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A New Family of Odd Nakagami Exponential (NE-G) Distributions

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Abstract — In this study, a new family of odd nakagami exponential (NE-G) distributions is introduced and investigated as a new generator of continuous distributions. Quantile, hazard rate function, moments, incomplete moments, order statistics, and entropies are only a few of the statistical features that are investigated. A unique model is presented and thoroughly examined. To estimate model parameters based on describing real-life data sets, the maximum likelihood method is applied. The bias and mean square error of maximum likelihood estimators are investigated using a comprehensive simulation exercise. Finally, the new family adaptability is demonstrated via application to real-world data sets.

Keywords — *Nakagami exponential, order statistics, moment, quantile functions*

Mathematics Subject Classification (2020) — 62E99, 62F99

1. Introduction

There is a great need for expanded variants of classical distributions in many applied disciplines, such as lifetime analysis, finance, and insurance. As a result, various efforts have been made to define new families of probability distributions that expand well-known distributions and allow a high level of flexibility in modelling data in practise.. Many researchers have suggested a variety of methods for producing new families of distributions [1]. The beta-generalized family of distributions was created by Kumaraswamy [2], the Exponentiated-G by [3], the Gamma-G (type I) by [4], the Gamma-G (type II) by [5], the Generalised Gamma-G by [6], the Log-Gamma-G by [7], Additive Weibull-G by [8], Beta Marshall-Olkin-G by [9], Logistic-G family by [4], the Generalized Odd Gamma-G by [10],odd-Gamma-G (type III) by [11], new Weibull-G by [12], the Marshall-Olkin Odd Burr III-G by [13] and Exponentiated Generalized Power Function Distribution by [14] among others.
[15] has developed and investigated a class of the Nakagami-G family of distributions. The goal of this study is to introduce the odd Nakagami Exponential-G (NE-G) family of distributions. This family of distributions is an extension of the Nakagami-G family proposed by [15]. We investigate some of the new distribution's statistical properties in depth, use the MLE method to estimate the parameters of the proposed distribution, and finally fit real-life data sets to the new distribution and some of the existing families of distributions to compare the NE-G family's performance.
The cumulative distribution function (cdf) and probability density function (pdf) of Nakagami Ex-

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ponential distribution (see [15]) is given by

$$F(x) = \gamma_* \left(\lambda, \frac{\lambda}{\beta} \left(\frac{1 - e^{-\alpha x}}{e^{-\alpha x}} \right)^2 \right) \tag{1}$$

$$f(x) = \frac{2\lambda^\lambda \alpha e^{-\alpha x} \left(1 - e^{-\alpha x}\right)^{2\lambda - 1}}{\Gamma(\lambda)\beta^\lambda \left(e^{-\alpha x}\right)^{2\lambda+1}} e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}}\right)^2} \tag{2}$$

The following is how the paper might look: In Section 2, a construction of the NE-G family of distributions is introduced. A special model is provided, along with plots of pdfs, cdfs, survival functions, and harzad rate functions to demonstrate the models' flexibility. Our applied study will be centred on the uniform distribution as a baseline. The parameter estimation and main statistical properties of the NE-G are presented. In Section 3, a Monte Carlo simulation study is presented. The proposed model is fitted based on two real data sets in Section 4 and compared to other well-known models.

2. Constructions of the NE-G Distributions

Equation 2 can be used to calculate the cumulative distribution of a random variable.

$$P\left(X \le x\right) = F(x) = \int_0^{\frac{G(x)}{1-G(x)}} \frac{2\lambda^\lambda \alpha e^{-\alpha x} \left(1 - e^{-\alpha t}\right)^{2\lambda - 1}}{\Gamma(\lambda)\beta^\lambda \left(e^{-\alpha t}\right)^{2\lambda+1}} e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha t}}{e^{-\alpha t}}\right)^2} \partial t \tag{3}$$

Therefore,

$$F(x) = \frac{1}{\Gamma(\lambda)} \gamma \left[\lambda, \frac{\lambda}{\beta} \left(\frac{1 - e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}} \right)^2 \right] \tag{4}$$

Equation 4 the pdf of Nakagami Exponential G family becomes

$$f(x) = \frac{2\lambda^\lambda \alpha g(x) \left[1 - e^{-\alpha \frac{G(x)}{1-G(x)}}\right]^{2\lambda-1}}{\Gamma(\lambda)\beta^\lambda \left[1 - G(x)\right]^2 e^{-2\lambda\alpha\left(\frac{G(x)}{1-G(x)}\right)}} e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}\right)^2} \tag{5}$$

The survival and hazard rate functions of the NE-G family are, respectively, given by:

$$R(x) = 1 - \frac{1}{\Gamma(\lambda)} \gamma \left[\lambda, \frac{\lambda}{\beta} \left(\frac{1 - e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}} \right)^2 \right] \tag{6}$$

and

$$hr(x) = \frac{\frac{2\lambda^\lambda \alpha g(x) \left[1 - e^{-\alpha \frac{G(x)}{1-G(x)}}\right]^{2\lambda-1}}{\Gamma(\lambda)\beta^\lambda \left[1 - G(x)\right]^2 e^{-2\lambda\alpha\left(\frac{G(x)}{1-G(x)}\right)}} e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}\right)^2}}{1 - \frac{1}{\Gamma(\lambda)} \gamma \left[\lambda, \frac{\lambda}{\beta} \left(\frac{1 - e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}} \right)^2 \right]} \tag{7}$$

Figures (1), (2), (3), and (4) show that the shapes of the pdf and cdf are flexible for certain parameter values. Plots of the hazard rate and survival functions of the NE-U distribution for some parameter values are shown.

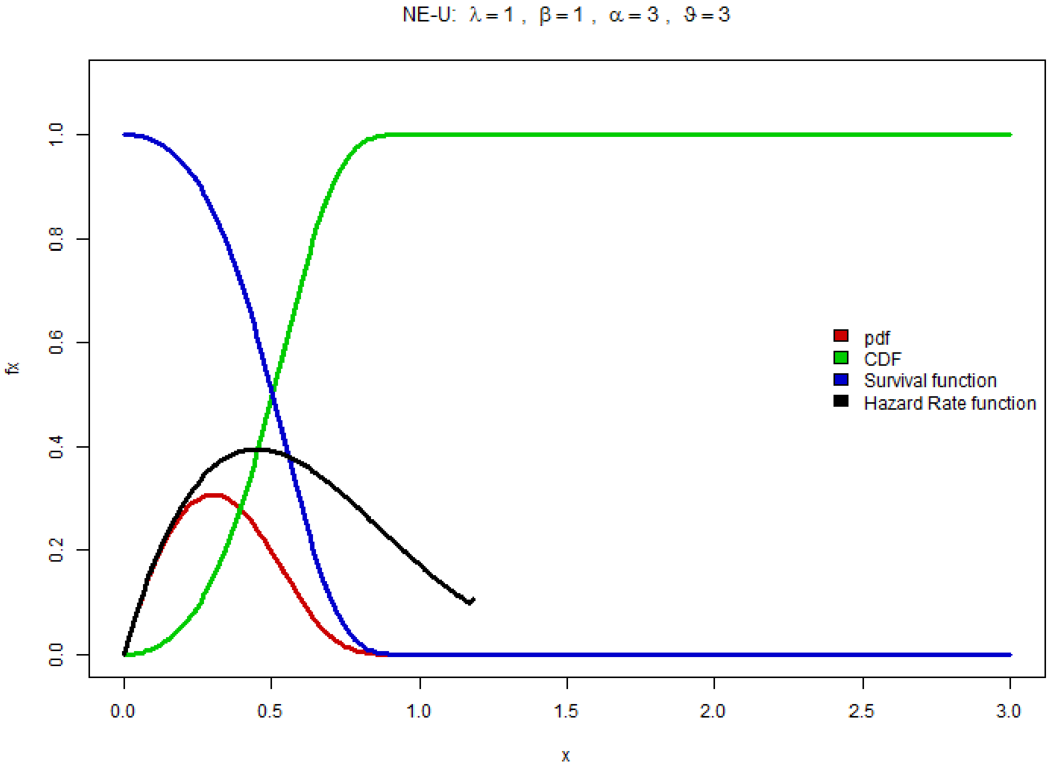

Fig. 1. NE-U function for $\lambda = 1, \beta = 1, \alpha = 3,$ and $\vartheta = 3$

