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Research Article

The disconnectedness of certain sets defined after uni-variate polynomials

VLADIMIR PETROV KOSTOV*

ABSTRACT. We consider the set of monic real uni-variate polynomials of a given degree d with non-vanishing coefficients, with given signs of the coefficients and with given quantities pos of their positive and neg of their negative roots (all roots are distinct). For $d \geq 6$ and for signs of the coefficients $(+, -, +, +, \ldots, +, +, -, +)$, we prove that the set of such polynomials having two positive, $d-4$ negative and two complex conjugate roots, is not connected. For $pos + neg \leq 3$ and for any d, we give the exhaustive answer to the question for which signs of the coefficients there exist polynomials with such values of pos and neg.

Keywords: Real polynomial in one variable, hyperbolic polynomial, Descartes' rule of signs, discriminant set.

2020 Mathematics Subject Classification: 26C10, 30C15.

1. INTRODUCTION

We consider questions about the general family of monic uni-variate real degree d polynomials: $Q_d := x^d + \sum_{j=0}^{d-1} a_j x^j$. In the space $\mathbb{R}^d$ of the coefficients a_j, one defines the *discriminant set* Δ_d as the set of their values for which the polynomial Q_d has a multiple real root. More precisely, if Δ_d^1 is the set of values of the coefficients for which Q_d has a multiple root (real or complex), then this is the set of the zeros of the determinant of the Sylvester matrix of the polynomials Q_d and Q_d'. One has to set $\Delta_d := \Delta_d^1 \setminus \Delta_d^2$, where Δ_d^2 is the set of values of the coefficients a_j for which there is a multiple complex conjugate pair of roots of Q_d and no multiple real root. It is true that $\dim(\Delta_d) = \dim(\Delta_d^1) = d-1$ and $\dim(\Delta_d^2) = d-2$.

The set

$$R_{1,d} := \mathbb{R}^d \setminus \Delta_d$$

consists of $[d/2]+1$ open components of dimension d ($[\cdot]$ stands for the integer part of $\cdot$). The polynomials Q_d from a given component have one and the same number μ of real roots (which are all distinct); the number ν of complex conjugate pairs can range from 0 to $[d/2]$, because $\mu + 2\nu = d$. Given two polynomials with one and the same number ν, one can continuously deform the roots of the first polynomial into the roots of the second one by keeping the real roots distinct throughout the deformation. This proves that to any possible number ν corresponds exactly one component of the set $R_{1,d}$.

In the same way one can consider the components of the set

$$R_{2,d} := \mathbb{R}^d \setminus (\Delta_d \cup \{a_0 = 0\}) .$$

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*Corresponding author: Vladimir Petrov Kostov; vladimir.kostov@unice.fr
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The polynomials from one and the same open component (also of dimension d) have one and the same numbers pos of positive and neg of negative roots (and no vanishing roots). When deforming the roots of one polynomial into the roots of another one, one has to keep the same numbers pos and neg throughout the deformation. To each pair (pos, neg) corresponds exactly one component of the set $R_{2,d}$. As $pos + neg = \mu$, $0 \leq pos, neg \leq \mu$ and $\mu + 2\nu = d$, there are

$$(d+1) + (d-1) + (d-3) + \cdots = ([d/2]+1)([(d+1)/2]+1)$$

components of the set $R_{2,d}$.

A more complicated task is to study the components of the set

$$R_{3,d} := \mathbb{R}^d \setminus (\Delta_d \cup \{a_0 = 0\} \cup \{a_1 = 0\} \cup \cdots \cup \{a_{d-1} = 0\})$$

of monic uni-variate polynomials with no multiple real roots and no zero coefficients.

Definition 1. A *sign pattern* of length $d+1$ is a sequence of $d+1$ symbols $+$ and/or $-$ beginning with a $+$. We say that a polynomial Q_d with no vanishing coefficients defines the sign pattern $\sigma_0 := (+, \beta_{d-1}, \beta_{d-2}, \ldots, \beta_0)$, $\beta_j = +$ or $-$, (notation: $\sigma(Q_d) = \sigma_0$), if sign$(a_j) = \beta_j$, $j = 0, \ldots, d-1$.

One can ask the question to which couple (sign pattern, pair (pos, neg)) (we call them *couples* for short) corresponds at least one component of the set $R_{3,d}$. The polynomials from a given component of $R_{3,d}$ have one and the same couple. All components are of dimension d.

When considering the set $R_{3,d}$, it is self-understood that the couples have to be defined in accordance with Descartes' rule of signs. This rule states that a real uni-variate polynomial Q_d has not more positive roots counted with multiplicity than the number c of sign changes in the sequence of its coefficients; the difference $c - pos$ is even, see [4, 9, 10, 11, 15, 16, 19, 20, 28] or [30]. Hence the sign of the constant term is $(-1)^{pos}$. When the polynomial has no zero coefficients, Descartes' rule of signs applied to $Q_d(-x)$ implies that Q_d has not more negative roots counted with multiplicity than the number p of sign preservations in that sequence (hence $c+p = d+1$), and the difference $p - neg$ is also even.

Definition 2. A pair (pos, neg) satisfying these conditions w.r.t. a given sign pattern σ_0 is called *compatible* with σ_0 (and vice versa), and the couple $(\sigma_0, (pos, neg))$ is also called *compatible*. For a monic polynomial Q_d with no vanishing coefficients, with pos positive simple and neg negative simple roots and no other real roots, we say that Q_d *realizes* the couple $(\sigma(Q_d), (pos, neg))$.

Yet this compatibility is just a necessary condition which turns out not to be sufficient. That is, there exist cases when to certain compatible couples correspond no components of $R_{3,d}$. So we formulate the first problem which we consider in the present paper:

Problem 1. *For a given degree d, for which compatible couples do there exist monic polynomials realizing these couples? In other words, to which of the compatible couples there corresponds at least one component of the set $R_{3,d}$?*

Some results in relationship with Problem 1 are formulated in the next section. The problem seems to have been stated for the first time in [2]. The first example when to a compatible couple corresponds no component of the set $R_{3,d}$ (this is an example with $d = 4$), and the exhaustive answer to the problem for $d = 4$, are to be found in [18]. For $d = 5$ and $d = 6$, the result is given in [1]. For $d = 7$ and partially for $d = 8$ (resp. completely for $d = 8$), the answer is formulated and proved in [12] and [13] (resp. in [22]). Different aspects concerning Descartes' rule of signs are treated in papers [23, 5, 6, 7, 8] and [14].

Of particular importance is the class of *hyperbolic polynomials*, i. e. real polynomials whose roots are all real. The *hyperbolicity domain* Π_d is the set of values of the coefficients a_j for which the polynomial Q_d is hyperbolic. For properties of hyperbolic polynomials and the domain Π_d see [3, 17, 21, 29] and [24].

