\eta)\right)^{1/2}\right)^2 \\
&-\frac{1}{2}\inf_{\eta\in Q}\left(\left(\widetilde{|B|^{2\gamma}}(\eta)\right)^{1/2}-\left(\widetilde{|B^*|^{2\delta}}(\eta)\right)^{1/2}\right)^2.
\end{aligned}$$

By taking supremum over $\eta\in Q$ above inequality, we have

$$\begin{aligned}
\sup_{\eta\in Q}\left|\left(\widetilde{A|A|^{\alpha+\beta-1}+B|B|^{\gamma+\delta-1}}(\eta)\right)\right| \leq\ & \frac{1}{2^{1/r}}\sup_{\eta\in Q}\left(\widetilde{|A|^{2r\alpha}}(\eta)+\widetilde{|A^*|^{2r\beta}}(\eta)\right)^{1/r} \\
&+\frac{1}{2^{1/r}}\sup_{\eta\in Q}\left(\widetilde{|B|^{2r\gamma}}(\eta)+\widetilde{|B^*|^{2r\delta}}(\eta)\right)^{1/r} \\
&-\frac{1}{2}\inf_{\eta\in Q}\left(\left(\widetilde{|A|^{2\alpha}}(\eta)\right)^{1/2}-\left(\widetilde{|A^*|^{2\beta}}(\eta)\right)^{1/2}\right)^2 \\
&-\frac{1}{2}\inf_{\eta\in Q}\left(\left(\widetilde{|B|^{2\gamma}}(\eta)\right)^{1/2}-\left(\widetilde{|B^*|^{2\delta}}(\eta)\right)^{1/2}\right)^2
\end{aligned}$$

which clearly implies that

$$\begin{aligned}
\operatorname{ber}\left(A|A|^{\alpha+\beta-1}+B|B|^{\gamma+\delta-1}\right) \leq\ & \frac{1}{2^{1/r}}\left\||A|^{2r\alpha}+|A^*|^{2r\beta}\right\|_{\text{ber}}^{1/r} \\
&+\frac{1}{2^{1/r}}\left\||B|^{2r\gamma}+|B^*|^{2r\delta}\right\|_{\text{ber}}^{1/r} \\
&-\frac{1}{2}\inf_{\eta\in Q}\left(\left(\widetilde{|A|^{2\alpha}}(\eta)\right)^{1/2}-\left(\widetilde{|A^*|^{2\beta}}(\eta)\right)^{1/2}\right)^2 \\
&-\frac{1}{2}\inf_{\eta\in Q}\left(\left(\widetilde{|B|^{2\gamma}}(\eta)\right)^{1/2}-\left(\widetilde{|B^*|^{2\delta}}(\eta)\right)^{1/2}\right)^2.
\end{aligned}$$

Then the desired result has been obtained. □

Using $r = 1$ in the proof of Theorem 3.4, we achieve the desired result.

Corollary 3.3. *If $A, B \in \mathcal{L}(\mathcal{H}(Q))$ and $\alpha, \beta, \gamma, \delta \geq 0$ such that $\alpha + \beta \geq 1$ and $\gamma + \delta \geq 1$, then we have*

$$\begin{aligned} &\operatorname{ber}\left(A|A|^{\alpha+\beta-1} + B|B|^{\gamma+\delta-1}\right) \\ \leq &\frac{1}{2}\left\| |A|^{2\alpha} + |A^*|^{2\beta} + |B|^{2\gamma} + |B^*|^{2\delta} \right\|_{\text{ber}} \\ &-\frac{1}{2}\inf_{\eta \in Q}\left(\left(\widetilde{|A|^{2\alpha}}(\eta)\right)^{1/2} - \left(\widetilde{|A^*|^{2\beta}}(\eta)\right)^{1/2}\right)^2 \\ &-\frac{1}{2}\inf_{\eta \in Q}\left(\left(\widetilde{|B|^{2\gamma}}(\eta)\right)^{1/2} - \left(\widetilde{|B^*|^{2\delta}}(\eta)\right)^{1/2}\right)^2. \end{aligned} \tag{3.24}$$

Remark 3.3. *(i) Setting $\alpha = \beta = \gamma = \delta = \frac{1}{2}$ in (3.24), we get*

$$\begin{aligned} \operatorname{ber}(A+B) \leq \frac{1}{2}\||A| + |A^*| + |B| + |B^*|\|_{\text{ber}} - \frac{1}{2}\inf_{\eta \in Q}\left(\left(\widetilde{|A|}(\eta)\right)^{1/2} - \left(\widetilde{|A^*|}(\eta)\right)^{1/2}\right)^2 \\ - \frac{1}{2}\inf_{\eta \in Q}\left(\left(\widetilde{|B|}(\eta)\right)^{1/2} - \left(\widetilde{|B^*|}(\eta)\right)^{1/2}\right)^2. \end{aligned}$$

(ii) In particular, take $B = A$, we get

$$\operatorname{ber}(A) \leq \frac{1}{2}\||A| + |A^*|\|_{\text{ber}} - \frac{1}{2}\inf_{\eta \in Q}\left(\left(\widetilde{|A|}(\eta)\right)^{1/2} - \left(\widetilde{|A^*|}(\eta)\right)^{1/2}\right)^2.$$

(iii) Setting $\alpha = \beta = \gamma = \delta = 1$ in (3.24), we get

$$\begin{aligned} \operatorname{ber}(A|A| + B|B|) \leq &\frac{1}{2}\left\| |A|^2 + |A^*|^2 + |B|^2 + |B^*|^2 \right\|_{\text{ber}} \\ &- \frac{1}{2}\inf_{\eta \in Q}\left(\left(\widetilde{|A|^2}(\eta)\right)^{1/2} - \left(\widetilde{|A^*|^2}(\eta)\right)^{1/2}\right)^2 \\ &- \frac{1}{2}\inf_{\eta \in Q}\left(\left(\widetilde{|B|^2}(\eta)\right)^{1/2} - \left(\widetilde{|B^*|^2}(\eta)\right)^{1/2}\right)^2. \end{aligned}$$

(iv) In particular, take $B = A$, we get

$$\begin{aligned} \operatorname{ber}(A|A|) &\leq \frac{1}{2}\left\| |A|^2 + |A^*|^2 \right\|_{\text{ber}} - \frac{1}{2}\inf_{\eta \in Q}\left(\left(\widetilde{|A|^2}(\eta)\right)^{1/2} - \left(\widetilde{|A^*|^2}(\eta)\right)^{1/2}\right)^2 \\ &= \frac{1}{2}\|A^*A + AA^*\|_{\text{ber}} - \frac{1}{2}\inf_{\eta \in Q}\left(\left(\widetilde{|A|^2}(\eta)\right)^{1/2} - \left(\widetilde{|A^*|^2}(\eta)\right)^{1/2}\right)^2. \end{aligned}$$

For more recent research on Berezin radius inequalities for operators and other relevant results, we recommend [4, 6, 13, 14, 15, 16, 18, 29, 33, 34].

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