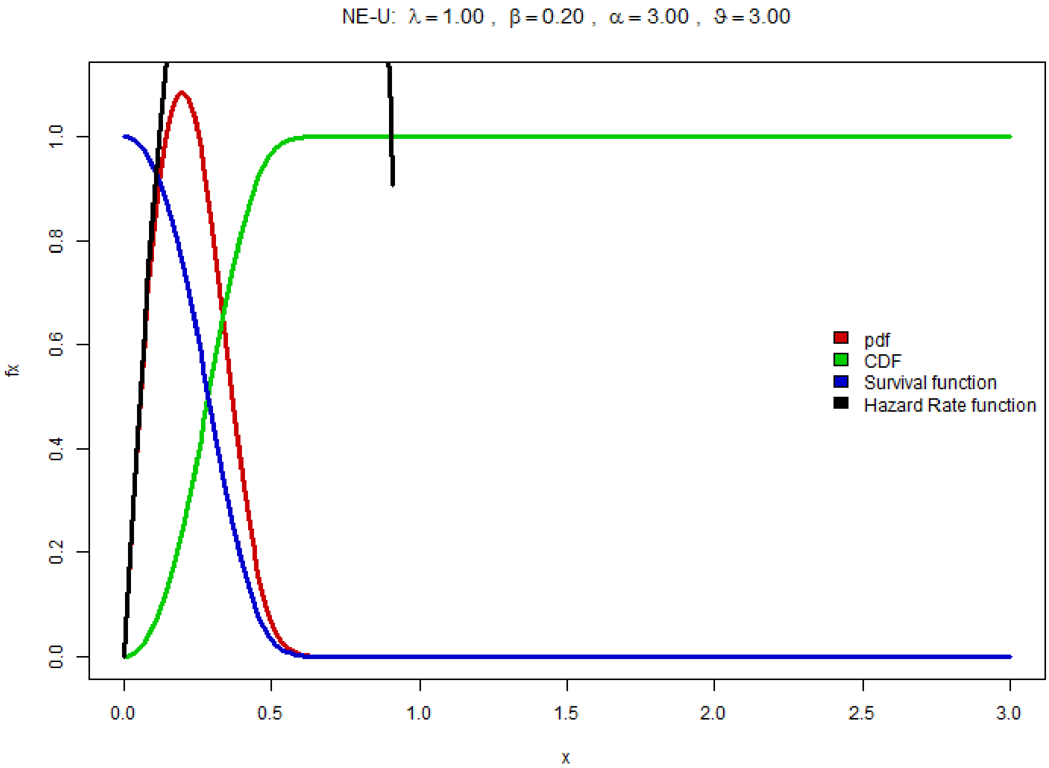

Fig. 2. NE-U function for $\lambda = 1, \beta = 0.2, \alpha = 3,$ and $\vartheta = 3$

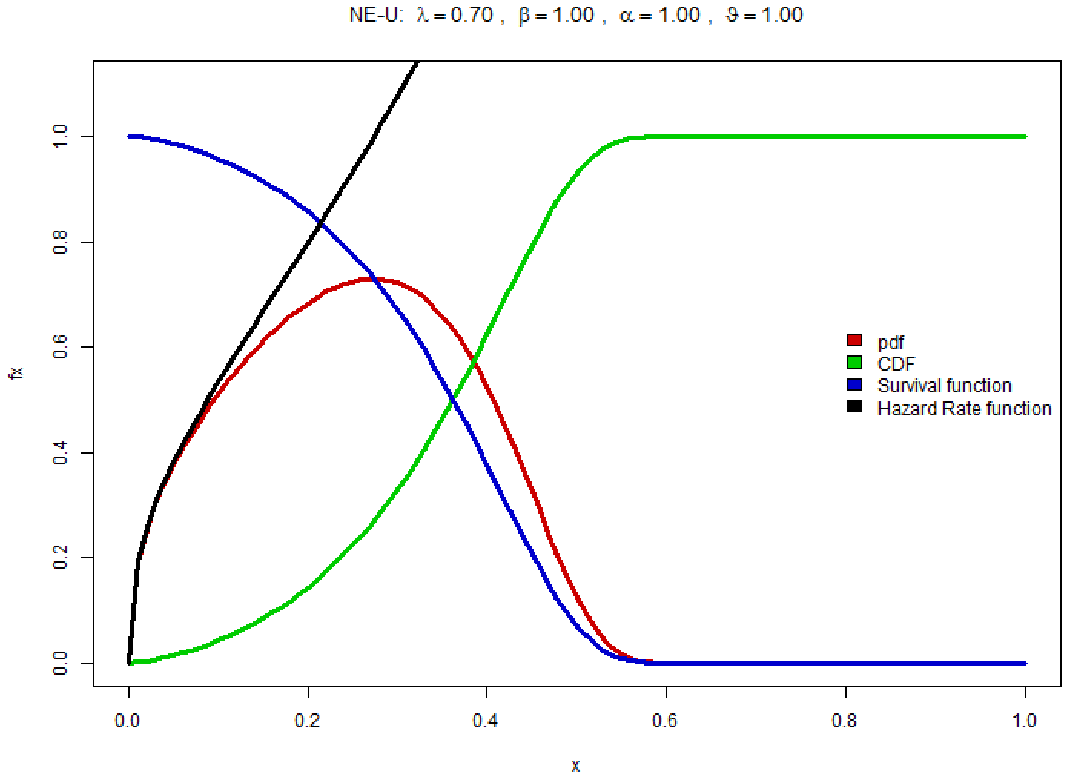

Fig. 3. NE-U function for $\lambda = 0.7, \beta = 1, \alpha = 1,$ and $\vartheta = 1$

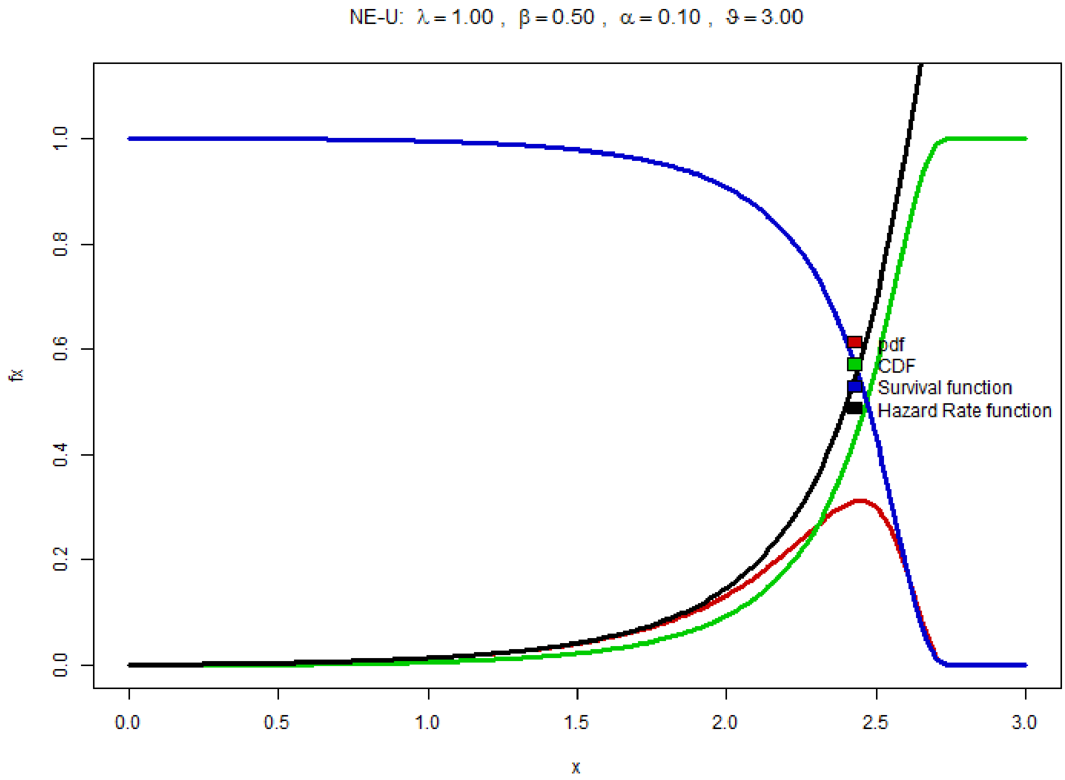

Fig. 4. NE-U function for $\lambda = 1, \beta = 0.5, \alpha = 1,$ and $\vartheta = 3$

2.1. Limiting Behaviour

The cdf defined in Equation 4 must satisfy the Limiting behaviour whether the Nakagami Exponential-G family of distributions is a valid family of distribution function. The behaviour of the NE-U distributions at $x \to 0$ and as $x \to 1$. In this case, what was considered is: when $\lim_{x\to 0} G(x) = 0$ since $\gamma(\lambda, 0) = \int_0^0 t^{\lambda-1} e^{-t} \partial t = 0$

$$\lim_{G(x)\to 0} F(x) = \frac{1}{\Gamma(\lambda)} \gamma \left[\lambda, \frac{\lambda}{\beta} \left(\frac{1 - e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}} \right)^2 \right] = 0 \tag{8}$$

when $\lim_{x\to\infty} G(x) = 1$

$$\lim_{G(x)\to \infty} F(x) = \frac{1}{\Gamma(\lambda)} \gamma \left[\lambda, \frac{\lambda}{\beta} \left(\frac{1 - e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}} \right)^2 \right] = 1 \tag{9}$$