In what follows we are also interested in another problem:

Problem 2. *For a given degree d, to which compatible couples correspond two or more components of the set $R_{3,d}$?*

To formulate our first result connected with Problem 2, we introduce the following notation:

Notation 1. For $d \geq 4$, we consider $\mathbb{R}^d$ as the set $\{(a_{d-1}, a_{d-2}, \ldots, a_0) | a_j \in \mathbb{R}\}$ of d-tuples of coefficients (excluding the leading one) of polynomials Q_d. We denote by $\sigma_\bullet$ the sign pattern $(+, -, +, +, \ldots, +, +, -, +)$ and by $\Pi_d^*(\sigma_\bullet)$ (resp. by $A(\sigma_\bullet, (2, d-4))$) the subset of $\mathbb{R}^d$ of polynomials with signs of the coefficients (all non-zero) as defined by $\sigma_\bullet$ and having four positive and $d-4$ negative distinct real roots (resp. two positive and $d-4$ negative distinct roots and one complex conjugate pair). Hence the polynomials of the set $\Pi_d^*(\sigma_\bullet)$ are hyperbolic while the ones of the set $A(\sigma_\bullet, (2, d-4))$ are not.

The following theorem is proved in Section 3.

Theorem 1. *(1) For $d \geq 6$, the set $A(\sigma_\bullet, (2, d-4))$ is non-empty and consists of more than one component of the set $R_{3,d}$. Hence the set $A(\sigma_\bullet, (2, d-4))$ is not connected.*

(2) For $d = 4$ and 5, the respective sets $A(\sigma_\bullet, (2, 0))$ and $A(\sigma_\bullet, (2, 1))$ are connected.

Remarks 1. (1) One can mention cases in which the components of the set $R_{3,d}$ are contractible and to each compatible couple corresponds exactly one component of the set $R_{3,d}$ (see [26]). Namely, such are the cases of hyperbolic polynomials and of polynomials having exactly one or no real roots at all.

(2) In the case of polynomials having exactly two real distinct roots (hence $pos + neg = 2$) to each compatible couple corresponds either one or no component of $R_{3,d}$, and all components are contractible. See more details in the next section or in [26]. Whether in the case of exactly three real roots to each compatible couple corresponds at most one component of the set $R_{3,d}$ is an open question.

(3) For $d = 4$ and $d = 5$, pictures of the set Δ_d^1 (from which one can deduce the form of the set $A(\sigma_\bullet, (2, d-4))$) can be found in [27] and [8] respectively.

2. Comments and further results

Given a sign pattern $\hat{\sigma}$ with c sign changes and p sign preservations (hence $c + p = d$), Descartes' rule of signs implies that any hyperbolic polynomial with sign pattern $\hat{\sigma}$ has exactly c positive and exactly p negative roots counted with multiplicity. We define the *canonical order of moduli* corresponding to $\hat{\sigma}$. The sign pattern $\hat{\sigma}$ is read from the right and to each sign change (resp. sign preservation) one puts in correspondence the letter P (resp. the letter N).

For example, for $\hat{\sigma} = \sigma_\dagger := (+, -, -, -, +, +)$ (resp. for $\hat{\sigma} = \sigma_\bullet$) this gives the string $NPNNP$ (resp. $PPNN \cdots NNPP$, $d-4$ times N). After this one inserts the symbol $<$ between any two consecutive letters which in the cases of $\sigma_\dagger$ and $\sigma_\bullet$ gives

$$N < P < N < N < P \quad \text{and} \quad P < P < N < N < \cdots < N < N < P < P$$

respectively. If one denotes by α_j and β_j the moduli of the positive and negative roots, then one replaces the letters P and N by these moduli which in the case of $\sigma_\dagger$ defines the canonical order

$$\beta_1 < \alpha_1 < \beta_2 < \beta_3 < \alpha_2$$

whereas the canonical order corresponding to $\sigma_\bullet$ is given by (3.1).

It is true that for any sign pattern σ_0 of length $d+1$, there exists a degree d monic hyperbolic polynomial T with $\sigma(T) = \sigma_0$ whose roots define the respective canonical order of moduli, see Proposition 1 in [25].

Our next step is to consider the cases when the polynomial Q_d has not more than three real roots, i. e. $pos + neg \leq 3$ (and hence in the case of equality the possible values of the pair (pos, neg) are $(3, 0)$, $(2, 1)$, $(1, 2)$ and $(0, 3)$). For the cases $pos = neg = 0$ and $pos + neg = 1$, see part (1) of Remarks 1. For $pos + neg = 2$ (hence d is even), we remind some of the results of [26].

Definition 3. For $pos + neg = 2$, we define *Case 1)* (resp. *Case 2)*) by the conditions the constant term to be positive, all coefficients of monomials of odd degree to be positive (resp. negative), the pair (pos, neg) to equal $(2, 0)$ (resp. $(0, 2)$) and the coefficient of at least one monomial of even degree to be negative.

Theorem 2. (see [26]). *For d even and $pos + neg = 2$,*

1. *A given compatible couple is realizable if and only if it does not correspond to Case 1) or 2).*
2. *If the constant term is positive (hence $(pos, neg) = (2, 0)$ or $(0, 2)$) and one is not in Case 1) or 2), a given compatible couple is realizable by polynomials having any ratio different from 1 between the moduli of the two real roots.*
3. *If the constant term is negative (hence $(pos, neg) = (1, 1)$) and there are two monomials of odd degree with coefficients of opposite signs, then such a compatible couple is realizable by polynomials with any ratio of the moduli α and β of its positive and negative root respectively.*
4. *If the constant term is negative and all coefficients of monomials of odd degree are positive (resp. negative), then such a compatible couple is realizable by polynomials with any ratio $\alpha/\beta < 1$ (resp. $\alpha/\beta > 1$) and not realizable by polynomials with $\alpha/\beta \geq 1$ (resp. $\alpha/\beta \leq 1$).*

To formulate the new results about the situation with $pos + neg = 3$, we introduce the following notion:

Definition 4. For a given degree d, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-*action* on the set of compatible couples is defined by two commuting involutions. The first of them maps a polynomial Q_d into $(-1)^d Q_d(-x)$ (this changes the pair (pos, neg) into (neg, pos), it changes the signs of the coefficients of x^{d-1}, x^{d-3}, ... and preserves the signs of the other coefficients). The second involution maps Q_d into $x^d Q_d(1/x)/Q_d(0)$ (the pair (pos, neg) is preserved and the sign pattern, eventually multiplied by -1, is read from the right; the roots of $x^d Q_d(1/x)/Q_d(0)$ are the reciprocals of the roots of Q_d). An *orbit* of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action consists of 2 or 4 compatible couples which are simultaneously realizable or not. This allows to formulate the results only for one of the 2 or 4 couples of a given orbit.

Theorem 3. *Suppose that the pair $(2, 1)$ is compatible with the sign pattern $\sigma_\triangle$ (hence the constant term is positive). Then*

1. *The couple $\mathcal{C} := (\sigma_\triangle, (2, 1))$ is realizable.*

   Denote by $-\beta < 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$ the three real roots of a polynomial realizing the couple $\mathcal{C}$.
2. *If there are monomials x^{2m} and x^{2n-1} with negative coefficients (one can have $2m < 2n - 1$ or $2n - 1 < 2m$), then for any of the five possibilities*

$$\beta < \alpha_1 < \alpha_2 \, , \quad \beta = \alpha_1 < \alpha_2 \, , \quad \alpha_1 < \beta < \alpha_2 \, , \quad \alpha_1 < \alpha_2 = \beta \quad \text{and} \quad \alpha_1 < \alpha_2 < \beta \, ,$$

   there exist polynomials realizing the couple $\mathcal{C}$.