2.2. Investigation of the Proposed NE-G Family of Distribution

The pdf defined in Equation 5 must satisfy the integral to show whether the Nakagami Exponential-G family of distributions is a valid family of probability density function.

$$\int_{-\infty}^{\infty} f(x) \partial x = 1 \tag{10}$$

$$\int_0^\infty \frac{2\lambda^\lambda \alpha g(x) \left[1 - e^{-\alpha \frac{G(x)}{1-G(x)}}\right]^{2\lambda - 1}}{\Gamma(\lambda)\beta^\lambda [1 - G(x)]^2 e^{-2\lambda\alpha\left(\frac{G(x)}{1-G(x)}\right)}} e^{-\frac{\lambda}{\beta}\left(\frac{1 - e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}} \right)^2} \partial x \tag{11}$$

let

$$u = \frac{\lambda}{\beta} \left(\frac{1 - e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}} \right)^2 \Rightarrow \left(\frac{u\beta}{\lambda}\right)^{\frac{1}{2}} = \left(\frac{1 - e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}} \right)$$

$$\partial x = \frac{\beta [1 - G(x)]^2 e^{-2\alpha\left(\frac{G(x)}{\bar{G}(x)}\right)}}{2\lambda\alpha g(x) \left(1 - e^{-\alpha\left(\frac{G(x)}{\bar{G}(x)}\right)}\right)} \partial u \tag{12}$$

substituting 12 into 11 yields

$$\frac{\lambda^{\lambda-1}}{\Gamma(\lambda)\beta^{\lambda-1}} \int_0^\infty \left(\frac{u\beta}{\lambda}\right)^{\lambda-1} e^{-u} \partial u \tag{13}$$

The density function defined in Equation 5 is clearly a valid family of distributions, as evidenced by this.

$$\frac{1}{\Gamma(\lambda)} \times \Gamma(\lambda) = 1 \tag{14}$$

The following are the primary reasons for employing the NE-G family in practice: (i) to give symmetrical distributions a skewness; (ii) to model real data with heavy-tailed distributions that aren't longer-tailed; (iii) under the same baseline distribution, to provide consistently better fits than other generated models

2.3. Expansion for NE-G Density and Distribution Function

Using generalized binomial and taylor expansion, one may obtain

$$f(x)=\frac{2\lambda^{\lambda}\alpha g(x)\left[1-e^{-\alpha\frac{G(x)}{1-G(x)}}\right]^{2\lambda-1}}{\Gamma(\lambda)\beta^{\lambda}\left[1-G(x)\right]^{2}e^{-2\lambda\alpha\left(\frac{G(x)}{1-G(x)}\right)}}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\left[\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}\right)^{2}\right]^{k} \quad (15)$$

$$=\frac{2\lambda^{\lambda+k}\alpha g(x)\left[1-e^{-\alpha\frac{G(x)}{1-G(x)}}\right]^{2\lambda+2k-1}}{\Gamma(\lambda)\beta^{\lambda+k}\left[1-G(x)\right]^{2}}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\mathrm{e}^{\alpha[2\lambda+2k]\left(\frac{G(x)}{1-G(x)}\right)} \quad (16)$$

$$\left[1-e^{-\alpha\frac{G(x)}{1-G(x)}}\right]^{2\lambda+2k-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{2\lambda+2k-1}{i}e^{-i\alpha\left(\frac{G(x)}{1-G(x)}\right)}$$

$$=\frac{2\lambda^{\lambda+k}\alpha g(x)}{\Gamma(\lambda)\beta^{\lambda+k}\left[1-G(x)\right]^{2}}\sum_{k,i=0}^{\infty}\frac{(-1)^{k+i}}{k!}\binom{2\lambda+2k-1}{i}\mathrm{e}^{\alpha[2\lambda+2k-i]\left(\frac{G(x)}{1-G(x)}\right)} \quad (17)$$

$$=\frac{2\lambda^{\lambda+k}\alpha g(x)}{\Gamma(\lambda)\beta^{\lambda+k}}\sum_{k,i,j,p=0}^{\infty}\frac{(-1)^{k+i+p}}{k!p!}\binom{2\lambda+2k-1}{i}\binom{j+p+1}{p}\alpha^{j}[2\lambda+2k-i]^{j}G(x)^{j+p} \quad (18)$$

reduced

$$f(x)=\sum_{k,i,j,p=0}^{\infty}\kappa_{k,i,j,p}h_{j+p+1}(x) \quad (19)$$

where

$$\kappa_{k,i,j,p}=\frac{2\lambda^{\lambda+k}\alpha}{\Gamma(\lambda)\beta^{\lambda+k}}\frac{(-1)^{k+i+p}}{k!p!}\binom{2\lambda+2k-1}{i}\binom{j+p+1}{p}\frac{\alpha^{j}[2\lambda+2k-i]^{j}}{[j+p+1]}$$

and

$$h_{j+p+1}(x)=(j+p+1)g(x)G(x)^{j+p}$$

We get by integrating from equation 19 with respect to x.

$$F(x)=\sum_{k,i,j,p=0}^{\infty}\kappa_{k,i,j,p}H_{j+p+1}(x) \quad (20)$$

where

$$H_{j+p+1}(x)=G(x)^{j+p+1}$$

2.4. Properties of the NE-G Distribution

We establish certain mathematical features of the NE-G family in this section.

2.4.1. Quantile Function

The inverse distribution function is the most popular and simplest method for producing random variates. The qf of the NE-G family can be obtained by inverting Equation 4 and then solve the equation numerically. We can utilize the following method:

$$u = F(x) = \frac{1}{\Gamma(\lambda)}\gamma\left[\lambda, \frac{\lambda}{\beta}\left(\frac{1 - e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}\right)^2\right] \tag{21}$$

$$G(x) = \frac{\frac{1}{\alpha}log\left\{\left\{\frac{\beta}{\lambda}\gamma^{-1}\left[\lambda, u\Gamma(\lambda)\right]\right\}^{1/2} + 1\right\}}{1 + \frac{1}{\alpha}log\left\{\left\{\frac{\beta}{\lambda}\gamma^{-1}\left[\lambda, u\Gamma(\lambda)\right]\right\}^{1/2} + 1\right\}} \tag{22}$$

2.5. Moment and Moment Generating Function

Moments are crucial in statistical analysis. Moments can be used to investigate some of the most important characteristics of a distribution.

2.5.1. Moment

The r^{th} ordinary moment of X is given by

$$E(X^r) = \mu_r' = \int_0^\infty x^r f(x)\partial x \tag{23}$$

The following can be derived from equation 19:

$$E(X^r) = \mu_r' = \sum_{k,i,j,p=0}^{\infty} \kappa_{k,i,j,p} E(Y_{j+p+1}^r) \tag{24}$$

Y_{j+p+1} represents the Exp-G distribution with power parameter (j+p + 1). Choosing r = 1 in equation 24, We now have the mean of X.

2.5.2. Moment Generating Function

The moment generating function of X say, $M_X(t) = E(e^{tX})$ is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' = \sum_{k,i,j,p,r=0}^{\infty} \frac{t^r \kappa_{k,i,j,p}}{r!} E(Y_{j+p+1}^r) \tag{25}$$

2.5.3. Incomplete Moments

The k^{th} incomplete moments, say $\rho_k(t)$, is given by

$$\rho_k(t) = \sum_{r=0}^{\infty} \kappa_{k,i,j,p} \int_{-\infty}^{t} h_{j+p+1}(x)\partial x \tag{26}$$

2.5.4. Entropy

The Rényi entropy of a random variable X is defined mathematically as follows:

$$I_R(\sigma) = \frac{1}{1-\sigma} log \left(\int_0^\infty f^\sigma(x)\partial x \right) \tag{27}$$

Where $\sigma > 0$ and $\sigma \neq 1$. Based on $f(x)$ of any distribution. From equation 5

$$f^\sigma(x) = \left\{ \frac{2\lambda^\lambda \alpha g(x) \left[1 - e^{-\alpha \frac{G(x)}{1-G(x)}}\right]^{2\lambda-1}}{\Gamma(\lambda)\beta^\lambda \left[1-G(x)\right]^2 e^{-2\lambda\alpha\left(\frac{G(x)}{1-G(x)}\right)}} e^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}{e^{-\alpha\left(\frac{G(x)}{1-G(x)}\right)}}\right)^2} \right\}^\sigma \tag{28}$$

$$f^\sigma(x) = \left(\frac{2\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda}\right)^\sigma \left(\frac{\sigma\lambda}{\beta}\right)^k \sum_{k,i,j,p=0}^{\infty} \binom{(2\lambda-1)\sigma+2k}{i} \binom{2\sigma+j+p-1}{p} \alpha^j \left(i - (2\lambda\sigma+2k)\right)^j g(x)^\sigma G(x)^{j+p} \tag{29}$$

reduced

$$f^\sigma(x) = \sum_{k,i,j,p=0}^{\infty} \varLambda_{k,i,j,p} g(x)^\sigma G(x)^{j+p} \tag{30}$$

where

$$\varLambda_{k,i,j,p} = \left(\frac{2\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda}\right)^\sigma \left(\frac{\sigma\lambda}{\beta}\right)^k \binom{(2\lambda-1)\sigma+2k}{i} \binom{2\sigma+j+p-1}{p} \alpha^j \left(i - (2\lambda\sigma+2k)\right)^j$$

$$I_R(\sigma) = \frac{1}{1-\sigma} log \left(\int_0^\infty \sum_{k,i,j,p=0}^{\infty} \varLambda_{k,i,j,p} g(x)^\sigma G(x)^{j+p} \partial x \right) \tag{31}$$

2.5.5. Order Statistics

Let $x_1, x_2 \ldots, x_n$ be independent random sample from a distribution function, $F(x)$, with an associated probability density function, $f(x)$. Then, the probability density function of the i^{th} order statistics, $x_{(i)}$, is given by:

$$f_{x_{(j)}}(x) = \frac{n!}{(r-1)!(n-r)!} \sum_{z=0}^{n-r} (-1)^z \binom{n-r}{z} f_X(x) \left[F_X(x)\right]^{z+r-1} \tag{32}$$

The *pdf* of i^{th} order statistic from NE-G distribution is obtained by substituting equation 19 and 20 into 32

$$f_{x_{(j)}}(x) = \frac{n!}{(r-1)!(n-r)!} \sum_{z=0}^{n-r} \sum_{k,i,j,p=0}^{\infty} (-1)^z \binom{n-r}{z} \kappa_{k,i,j,p} h_{j+p+1}(x) \left[\sum_{k,i,j,p=0}^{\infty} \kappa_{k,i,j,p} H_{j+p+1}(x)\right]^{z+r-1} \tag{33}$$

2.6. Maximum Likelihood Estimator

The maximum likelihood method is used here to estimate the unknown parameters of the NE-G family. When constructing confidence intervals, MLEs have desirable properties and provide simple approximations that work well in finite samples.
The log-likelihood function $\ell(\Theta)$ for the vector of parameters $\Theta = (\lambda, \beta, \xi)^T$ from n observations $(x_1, \cdots, x_n)$ has the form

$$\ell(\Theta) = n\ln(2) + n\lambda\ln(\lambda) + \sum_{i=0}^{n}\ln(g(x_i,\xi)) + (2\lambda-1)\sum_{i=0}^{n}\ln\left[1-\mathrm{e}^{-\alpha\left(\frac{G(x_i,\xi)}{1-G(x_i,\xi)}\right)}\right] - \frac{\lambda}{\beta}\sum_{i=0}^{n}\left(\frac{1-\mathrm{e}^{-\alpha\frac{G(x_i,\xi)}{1-G(x_i,\xi)}}}{\mathrm{e}^{-\alpha\frac{G(x_i,\xi)}{1-G(x_i,\xi)}}}\right)^2$$
$$- n\ln(\Gamma(\lambda)) - n\lambda\ln(\beta) - 2\sum_{i=0}^{n}\ln\left[1-G(x_i,\xi)\right] + 2\lambda\alpha\sum_{i=0}^{n}\left(\frac{G(x_i,\xi)}{1-G(x_i,\xi)}\right) \tag{34}$$

The maximum likelihood estimators can be obtained by numerically solving the following equations

$$\frac{\partial\ell(\Theta)}{\partial\lambda} = n + n\ln(\lambda) + 2\sum_{i=0}^{n}\ln\left[1-\mathrm{e}^{-\alpha\left(\frac{G(x_i,\xi)}{1-G(x_i,\xi)}\right)}\right] - \frac{1}{\beta}\sum_{i=0}^{n}\left(\frac{1-\mathrm{e}^{-\alpha\frac{G(x_i,\xi)}{1-G(x_i,\xi)}}}{\mathrm{e}^{-\alpha\frac{G(x_i,\xi)}{1-G(x_i,\xi)}}}\right)^2 - n\frac{\Gamma'(\lambda)}{\Gamma(\lambda)} - n\ln(\beta)$$
$$+ 2\alpha\sum_{i=0}^{n}\left(\frac{G(x_i,\xi)}{1-G(x_i,\xi)}\right) \tag{35}$$

$$\frac{\partial\ell(\Theta)}{\partial\beta} = \frac{\lambda}{\beta^2}\sum_{i=0}^{n}\left(\frac{1-\mathrm{e}^{-\alpha\frac{G(x_i,\xi)}{1-G(x_i,\xi)}}}{\mathrm{e}^{-\alpha\frac{G(x_i,\xi)}{1-G(x_i,\xi)}}}\right)^2 - \frac{n\lambda}{\beta} \tag{36}$$

$$\frac{\partial\ell(\Theta)}{\partial\xi} = \sum_{i=0}^{n}\frac{g(xi,\xi)'}{g(xi,\xi)} + (2\lambda-1)\sum_{i=0}^{n}\frac{\alpha G(x_i,\xi)'\mathrm{e}^{-\alpha\left(\frac{G(x_i,\xi)}{1-G(x_i,\xi)}\right)}}{(1-G(x_i,\xi))^2\left[1-\mathrm{e}^{-\alpha\left(\frac{G(x_i,\xi)}{1-G(x_i,\xi)}\right)}\right]} - \frac{2\alpha\lambda}{\beta}\sum_{i=0}^{n}\frac{G(x_i,\xi)'}{(1-G(x_i,\xi))^2}\times$$
$$\left[\frac{1-\mathrm{e}^{-\alpha\left(\frac{G(x_i,\xi)}{1-G(x_i,\xi)}\right)}}{\mathrm{e}^{-2\alpha\left(\frac{G(x_i,\xi)}{1-G(x_i,\xi)}\right)}}\right] + 2\sum_{1=0}^{n}\frac{G(x_i,\xi)'}{1-G(x_i,\xi)} + 2\lambda\alpha\sum_{i=0}^{n}\frac{G(x_i,\xi)'}{(1-G(x_i,\xi))^2} \tag{37}$$

where $g(x_i,\xi)' = \frac{\partial g(x_i,\xi)}{\partial\xi}$ and $G(x_i,\xi)' = \frac{\partial G(x_i,\xi)}{\partial\xi}$

2.7. The Nakagami Exponential Uniform (NE-U) Distribution

The cdf and pdf of our baseline distribution, the uniform distribution with parameters $(0, \vartheta)$ are given by:

$$G(x;\vartheta) = \frac{x}{\vartheta}$$

and

$$g(x;\vartheta) = \frac{1}{\vartheta}, 0 < x < \vartheta$$

By plugging $g(x;\vartheta)$ and $G(x;\vartheta)$ into Equation 5, we get the pdf of the NE-U distribution.

$$f(x) = \frac{2\lambda^\lambda\alpha}{\vartheta\Gamma(\lambda)\beta^\lambda\left(\frac{\vartheta-x}{\vartheta}\right)^2}\frac{\left[1-e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right]^{2\lambda-1}}{e^{-2\lambda\alpha\left(\frac{x}{\vartheta-x}\right)}}e^{\frac{-\lambda}{\beta}\left(\frac{1-e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}{e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}\right)^2} \tag{38}$$

2.8. Investigation of the Proposed NE-U Distribution

$$\int_0^\infty \frac{2\lambda^\lambda\alpha}{\vartheta\Gamma(\lambda)\beta^\lambda\left(\frac{\vartheta-x}{\vartheta}\right)^2}\frac{\left[1-e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right]^{2\lambda-1}}{e^{-2\lambda\alpha\left(\frac{x}{\vartheta-x}\right)}}e^{\frac{-\lambda}{\beta}\left(\frac{1-e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}{e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}\right)^2}\partial x=1 \tag{39}$$

let

$$u=\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}{e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}\right)^2 \tag{40}$$

$$\partial x=\frac{\beta(\vartheta-x)^2\mathrm{e}^{-\frac{2\alpha x}{\vartheta-x}}}{2\lambda\alpha\vartheta\left(1-\mathrm{e}^{-\frac{\alpha x}{\vartheta-x}}\right)}\partial u \tag{41}$$

substituting equation 41 and 40 into equation 39 yield

$$\frac{\lambda^{\lambda-1}}{\Gamma(\lambda)\beta^{\lambda-1}}\int_0^\infty\left(\frac{\beta u}{\lambda}\right)^{\lambda-1}\mathrm{e}^{-u}\partial u \tag{42}$$

$$\int_0^\infty f(x)\partial x=\frac{1}{\Gamma(\lambda)}\times\Gamma(\lambda)=1 \tag{43}$$

Hence Nakagami Exponential Uniform Distribution is pdf.