(3) *If all odd monomials have positive coefficients, then only the possibility $\beta < \alpha_1 < \alpha_2$ is realizable.*

(4) *If all even monomials have positive coefficients, then only the possibility $\alpha_1 < \alpha_2 < \beta$ is realizable.*

The theorem is proved in Section 4. The compatibility of the sign pattern with the pair $(2,1)$ implies that in part (3) (resp. in part (4)) of the theorem there is at least one even (resp. odd) monomial whose coefficient is negative.

Notation 2. For d odd, we denote by $D(a,b,c)$ the sign pattern consisting of $2a$ pluses followed by b pairs "$-,+$" followed by $2c$ minuses, where $1 \leq a$, $1 \leq b$, $1 \leq c$ and $2a+2b+2c = d+1$.

Theorem 4. *Suppose that the pair $(3,0)$ is compatible with the sign pattern $\sigma_\diamond$ which is not of the form $D(a,b,c)$. Then the couple $(\sigma_\diamond, (3,0))$ is realizable.*

The theorem is proved in Section 5.

Theorem 5. *For $j = 1, 2, \ldots, b$, the couple $(D(a,b,c), (2j+1, 0))$ is not realizable.*

The theorem is proved in Section 6. Its proof resembles the proof of part (i) of Theorem 4 in [27] which treats a particular case of Theorem 5. However the proof of Lemma 1 (used in the proof of Theorem 5) is more complicated than the proof of its analog which is Lemma 6 of [27]. This renders indispensable giving the whole proof of Theorem 5.

Remark 1. *For the sign pattern $D(a,b,c)$, compatible are the following pairs (pos, neg):*

1) *the ones mentioned in Theorem 5;*
2) *the pair $(1,0)$;*
3) *the pairs $(2j+1, 2r)$, $r = 1, 2, \ldots, a+c-1$, $j = 0, 1, \ldots, b$.*

Realizability of the couples $(D(a,b,c), (pos, neg))$ with (pos, neg) as in 2) and 3) can be proved by analogy with the proof of parts (ii) and (iii) of Theorem 4 in [27].

3. Proof of Theorem 1

Part (1). A) For $d \geq 6$, the set $\Pi_d^*(\sigma_\bullet)$ is non-empty, see Proposition 1 in [25]. Fix a polynomial $Q^* \in \Pi_d^*(\sigma_\bullet)$. By Proposition 1 of [25], one can choose Q^* such that the moduli of its positive and negative roots (denoted by $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ and $\beta_1 < \beta_2 < \cdots < \beta_{d-5} < \beta_{d-4}$ respectively) satisfy the string of inequalities

$$\alpha_1 < \alpha_2 < \beta_1 < \beta_2 < \cdots < \beta_{d-5} < \beta_{d-4} < \alpha_3 < \alpha_4 \, . \tag{3.1}$$

So the negative roots of Q^* are $-\beta_{d-4} < -\beta_{d-5} < \cdots < -\beta_1 < 0$. Starting with Q^*, we construct two polynomials Q^1 and Q^2 of the set $A(\sigma_\bullet, (2, d-4))$ (so this set is non-empty) about which we show that they belong to different components of $R_{3,d}$. This implies the theorem.

B) We consider the one-parameter family of polynomials

$$\tilde{Q}_t := Q^* + tx^2(x+\beta_1)(x+\beta_2)\cdots(x+\beta_{d-4}) \, , \quad t \geq 0 \, .$$

For any $t \geq 0$, one has $\sigma(\tilde{Q}_t) = \sigma_\bullet$. As t increases, the roots $-\beta_1, -\beta_2, \ldots, -\beta_{d-4}$ of $\tilde{Q}_t$ do not move. The roots α_1 and α_3 move to the right while α_2 and α_4 move to the left. For some $t_0 > 0$, either α_1 coalesces with α_2 or α_3 coalesces with α_4 or both these things take place. Indeed, the values of $\tilde{Q}_t$ for each fixed $x \geq \alpha_1$ increase at least as fast as $t\alpha_1^2 \prod_{i=1}^{d-4}(\alpha_1 + \beta_i)$.

If for $t = t_0$, α_1 and α_2 coalesce and α_3 and α_4 remain positive and distinct, then one can fix $t_1 > t_0$ sufficiently close to t_0 for which the roots α_1 and α_2 have given birth to a complex conjugate pair while α_3 and α_4 are still positive and distinct. We set $Q^1 := \tilde{Q}_{t_1}$. Hence the polynomial Q^1 has $d-2$ real roots

$$(3.2)\qquad \begin{array}{l} -\beta_{d-4} < -\beta_{d-5} < \cdots < -\beta_1 < 0 < \alpha_3 < \alpha_4 \quad \text{such that} \\ \\ 0 < \beta_1 < \beta_2 < \cdots < \beta_{d-4} < \alpha_3 < \alpha_4 \end{array}$$

and a complex conjugate pair. After this we set $Q^2_* := x^d Q^1(1/x)$. The sequence of coefficients of Q^1, when read from the right, is the string of coefficients of Q^2_*. After this we set $Q^2 := Q^2_*/Q^1(0)$, so Q^2 is monic. The sign pattern $\sigma_\bullet$ is center-symmetric, therefore $\sigma(Q^2) = \sigma_\bullet = \sigma(Q^1)$. The roots of the polynomial Q^2 are the reciprocals of the roots of Q^1. The real roots of Q^2 satisfy the conditions

$$(3.3)\qquad \begin{array}{l} -\beta_{d-4} < -\beta_{d-5} < \cdots < -\beta_1 < 0 < \alpha_3 < \alpha_4 \quad \text{and} \\ \\ 0 < \alpha_1 < \alpha_2 < \beta_1 < \beta_2 < \cdots < \beta_{d-4}; \end{array}$$

the polynomial Q^2 has also a complex conjugate pair.

If for $t = t_0$, α_3 and α_4 coalesce while α_1 and α_2 remain positive and distinct, then for some $t_1 > t_0$ sufficiently close to t_0 we obtain the polynomial Q^2 with exactly two positive and $d-4$ negative roots which satisfy conditions (3.3). After this we set $Q^1_* = x^d Q^2(1/x)$ and $Q^1 := Q^1_*/Q^2(0)$. The real roots of Q^1 satisfy conditions (3.2).

Finally, if for $t = t_0$, one has $\alpha_1 = \alpha_2 = a > 0$ and $\alpha_3 = \alpha_4 = b > a$, then one constructs the polynomials

$$Q^\pm := \tilde{Q}_{t_0} \pm \varepsilon(x - (a+b)/2)\ , \quad \varepsilon > 0\ .$$

For ε small enough,

1) the coefficients of $Q^\pm$ are non-zero and $\sigma(Q^\pm) = \sigma_\bullet$;
2) each of the polynomials $Q^\pm$ has $d-4$ distinct negative roots close to $-\beta_i$;
3) Q^+ has two distinct positive roots close to a and a complex conjugate pair close to b;
4) and vice versa for Q^-.

We set $Q^1 := Q^-$ and $Q^2 := Q^+$.