2.8.1. Cdf and Survival and Hazard Function of the NE-U Distribution

The distribution function of NE-U has the form

$$F(x)=\gamma_1\left(\lambda,\frac{\lambda}{\beta}\left(\mathrm{e}^{\alpha\left(\frac{x}{\vartheta-x}\right)}-1\right)^2\right) \tag{44}$$

The survival function of NE-U has the form

$$R(x)=1-\gamma_1\left(\lambda,\frac{\lambda}{\beta}\left(\mathrm{e}^{\alpha\left(\frac{x}{\vartheta-x}\right)}-1\right)^2\right) \tag{45}$$

The hazard rate function of NE-U has the form

$$hrf(x)=\frac{\frac{2\lambda^\lambda\alpha}{\vartheta\Gamma(\lambda)\beta^\lambda\left(\frac{\vartheta-x}{\vartheta}\right)^2}\frac{\left[1-e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right]^{2\lambda-1}}{e^{-2\lambda\alpha\left(\frac{x}{\vartheta-x}\right)}}e^{\frac{-\lambda}{\beta}\left(\frac{1-e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}{e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}\right)^2}}{1-\gamma_1\left(\lambda,\frac{\lambda}{\beta}\left(\mathrm{e}^{\alpha\left(\frac{x}{\vartheta-x}\right)}-1\right)^2\right)} \tag{46}$$

2.9. Expansion for NE-U Distribution

In this part a simple form for the probability density function of NE-U distribution is derived. Using generalized binomial and taylor expansion in the equation 38 one can obtain

$$\mathrm{e}^{-\frac{\lambda}{\beta}\left(\frac{1-e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}{e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}\right)^2}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{\lambda}{\beta}\right)^i\left(\frac{1-e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}{e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}\right)^{2i} \tag{47}$$

Substituting 47 in pdf 38 then, the pdf of NE-U can be written as

$$f(x) = \frac{2\lambda^\lambda \alpha}{\vartheta \Gamma(\lambda) \beta^\lambda \left(\frac{\vartheta - x}{\vartheta}\right)^2} \frac{\left[1 - e^{-\alpha\left(\frac{x}{\vartheta - x}\right)}\right]^{2\lambda - 1}}{e^{-2\lambda\alpha\left(\frac{x}{\vartheta - x}\right)}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\lambda}{\beta}\right)^i \left(\frac{1 - e^{-\alpha\left(\frac{x}{\vartheta - x}\right)}}{e^{-\alpha\left(\frac{x}{\vartheta - x}\right)}}\right)^{2i} \tag{48}$$

$$\left[1 - \left(1 - e^{-\alpha\left(\frac{x}{\vartheta - x}\right)}\right)\right]^{-2(\lambda + i)} = \sum_{j=0}^{\infty} \binom{2(\lambda + i) + j - 1}{j} \left(1 - e^{-\alpha\left(\frac{x}{\vartheta - x}\right)}\right)^j \tag{49}$$

By inserting 49 in pdf 48 then, the pdf of NE-U can be written as

$$f(x) = \frac{2\lambda^\lambda \alpha}{\vartheta \Gamma(\lambda) \beta^\lambda \left(\frac{\vartheta - x}{\vartheta}\right)^2} \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\lambda}{\beta}\right)^i \binom{2(\lambda + i) + j - 1}{j} \left(1 - e^{-\alpha\left(\frac{x}{\vartheta - x}\right)}\right)^{2\lambda + 2i + j - 1} \tag{50}$$

$$\left(1 - e^{-\alpha\left(\frac{x}{\vartheta - x}\right)}\right)^{2\lambda + 2i + j - 1} = \sum_{k=0}^{\infty} (-1)^k \binom{2\lambda + 2i + j - 1}{k} e^{-k\alpha\left(\frac{x}{\vartheta - x}\right)} \tag{51}$$

Substituting 51 in pdf 50 then, the pdf of NE-U can be written as

$$f(x) = \frac{2\lambda^\lambda \alpha}{\vartheta \Gamma(\lambda) \beta^\lambda \left(\frac{\vartheta - x}{\vartheta}\right)^2} \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+k}}{i!} \left(\frac{\lambda}{\beta}\right)^i \binom{2(\lambda + i) + j - 1}{j} \binom{2\lambda + 2i + j - 1}{k} e^{-k\alpha\left(\frac{x}{\vartheta - x}\right)} \tag{52}$$

Therefore, the NE-U distribution reduced to

$$f(x) = \sum_{i,j,k=0}^{\infty} \frac{\amalg_{i,j,k}}{\left(\frac{\vartheta - x}{\vartheta}\right)^2} \mathrm{e}^{\frac{-k\alpha x}{\vartheta - x}} \tag{53}$$

where

$$\amalg_{i,j,k} = \frac{2\lambda^\lambda \alpha}{\vartheta \Gamma(\lambda) \beta^\lambda} \frac{(-1)^{i+k}}{i!} \left(\frac{\lambda}{\beta}\right)^i \binom{2(\lambda + i) + j - 1}{j} \binom{2(\lambda + i) + j - 1}{k}$$

2.10. Quantile Function

It is not possible to obtain the qf of the NE-U distribution explicitly.

$$x = \frac{\frac{\vartheta}{\alpha} log \left\{ \left\{ \frac{\beta}{\lambda} \gamma^{-1} \left[\lambda, u\Gamma(\lambda)\right] \right\}^{1/2} + 1 \right\}}{1 + \frac{1}{\alpha} log \left\{ \left\{ \frac{\beta}{\lambda} \gamma^{-1} \left[\lambda, u\Gamma(\lambda)\right] \right\}^{1/2} + 1 \right\}} \tag{54}$$

2.11. Parameters Estimation

In this section, the method of estimation employed was Maximum Likelihood estimation to estimate the parameters of the NE-U distribution. Let $x_1 \cdots, x_n$ be a random sample from the NE-U, and the likelihood function be expressed as follows:

$$\prod_{i=1}^{n} f(x) = \prod_{i=1}^{n} \frac{2\lambda^\lambda \alpha}{\vartheta \Gamma(\lambda) \beta^\lambda \left(\frac{\vartheta - x}{\vartheta}\right)^2} \frac{\left[1 - e^{-\alpha\left(\frac{x}{\vartheta - x}\right)}\right]^{2\lambda - 1}}{e^{-2\lambda\alpha\left(\frac{x}{\vartheta - x}\right)}} e^{\frac{-\lambda}{\beta}\left(\frac{1 - e^{-\alpha\left(\frac{x}{\vartheta - x}\right)}}{e^{-\alpha\left(\frac{x}{\vartheta - x}\right)}}\right)^2} \tag{55}$$

Now, in the equation 55, take the log of the likelihood function.

$$\ell = n\ln(2) + n\lambda\ln(\lambda) + n\ln(\alpha) + (2\lambda-1)\sum_{i=1}^{n}\ln\left[1-e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right] - \frac{\lambda}{\beta}\sum_{i=1}^{n}\left\{\frac{\left[1-\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right]}{\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}\right\}^2$$
$$- n\ln(\vartheta) - n\ln(\Gamma(\lambda)) - n\lambda\ln(\beta) - 2\sum_{i=1}^{n}\ln\left(\frac{x}{\vartheta-x}\right) + 2\lambda\alpha\sum_{i=1}^{n}\ln\left(\frac{x}{\vartheta-x}\right) \tag{56}$$

By obtaining the derivative of the equation 56 with respect to $\lambda, \beta, \alpha, \vartheta$ respectively,

$$\frac{\partial\ell}{\partial\lambda} = n\ln(\lambda) + n + 2\sum_{i=1}^{n}\ln\left[1-e^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right] - \frac{1}{\beta}\sum_{i=1}^{n}\left\{\frac{\left[1-\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right]}{\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}\right\}^2 - \frac{n\Gamma'(\lambda)}{\Gamma(\lambda)} - n\ln(\beta)$$
$$+ 2\alpha\sum_{i=1}^{n}\ln\left(\frac{x}{\vartheta-x}\right) \tag{57}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{\lambda}{\beta^2}\sum_{i=1}^{n}\left\{\frac{\left[1-\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right]}{\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}\right\}^2 - \frac{n\lambda}{\beta} \tag{58}$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + (2\lambda-1)\sum_{i=1}^{n}\frac{x\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}{(\vartheta-x)\ln\left(1-\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right)} - \frac{2\lambda}{\beta}\sum_{i=1}^{n}\left\{\frac{x\left[1-\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right]}{(\vartheta-x)\mathrm{e}^{-2\alpha\left(\frac{x}{\vartheta-x}\right)}}\right\}$$
$$+ 2\lambda\sum_{i=1}^{n}\ln\left(\frac{x}{\vartheta-x}\right) \tag{59}$$

$$\frac{\partial\ell}{\partial\vartheta} = -(2\lambda-1)\sum_{i=1}^{n}\frac{x\alpha\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}}{(\vartheta-x)^2\ln\left(1-\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right)} + \frac{2\lambda\alpha}{\beta}\sum_{i=1}^{n}\frac{x\left[1-\mathrm{e}^{-\alpha\left(\frac{x}{\vartheta-x}\right)}\right]}{(\vartheta-x)^2\mathrm{e}^{-2\alpha\left(\frac{x}{\vartheta-x}\right)}} + \sum_{i=1}^{n}\left(\frac{2}{\vartheta-x}\right)$$
$$-\sum_{i=1}^{n}\left(\frac{2\lambda\alpha}{\vartheta-x}\right) \tag{60}$$

equate equations 57 to 60 to zero and solve them using any numerically iterative techniques.