C) Suppose that the two polynomials Q^1 and Q^2 belong to one and the same component of the set $R_{3,6}$. Then it is possible to connect them by a continuous path (homotopy) within this component: Q^s, $s \in [1,2]$. Along the path the two positive, the $d-4$ negative and the two complex conjugate roots of Q^s depend continuously on s while remaining distinct throughout the homotopy. We denote the negative roots by $-\tilde{\beta}_j$, $j = 1, \ldots, d-4$, and the two positive roots by $\tilde{\gamma}_j$, $j = 1, 2$, where

$$\begin{array}{llll} \text{for} & s = 1\ , & \text{one has} & \tilde{\beta}_j = \beta_j\ , \quad \tilde{\gamma}_j = \alpha_{2+j}\ ; \\ \\ \text{for} & s = 2\ , & \text{one has} & \tilde{\beta}_j = \beta_j\ , \quad \tilde{\gamma}_j = \alpha_j\ . \end{array}$$

Hence there exists $s = s_0 \in (1,2)$ such that for $s = s_0$, $\tilde{\beta}_{d-4} = \tilde{\gamma}_2$. This means that the polynomial Q^{s_0} has exactly $d-2$ real roots such that

$$-\tilde{\beta}_{d-4} < \cdots < -\tilde{\beta}_1 < 0 < \tilde{\gamma}_1 < \tilde{\gamma}_2 \quad , \quad \tilde{\beta}_{d-4} = \tilde{\gamma}_2\ .$$

Using a linear change $x \mapsto hx$, $h > 0$, we achieve the condition $\tilde{\beta}_{d-4} = \tilde{\gamma}_2 = 1$.

D) Suppose that d is even. The fact that ± 1 are roots of Q_d implies the two conditions:

$$a_d + a_{d-2} + a_{d-4} + \cdots + a_2 + a_0 = 0 \quad \text{and} \quad a_{d-1} + a_{d-3} + \cdots + a_3 + a_1 = 0\ .$$

The first of them is possible only if all even coefficients are 0, because in the corresponding positions the sign pattern $\sigma_\bullet$ contains $(+)$-signs. However $a_d = 1$. This contradiction means that the homotopy Q^s does not exist, so Q^1 and Q^2 belong to different components of the set $R_{3,d}$ and the set $A(\sigma_\bullet, (2, d-4))$ is not connected. One can observe that this reasoning is not valid for $d = 2$ or $d = 4$, because in these cases there are no negative roots at all.

E) Suppose that $d \geq 7$ is odd. Set $\delta := \tilde{\beta}_{d-5} > 0$ and $Q_d^{s_0} = (x+\delta)U(x)$, where $U = x^{d-1} + \sum_{j=0}^{d-2} u_j x^j$. The polynomial U has an even number of positive roots, so $u_0 > 0$. The conditions

$$0 > a_{d-1} = \delta + u_{d-2} \quad \text{and} \quad \delta > 0$$

imply $u_{d-2} < 0$ whereas from

$$0 < a_{d-2} = \delta u_{d-2} + u_{d-3}\ , \quad \delta > 0 \quad \text{and} \quad u_{d-2} < 0$$

one deduces that $u_{d-3} > 0$. In the same way one has

$$0 > a_1 = \delta u_1 + u_0\ , \quad \delta > 0\ , \quad u_0 > 0\ , \quad \text{so} \quad u_1 < 0 \quad \text{and}$$

$$0 < a_2 = \delta u_2 + u_1\ , \quad \delta > 0\ , \quad u_1 < 0\ , \quad \text{so} \quad u_2 > 0\ .$$

The first three and the last three of the coefficients of the polynomial $U(-x)$ are positive. By Descartes' rule of signs it has not more than $d-5$ positive roots, and it has exactly $d-5$ positive roots only if it has $d-5$ sign changes. On the other hand, one knows that $U(-x)$ has exactly $d-5$ positive roots $-\tilde{\beta}_j$, $j = 1, 2, \ldots, d-6, d-4$. Hence $U(x)$ has $d-5$ sign preservations, therefore $u_k > 0$ for $2 \leq k \leq d-3$.

Thus $\sigma(U) = \sigma_\bullet$ (but here the sign pattern $\sigma_\bullet$ is meant to be of length d, not $d+1$). Suppose that the homotopy Q^s exists. Along this homotopy the root $-\tilde{\beta}_{d-5}$ is a continuous negative-valued function. As division of Q^s by $x+\delta$ gives the polynomials U, there exists a homotopy between the polynomial U corresponding to Q^1 and the one corresponding to Q^2. We denote them by U^1 and U^2. They are of even degree $d-1 \geq 6$, each of them has exactly two positive roots $\tilde{\gamma}_1 < \tilde{\gamma}_2$, exactly $d-5$ negative roots and one complex conjugate pair. For the moduli of the real roots one has

$$\tilde{\beta}_j < \tilde{\gamma}_1 \quad \text{for} \quad U^1 \quad \text{and} \quad \tilde{\gamma}_2 < \tilde{\beta}_j \quad \text{for} \quad U^2\ , \quad j = 1,\ 2, \ldots\ ,\ d-6,\ d-4$$

(see (3.2) and (3.3)). This, however, is impossible, see D). □

Part (2). F) For $d = 4$, for each polynomial $Q \in A(\sigma_\bullet, (2,0))$, there exists a unique quantity $g > 0$ such that for $g' \in [0, g)$, one has $Q + g' \in A(\sigma_\bullet, (2,0))$ and for $g' = g$, the polynomial $Q + g'$ has a multiple positive root.

On the other hand, for each polynomial $Q \in A(\sigma_\bullet, (2,0))$, there exists a unique quantity $h > 0$ such that for $h' \in [0, h)$, one has $Q - h' \in A(\sigma_\bullet, (2,0))$ and for $h' = h$, Q has either a zero root or a multiple positive root. The quantities g and h are continuous functions of the coefficients of Q.

Denote by $A^*(\sigma_\bullet)$ the set of monic polynomials whose coefficients have signs as defined by the sign pattern $\sigma_\bullet$ and which have a multiple positive root and a complex conjugate pair. Hence the set $A(\sigma_\bullet, (2,0))$ is homeomorphic to the direct product of the set $A^*(\sigma_\bullet)$ and an open interval. Therefore if $A^*(\sigma_\bullet)$ is connected, then such is $A(\sigma_\bullet, (2,0))$ as well.

Denote by $A_0^*(\sigma_\bullet)$ the subset of $A^*(\sigma_\bullet)$ for which the multiple root of Q is at 1. Each polynomial $Q \in A^*(\sigma_\bullet)$ can be transformed into a polynomial of $A_0^*(\sigma_\bullet)$ by a linear change of the variable x followed by a multiplication with a non-zero constant. Hence $A^*(\sigma_\bullet)$ is homeomorphic to $A_0^*(\sigma_\bullet) \times (0, \infty)$.

Any polynomial $Q \in A_0^*(\sigma_\bullet)$ is of the form

$$(x-1)^2(x^2+Ax+B) = x^4 + (A-2)x^3 + (B-2A+1)x^2 + (A-2B)x + B \ ,$$

where $A^2 - 4B \leq 0$. The set $A_0^*(\sigma_\bullet)$ is defined by the conditions

$$A < 2 \ , \quad B - 2A + 1 > 0 \ , \quad A - 2B < 0 \quad \text{and} \quad B \geq A^2/4 \ .$$

This is the set of points in the plane (A, B) which are to the left of the vertical line $A = 2$ and above or on the graph of the function (of the argument $A \in (-\infty, 2)$) $\max(2A-1, A/2, A^2/4)$; strictly above for $A \in [0, 2)$ and above or on the graph for $A < 0$. This is a contractible set.

G) For $d = 5$, we denote by $A^\dagger(\sigma_\bullet)$ the set of monic polynomials the signs of whose coefficients are defined by the sign pattern $\sigma_\bullet$ and which have a simple negative root, a double positive root and a complex conjugate pair. Denote by $A_0^\dagger(\sigma_\bullet)$ its subset for which the double root is at 1. By complete analogy with part F) of the proof we show that connectedness of $A_0^\dagger(\sigma_\bullet)$ implies the one of $A(\sigma_\bullet, (2, 1))$.

Any polynomial $Q \in A_0^\dagger(\sigma_\bullet)$ is of the form

$$(x-1)^2(x+A)(x^2+Bx+C) = x^5 + \sum_{j=0}^{4} f_j x^j \ , \quad \text{where}$$

$$f_4 = A + B - 2 \ , \qquad f_3 = AB - 2A - 2B + C + 1 \ ,$$

$$f_2 = -2AB + AC + A + B - 2C \ , \quad f_1 = AB - 2AC + C$$

$$\text{and} \qquad f_0 = AC \ ,$$

with $A > 0$ and $B^2 - 4C < 0$. For any $\rho > 0$ and $r > 0$, the polynomial $Q_{\rho,r} := Q + \rho(x-1)^2 + rx^3(x-1)^2$ defines the sign pattern $\sigma_\bullet$ and belongs to the set $A_0^\dagger(\sigma_\bullet)$. Indeed, it is non-negative for $x \geq 0$, with equality only for $x = 1$; its second derivative at $x = 1$ is positive, so $x = 1$ is a double root; the sign pattern $\sigma_\bullet$ and Descartes' rule of sign imply that $Q_{\rho,r}$ has not more than one negative root, so it has exactly one such root. Hence one can choose ρ and r such that $f_2 = f_3$. The set $A_0^\dagger(\sigma_\bullet)$ is connected if and only if its subset defined by the condition $f_2 = f_3$ is connected.

H) The condition $f_2 = f_3$ allows to express A as a function of B and C:

$$A = T_0/D \ , \quad \text{where} \quad T_0 = 3B - 3C - 1 \quad \text{and} \quad D = 3B - C - 3 \ .$$

For the coefficients f_i with $A = T_0/D$ one finds

$$f_4 = T_4/D \ , \qquad T_4 = 3B^2 - BC - 6B - C + 5 \ ,$$

$$f_3 = f_2 = T_3/D \ , \quad T_3 = -3B^2 + 2BC - C^2 + 2B + 2C - 1 \ ,$$

$$f_1 = T_1/D \ , \qquad T_1 = 3B^2 - 6BC + 5C^2 - B - C \ .$$

In Fig. 1 and 2 we represent the following sets:

– $\mathcal{L}_0 : T_0 = 0$ (in solid line) and $\mathcal{L} : D = 0$ (in dashed line) are straight lines;

– $\mathcal{E}_3 : T_3 = 0$ (in dashed line) and $\mathcal{E}_1 : T_1 = 0$ (in dotted line) are ellipses;
– $\mathcal{H} : T_4 = 0$ (in solid line) is a hyperbola;
– $\mathcal{P} : C = B^2/4$ is a parabola (in dash-dotted line).

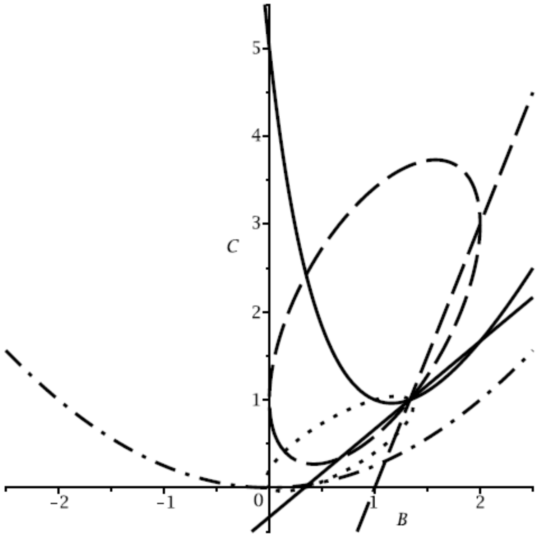

FIGURE 1. The set $A_0^{\dagger}(\sigma_\bullet)$ subdued to the condition $f_2 = f_3$ (global view).

Remark 2. As $C > 0$, only the branch of $\mathcal{H}$ belonging to the upper half-plane is represented in Fig. 1 and 2. The asymptotes of $\mathcal{H}$ are the lines $B = -1$ and $C = 3B - 6$. We denote by $\mathrm{Int}(\mathcal{E}_i)$ and $\mathrm{Out}(\mathcal{E}_i)$ the intersections with the half-plane $C > 0$ of the interior and the exterior of the ellipse $\mathcal{E}_i$. By $\mathrm{Int}(\mathcal{H})$ we denote the part of the upper half-plane which is above and by $\mathrm{Out}(\mathcal{H})$ the part which is below the branch of $\mathcal{H}$ with $C > 0$. Notice that

$$\mathrm{Int}(\mathcal{E}_3) :\ T_3 > 0,\ C > 0, \quad \mathrm{Int}(\mathcal{E}_1) :\ T_1 < 0,\ C > 0, \quad \mathrm{Int}(\mathcal{H}) :\ T_4 < 0,\ C > 0,$$

$$\mathrm{Out}(\mathcal{E}_3) :\ T_3 < 0,\ C > 0, \quad \mathrm{Out}(\mathcal{E}_1) :\ T_1 > 0,\ C > 0, \quad \mathrm{Out}(\mathcal{H}) :\ T_4 > 0,\ C > 0.$$

The ellipse $\mathcal{E}_1$ intersects the C-axis at $(0,0)$ and $(0,1/5)$ while $\mathcal{E}_3$ is tangent to the C-axis at $(0,1)$. The leftmost point of the ellipse $\mathcal{E}_1$ is at

$$((8-\sqrt{70})/12 = -0.030\ldots, (10-\sqrt{70})/20 = 0.081\ldots)\ .$$

The point $(4/3, 1)$ is a common point for $\mathcal{L}$, $\mathcal{L}_0$, $\mathcal{H}$, $\mathcal{E}_1$ and $\mathcal{E}_3$.

The intersecting lines $\mathcal{L}_0$ and $\mathcal{L}$ define two pairs of opposite sectors. The ones of opening $> \pi/2$ are denoted by $\mathcal{S}_u : T_0 < 0, D < 0$ (upper) and $\mathcal{S}_\ell : T_0 > 0, D > 0$ (lower). One has $A > 0$ exactly when the point (B, C) belongs to one of these two sectors.