3. NE-U Performance

The proposed NE-U model is evaluated in two ways in this section. The performance of the MLE's was examined using a simulation study. Secondly, the goodness of the fit of the NE-U was evaluated in relation to the other existing distributions.

3.1. Monte Carlo Simulation

The properties of maximum likelihood estimators for the parameters of the NE-U distribution were investigated using simulation in this section. The Average Bias and MSE of the parameters were measured. To generate random samples from the NE-U distribution, the quantile function given in equation 54 was employed. The simulation experiment was repeated N = 10000 times with sample sizes of n = 25, 50, 75, 100, 125, and parameter values of $\lambda, \beta, \alpha, \vartheta$ = (0.5, 0.9, 2, 1) and $\lambda, \beta, \alpha, \vartheta$ = (0.5, 0.9, 3, 1).

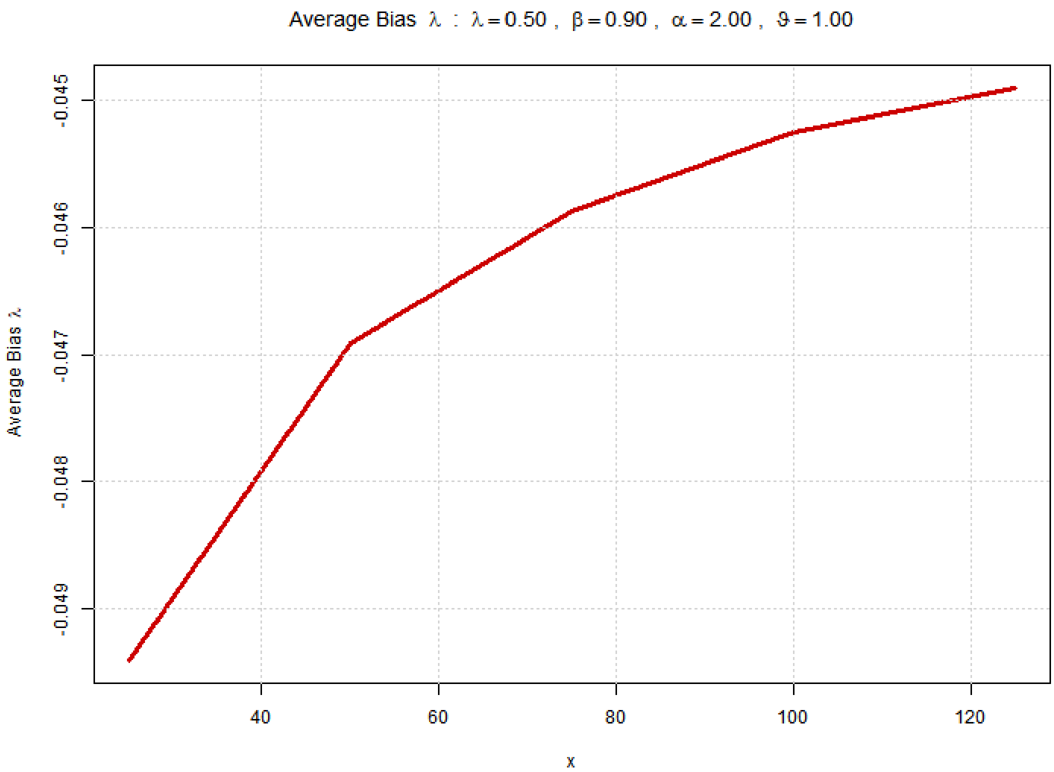

Fig. 5. Bias Lambda for $\lambda = 0.5, \beta = 0.9, \alpha = 2,$ and $\vartheta = 1$

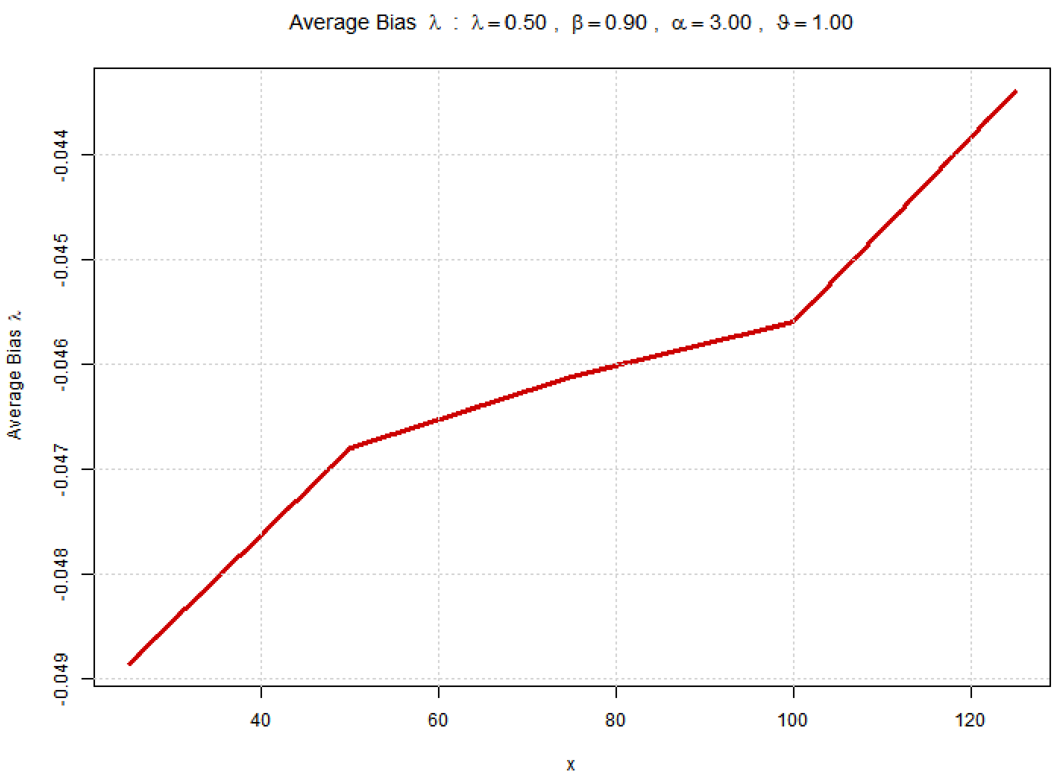

Fig. 6. Bias Lambda for $\lambda = 0.5, \beta = 0.9, \alpha = 3,$ and $\vartheta = 1$

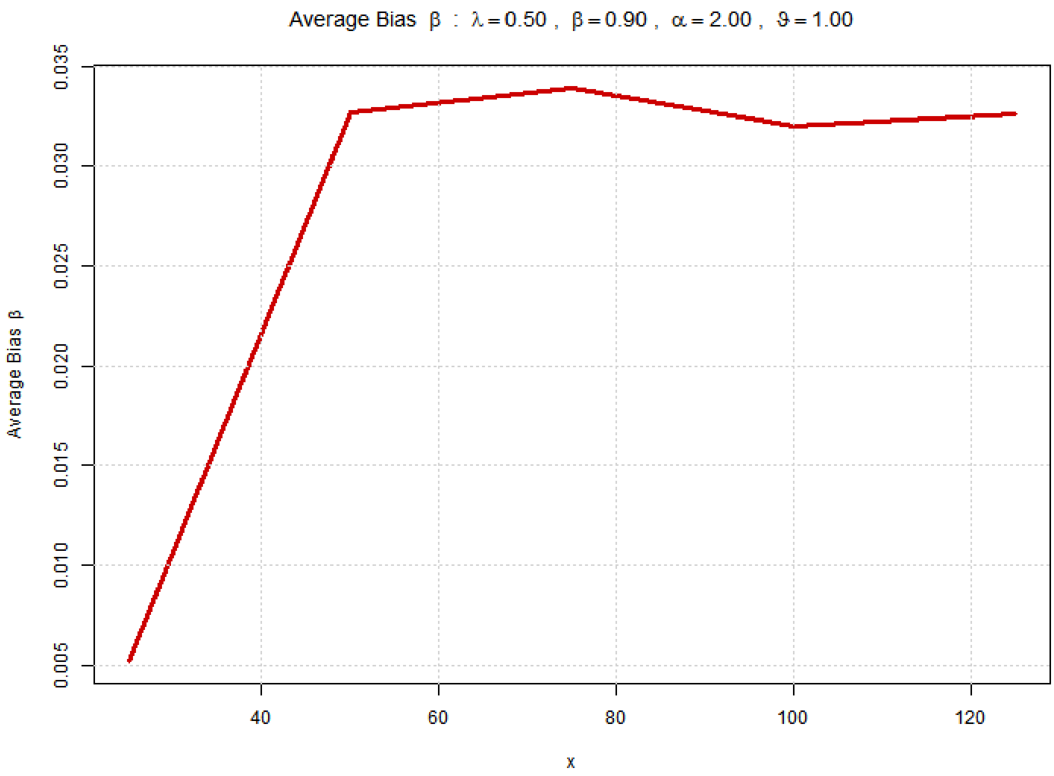

Fig. 7. Bias Beta for $\lambda = 0.5, \beta = 0.9, \alpha = 2,$ and $\vartheta = 1$