I) The signs of the coefficients f_i and of the quantities $A > 0$ and $C > 0$ imply that one must have one of the two systems of conditions:

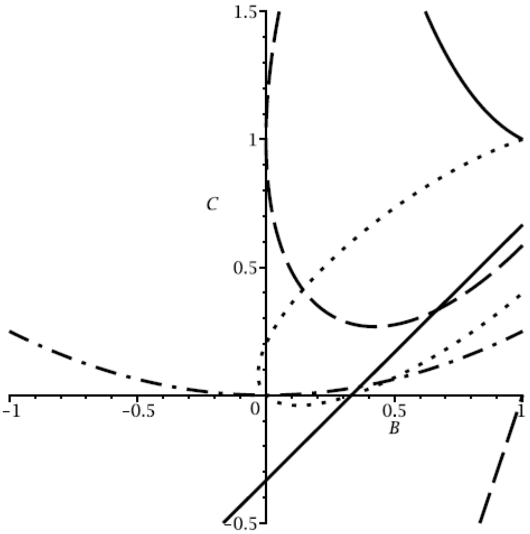

FIGURE 2. The set $A_0^\dagger(\sigma_\bullet)$ subdued to the condition $f_2 = f_3$ (local view).

$$(\mathrm{i})\ :\ (B,C) \in \mathcal{S}_\ell \cap \mathrm{Int}(\mathcal{E}_1) \cap \mathrm{Int}(\mathcal{E}_3) \cap \mathrm{Int}(\mathcal{H})\ , \quad \text{i.e.}$$

$$T_0 > 0\ , \quad D > 0\ , \quad T_1 < 0\ , \quad T_3 > 0 \quad \text{and} \quad T_4 < 0 \quad \text{or}$$

$$(\mathrm{ii})\ :\ (B,C) \in \mathcal{S}_u \cap \mathrm{Out}(\mathcal{E}_1) \cap \mathrm{Out}(\mathcal{E}_3) \cap \mathrm{Out}(\mathcal{H})\ , \quad \text{i.e.}$$

$$T_0 < 0\ , \quad D < 0\ , \quad T_1 > 0\ , \quad T_3 < 0 \quad \text{and} \quad T_4 > 0\ .$$

The possibility (i) is to be excluded. Indeed, one has

$$\mathcal{E}_3 \cap \mathcal{L}_0 = \{(2/3, 1/3)\ , \quad (4/3, 1)\} \quad \text{and} \quad \mathcal{E}_3 \cap \mathcal{L} = \{(4/3, 1)\ , \quad (2, 3)\}\ ,$$

see Fig. 1 and 2, so $\mathrm{Int}(\mathcal{E}_3)$ intersects with $\mathcal{S}_u$, but not with $\mathcal{S}_\ell$.

J) We describe the set obtained in case (ii). For $B \leq -1$, this is the part of the upper plane which is above the parabola $\mathcal{P}$. For $-1 < B < (8-\sqrt{70})/12$, this is its part between the parabola $\mathcal{P}$ from below and the hyperbola $\mathcal{H}$ from above, see Fig. 1. For each $(8-\sqrt{70})/12 \leq B < 0$, this is the union of two intervals whose endpoints belong to $\mathcal{H}$ and $\mathcal{E}_1$ for the upper and to $\mathcal{E}_1$ and $\mathcal{P}$ for the lower interval. For $B \geq 0$, this is the union of two curvilinear triangles, each with one rectilinear side which is part of the C-axis. The above triangle has vertices at $(0,1)$, $(0,5)$ and $(0.34\ldots, 2.42\ldots)$. The latter point, together with $(4/3, 1)$, is the intersection $\mathcal{H} \cap \mathcal{E}_3$.

The lower triangle has vertices at $(0, 1/5)$, $(0,1)$ and $(0.14\ldots, 0.41\ldots)$. The latter point, together with $(4/3, 1)$, is the intersection $\mathcal{E}_1 \cap \mathcal{E}_3$.

To see that there is no other point of the set defined in case (ii) with $B > 0$, one has to observe the order on $\mathcal{P}$ of the intersection points of

$$\mathcal{P} \cap \mathcal{L}_0 = \{(0.36\ldots,\ 0.03\ldots), (3.63\ldots,\ 3.29\ldots)\}$$

and

$$\mathcal{P} \cap \mathcal{E}_1 = \{(0,\ 0), (0.47\ldots,\ 0.22\ldots)\}.$$

The connectedness of the set obtained in case (ii) follows from its description. $\square$

4. Proof of Theorem 3

Part (1). The last component of $\sigma_\triangle$ is a $+$. Suppose that there is a minus sign in $\sigma_\triangle$ corresponding to x^{2m}, $1 \leq m \leq [d/2]$. The polynomial $-x^{2m}+1$ has exactly two real roots, namely ± 1, and they are simple. For $\varepsilon > 0$ small enough, the polynomial $P_0 := \varepsilon x^d - x^{2m} + 1$ has exactly three real roots two of which are close to ± 1 and the third is > 1. (One can notice that by Descartes' rule of signs it has not more than two positive and not more than one negative root.)

Fix a degree d polynomial P_1 with $\sigma(P_1) = \sigma_\triangle$. Then for $0 < \eta \ll \varepsilon$, the polynomial $P_0 + \eta P_1$ has signs of the coefficients as defined by $\sigma_\triangle$ and has exactly one negative and two positive simple roots and $(d-3)/2$ complex conjugate pairs counted with multiplicity. Thus $P_0 + \eta P_1$ realizes the couple $(\sigma_\triangle, (2,1))$.

Suppose now that there are $(+)$-signs in $\sigma_\triangle$ corresponding to all monomials of even degrees. Then there is a monomial x^{2m+1}, $1 \leq 2m+1 < d$, whose sign is negative. The polynomial $P_2 := x^d - x^{2m+1}$ has simple roots at ± 1 and a $(2m+1)$-fold root at 0. For $\varepsilon > 0$ small enough, the polynomial $P_2 + \varepsilon$ has exactly three real roots (two positive and one negative) all of which are simple. Then with P_1 and η as above, the polynomial $P_2 + \varepsilon + \eta P_1$ realizes the couple $(\sigma_\triangle, (2,1))$.

Part (2). We construct a polynomial of the form $V := x^d - Ax^{2m} - Bx^{2n-1} + C$, $A > 0$, $B > 0$, $C > 0$, such that $V(1) = V'(1) = V(-1) = 0$:

$$\begin{aligned} &1 - A - B + C = 0\,, \quad -1 - A + B + C = 0\,, \qquad d - 2mA - (2n-1)B = 0 \\ &\text{hence} \qquad\qquad A = C = (d - 2n + 1)/2m\,, \quad B = 1\,. \end{aligned} \tag{4.4}$$

By Descartes' rule of signs, V has no other real roots. After this one decreases C: $C \mapsto C - t$, $t \geq 0$. For $t = 0$, the root -1 moves with a finite speed to the right while the double root at 1 splits into two real roots moving for $t = 0$ with infinite speeds to the left and right respectively. Hence for $t > 0$ close to 0, one has $\alpha_1 < \beta < \alpha_2$. The linear system (4.4) with unknown variables A, B and C has non-zero determinant. Hence for $\varepsilon > 0$ small enough, one can obtain polynomials V satisfying the conditions

$$V(1) = V'(1) = V(-1 \pm \varepsilon) = 0 \quad (\text{resp.} \quad V(1) = V'(1) \pm \varepsilon = V(-1) = 0)$$

which after decreasing C yield polynomials satisfying the inequalities $\beta < \alpha_1 < \alpha_2$ or $\alpha_1 < \alpha_2 < \beta$ (resp. the conditions $\beta = \alpha_1 < \alpha_2$ or $\alpha_1 < \alpha_2 = \beta$). It remains to construct the polynomial $V + \eta P_1$, where $0 < \eta \ll \varepsilon$ and $\sigma(P_1) = \sigma_\triangle$.

Part (3). There exists a monomial x^{2m} with negative coefficient. Then for $\varepsilon > 0$ small enough, the polynomial $W := x^{2m-1}(x-1)(x-2) + \varepsilon$ has exactly one negative and two positive roots whose moduli satisfy the condition $\beta < \alpha_1 < \alpha_2$. Its four non-zero coefficients have the signs as defined by $\sigma_\triangle$. After this one constructs the polynomial $W + \eta P_1$ with η and P_1 as above.

The inequality $\beta \geq \alpha_1$ is impossible. Indeed, represent a polynomial W realizing the couple $\mathcal{C}$ in the form $W = W_o + W_e$, where W_o is the odd and W_e is the even part of W. Then for $x \in (-\beta, 0)$, one has $W_e(x) = W_e(-x)$ and $W_o(x) < W_o(-x)$. As $W(x) > 0$ for $x \in (-\beta, 0)$, one cannot have $W(\alpha_1) = 0$. This is a contradiction.

Part (4). Changing the polynomial $Y(x)$ with $\sigma(Y) = \sigma_\triangle$ which realizes the couple $\mathcal{C}$ to $Y_1 := x^d Y(1/x)/Y(0)$ (we set $\sigma^R_\triangle := \sigma(Y_1)$), one obtains a polynomial realizing the couple $(\sigma^R_\triangle, (2,1))$, where all odd monomials have positive signs, see Definition 4. The roots of Y_1 are the reciprocals of the roots of Y, so one deduces part (4) from part (3).

5. Proof of Theorem 4

The last sign of $\sigma_\diamond$ is a $-$. Suppose that there are two monomials x^{2m} and x^{2p}, $m > p > 0$, whose signs defined by $\sigma_\diamond$ are $-$ and $+$ respectively. Consider the polynomial $P_3 := -x^{2m} + Ax^{2p} - B$, $A > 0$, $B > 0$. By Descartes' rule of signs it has at most two positive and at most two negative roots. We define A and B such that P_3 has double roots at 1 and (-1):

$$-1 + A - B = 0\,, \quad -2m + 2pA = 0 \qquad \text{hence}$$

$$A = m/p > 0\,, \qquad B = (m-p)/p > 0\,.$$

Then for $\varepsilon > 0$ small enough, the polynomial $P_3 + \varepsilon x^d$ has exactly three real roots, all simple and positive. Suppose that P_4 is a degree d polynomial such that $\sigma(P_4) = \sigma_\diamond$. Then for $0 < \eta \ll \varepsilon$, the polynomial $P_3 + \varepsilon x^d + \eta P_4$ has sign pattern $\sigma_\diamond$ and has exactly three real roots, all simple and positive.

Suppose that there are no monomials x^{2m} and x^{2p} as above. Then the signs of the first a even monomials are positive and the ones of the last $(d + 1 - 2a)/2$ of them are negative, $0 \leq a \leq (d-1)/2$. Suppose that there are monomials $x^{2\nu}$, $x^{2\mu-1}$ and $x^{2\theta}$, $2\nu > 2\mu - 1 > 2\theta$, whose signs defined by $\sigma_\diamond$ are $-$, $+$ and $-$ respectively. By Descartes' rule of signs a polynomial of the form $P_5 := -x^{2\nu} + Cx^{2\mu-1} - Dx^{2\theta}$, $C > 0$, $D > 0$, has at most two positive roots and no negative roots; clearly it has a (2θ)-fold root at 0. One can choose C and D such that the positive roots are at 1 and 2:

$$-1 + C - D = 0\,, \qquad\qquad -2^{2\nu} + 2^{2\mu-1}C - 2^{2\theta}D = 0 \quad \text{hence}$$

$$D = (2^{2\nu} - 2^{2\mu-1})/(2^{2\mu-1} - 2^{2\theta}) > 0\,, \quad C = D + 1 > 0\,.$$

For $\varepsilon > 0$ small enough, the polynomial $P_5 + \varepsilon x^d$ has three positive simple roots and no other real roots, and the polynomial $P_6 := P_5 + \varepsilon x^d + \eta P_4$ with η and P_4 as above has three positive simple roots, no other real roots and $\sigma(P_6) = \sigma_\diamond$.

So now we suppose that there are no monomials x^{2m} and x^{2p}, and no monomials $x^{2\nu}$, $x^{2\mu-1}$ and $x^{2\theta}$ as above. Suppose that there are monomials x^{2u-1} and x^{2v-1}, $d > 2u - 1 > 2v - 1 > 0$, such that their signs are $-$ and $+$ respectively. One can construct a polynomial $P_7 := x^d - Ex^{2u-1} + Fx^{2v-1}$, $E > 0$, $F > 0$, having double roots at ± 1, a $(2v-1)$-fold root at 0 and no other real roots:

$$1 - E + F = 0\,, \qquad\qquad d - (2u-1)E + (2v-1)F = 0 \quad \text{hence}$$

$$F = (d - 2u + 1)/2(u - v) > 0\,, \quad E = F + 1 > 0\,.$$

The absence of other real roots is guaranteed by Descartes' rule of signs. Hence for $0 < \eta \ll \varepsilon \ll 1$, the polynomial $P_7 - \varepsilon + \eta P_4$ has sign pattern $\sigma_\diamond$, three simple positive roots and no other real roots (recall that $P_4(0) < 0$).

Suppose that there are no couples or triples of monomials x^{2m}, x^{2p} or $x^{2\nu}$, $x^{2\mu-1}$, $x^{2\theta}$ or x^{2u-1}, x^{2v-1}. Then the signs of the first $h_o \geq 1$ odd monomials (including x^d) are positive and the signs of the remaining $(d + 1 - 2h_o)/2$ odd monomials are negative. The signs of the first $h_e \geq 0$ even monomials are positive and the signs of the other $(d+1-2h_e)/2$ ones are negative. The absence of triples $x^{2\nu}$, $x^{2\mu-1}$, $x^{2\theta}$ implies $h_o \leq h_e + 1$. The cases $h_o = h_e + 1$ and $h_o = h_e$ are impossible, because there is only one sign change in the sign pattern. Therefore $1 \leq h_o \leq h_e - 1$. This means that the sign pattern is $D(a,b,c)$ with $a = h_o$, $b = h_e - h_o$ and $c = (d+1-2a-2b)/2$.