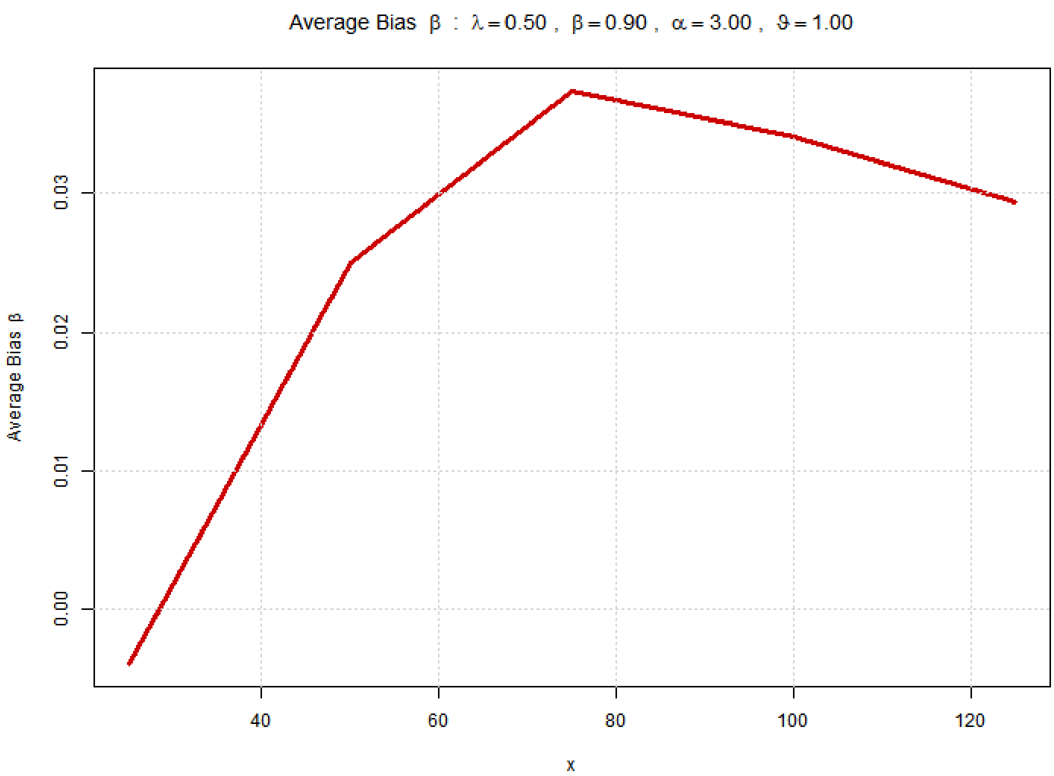

Fig. 8. Bias Beta for $\lambda = 0.5, \beta = 0.9, \alpha = 3,$ and $\vartheta = 1$

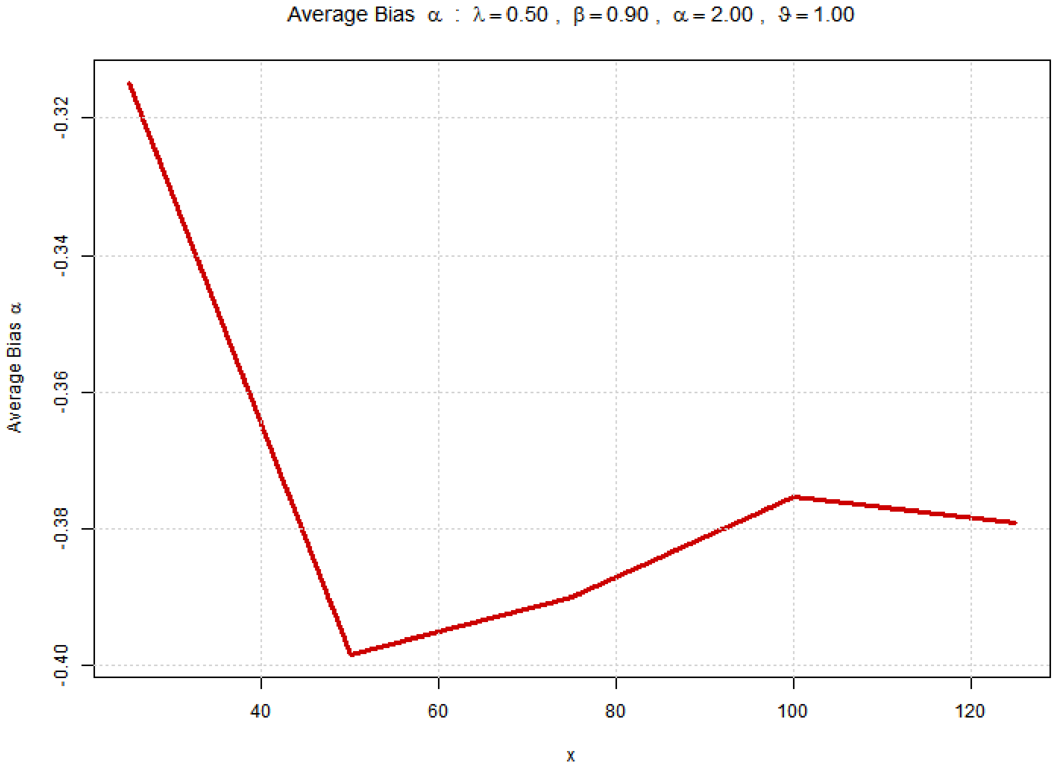

Fig. 9. Bias Alpha for $\lambda = 0.5, \beta = 0.9, \alpha = 2,$ and $\vartheta = 1$

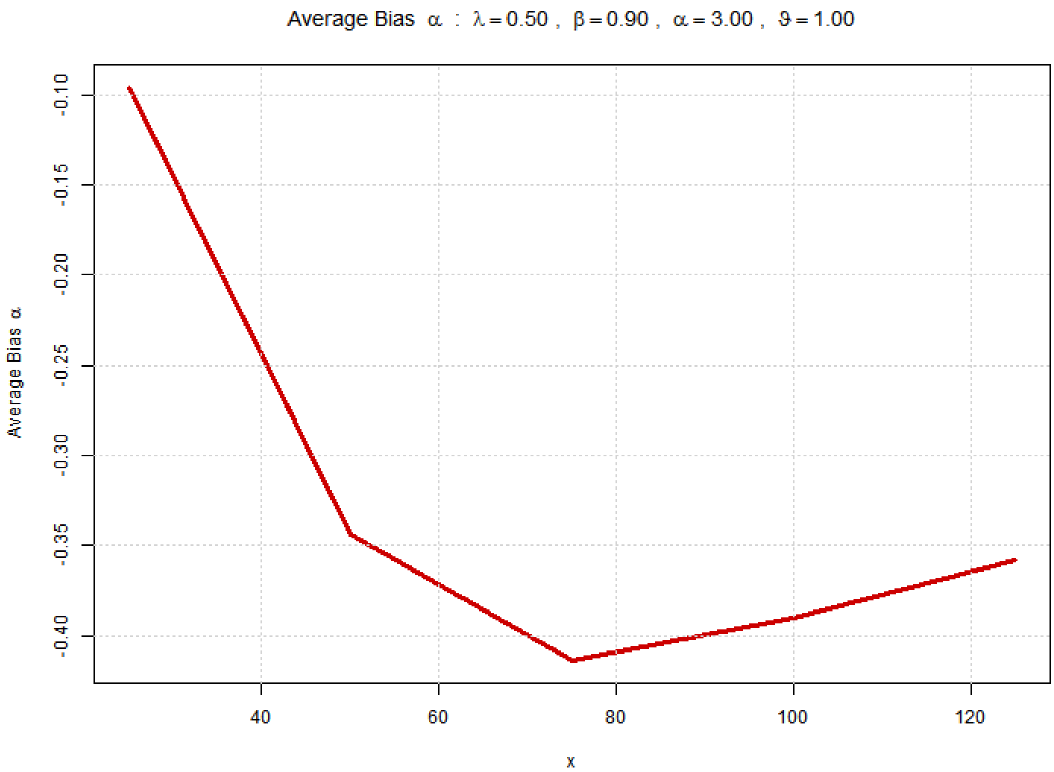

Fig. 10. Bias Alpha for $\lambda = 0.5, \beta = 0.9, \alpha = 3,$ and $\vartheta = 1$