6. Proof of Theorem 5

Suppose that a polynomial $P := \sum_{j=0}^{d} a_j x^j$ realizes the couple $(D(a,b,c),(3,0))$. Denote by

$$P_o := \sum_{\nu=0}^{(d-1)/2} a_{2\nu+1} x^{2\nu+1} \quad \text{and} \quad P_e := \sum_{\nu=0}^{(d-1)/2} a_{2\nu} x^{2\nu}$$

its odd and even parts respectively. In each of the sequences $\{a_{2\nu+1}\}_{\nu=0}^{(d-1)/2}$ and $\{a_{2\nu}\}_{\nu=0}^{(d-1)/2}$ there is exactly one sign change. Descartes' rule of signs implies that the polynomial P_o has exactly three real roots, namely $-x_o$, 0 and x_o, $x_o > 0$, while the polynomial P_e has exactly two real roots $\pm x_e$, $x_e > 0$; all these five roots are simple.

Remarks 2. (1) The polynomial P_e is positive and increasing on (x_e, ∞) and negative on $[0, x_e)$. The polynomial P_o is positive and increasing on (x_o, ∞) and negative on $(0, x_o)$.

(2) One has $x_o \neq x_e$, otherwise $P(-x_o) = 0$, i.e. P has a negative root which is a contradiction.

(3) One can assume that all positive roots of P are distinct. Indeed, if this is not the case, then one can perturb P to make all its positive roots distinct without changing the signs of its coefficients as follows. If P has an ℓ-fold root $\lambda > 0$ ($\ell > 1$), i.e. $P = (x-\lambda)^{\ell} P^0$, $P^0(\lambda) \neq 0$, then for $\varepsilon > 0$ small enough, the polynomial $(x-\lambda)^{\ell-1}(x-\lambda-\varepsilon)P^0$ has the same sign pattern and its ℓ-fold root has split into an $(\ell-1)$-fold and a simple real roots. It remains to iterate this construction sufficiently many times.

Notation 3. We denote by $0 < \xi_1 < \xi_2 < \xi_3$ the smallest three of the positive roots of P and by ζ a positive number different from x_o and x_e.

It is clear that $P(\zeta) > 0$ for $\zeta \in (\xi_1, \xi_2)$ and $P(\zeta) < 0$ for $\zeta \in (\xi_2, \xi_3)$. For $\zeta \in (\xi_1, \xi_2)$, it is impossible to have $P_e(\zeta) \leq 0$ and $P_o(\zeta) \leq 0$ (with at most one equality, see part (2) of Remarks 2). It is also impossible to have $P_e(\zeta) \geq 0$ and $P_o(\zeta) \geq 0$. Indeed, this would imply that $x_e \leq \zeta < \xi_2$ and $x_o \leq \zeta < \xi_2$ which means that for $x \in (\xi_2, \xi_3)$, one has $P_e(x) \geq 0$ and $P_o(x) \geq 0$, i.e. $P(x) > 0$. This is a contradiction.

Two possible situations are left:

a) $P_e(\zeta) > 0$, $P_o(\zeta) < 0$;

b) $P_e(\zeta) < 0$, $P_o(\zeta) > 0$

(we skip the cases of equalities, because they were already taken into account).

Situation a) cannot take place, because this would mean that

$$P(-\zeta) = P_e(\zeta) - P_o(\zeta) > 0 \, ,$$

and since $P(0) < 0$ and $P(x) \to -\infty$ for $x \to -\infty$, in each of the intervals $(-\infty, -\zeta)$ and $(-\zeta, 0)$ the polynomial P would have at least one root – a contradiction.

So suppose that we are in situation b), so $x_o < \zeta < x_e$. Without loss of generality one can assume that $\xi_1 = 1$; this can be achieved by a rescaling $x \mapsto \xi_1 x$. Hence $P_o(1) = \beta > 0$ and $P_e(1) = -\beta$. Considering the polynomial P/β instead of P, one can assume that $\beta = 1$. One deduces from Lemma 1 which follows that there are no real roots of P larger than 1 (one can use the Taylor series of P at 1); this contradiction completes the proof.

Lemma 1. *Under the above assumptions, $P^{(m)}(1) > 0$, for any $m = 1, 2, \ldots, d$.*

Proof of Lemma 1. In the proof we allow zero values of the coefficients as well. This is because we need to deal with compact sets on which minimization arguments are to be applied.

Suppose that the sum $\delta_1 := a_1 + a_3 + \cdots + a_{2b+2c-1}$ is fixed (recall that these are all the negative coefficients of P_o). Then for any $m = 1, 2, \ldots, d$, it is true that $P_o^{(m)}(1)$ is minimal for

$$a_{2b+2c-1} = \delta_1 \ , \quad a_1 = a_3 = \cdots = a_{2b+2c-3} = 0 \ .$$

Indeed, when computing the values of the derivatives at $x = 1$, monomials of larger degree in x are multiplied by larger factors (equal to these degrees). We apply here $(d-3)/2$ times the fact that for $A + B$ fixed, the inequalities $A \geq 0$, $B \geq 0$ and $\lambda > \mu > 0$ imply that the sum $\lambda A + \mu B$ is maximal when $B = 0$.

Similarly, if the sum $\delta_2 := a_{2b+2c+1} + a_{2b+2c+3} + \cdots + a_d$ of all positive coefficients of P_o is fixed, then $P_o^{(m)}(1)$ is minimal for $a_{2b+2c+1} = \delta_2$, $a_{2b+2c+3} = \cdots = a_d = 0$.

For the polynomial P_e, we obtain in the same way that if the sums

$$\delta_3 := a_0 + a_2 + \cdots + a_{2c-2} \quad \text{and} \quad \delta_4 := a_{2c} + \cdots + a_{d-1}$$

are fixed, then $P_e^{(m)}(1)$ is minimal for $a_{2c-2} = \delta_3$, $a_0 = a_2 = \cdots = a_{2c-4} = 0$, $a_{2c} = \delta_4$, $a_{2c+2} = \cdots = a_{d-1} = 0$. Thus the polynomials P_o and P_e are of the form

$$P_o = Ex^{2b+2c+1} - Fx^{2b+2c-1} \quad , \quad P_e = Gx^{2c} - Hx^{2c-2} \ ,$$

with $E := a_{2b+2c+1} \geq 0$, $-F := a_{2b+2c-1} \leq 0$, $G := a_{2c} \geq 0$ and $-H := a_{2c-2} \leq 0$. Recall that

$$P(1) = 0 \ , \quad P_o(1) = 1 \quad \text{and} \quad P_e(1) = -1 \ , \quad \text{i. e.} \quad E - F = 1 \quad \text{and} \quad G - H = -1 \ .$$

The values of the derivatives at $x = 1$ are of the form

$$P^{(m)}(1) = u_m E - v_m F + w_m G - t_m H \ , \quad u_m > v_m > w_m > t_m \ ,$$

with $u_m, \ v_m, w_m, \ t_m \in \mathbb{N}$. Hence

$$\begin{aligned} P^{(m)}(1) &= (u_m - v_m)E + v_m(E - F) + (w_m - t_m)G + t_m(G - H) \\ &= (u_m - v_m)E + (w_m - t_m)G + (v_m - t_m) > 0 \ . \end{aligned}$$

□

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VLADIMIR PETROV KOSTOV
UNIVERSITÉ CÔTE D'AZUR
DEPARTMENT OF MATHEMATICS
CNRS, LJAD, FRANCE
ORCID: 0000-0001-5836-2678
E-mail address: vladimir.kostov@unice.fr