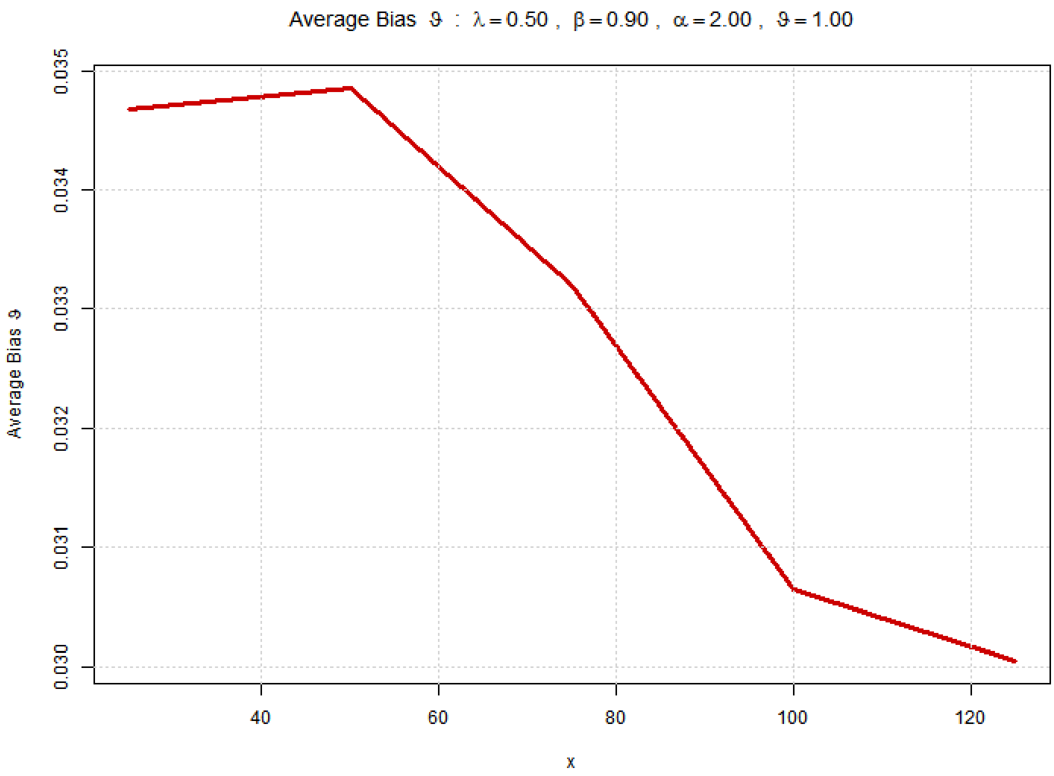

Fig. 11. Bias Vartheta for $\lambda = 0.5, \beta = 0.9, \alpha = 2,$ and $\vartheta = 1$

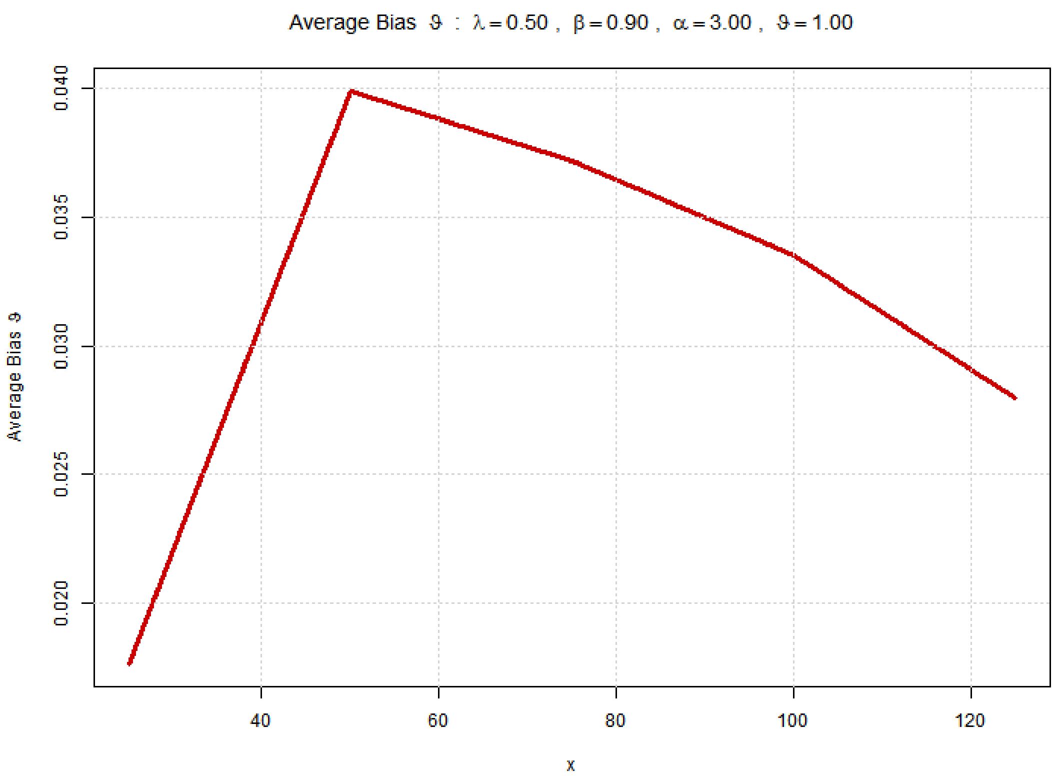

Fig. 12. Bias Vartheta for $\lambda = 0.5, \beta = 0.9, \alpha = 3,$ and $\vartheta = 1$

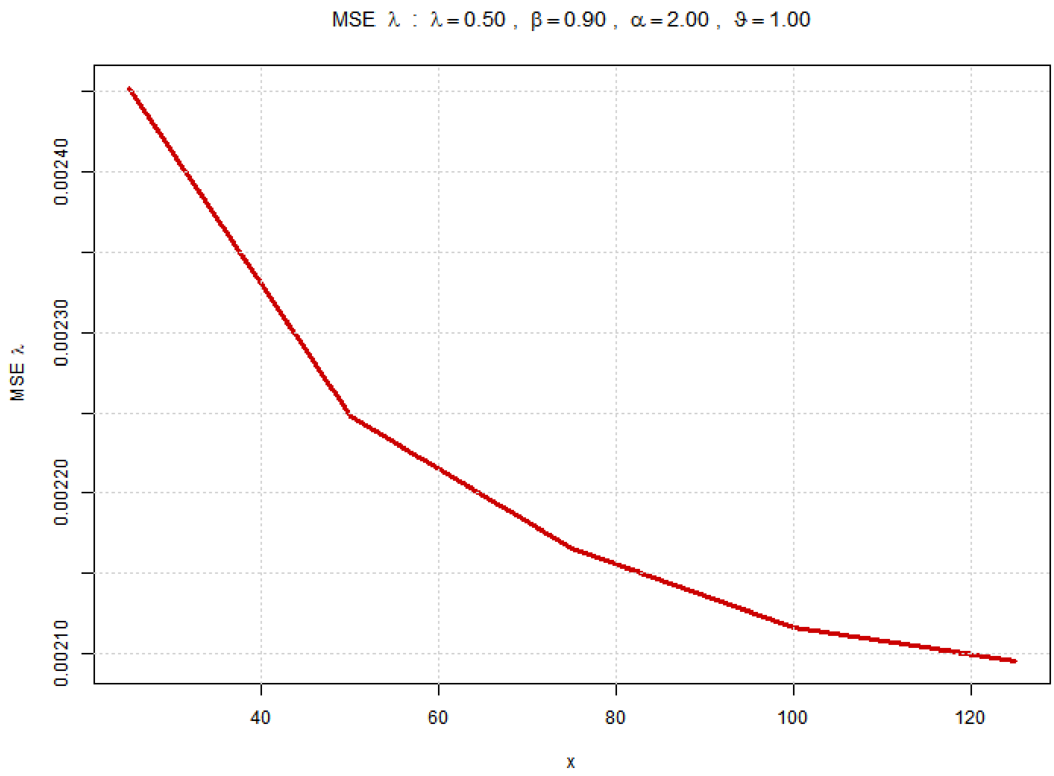

Fig. 13. MSE Lambda for $\lambda = 0.5, \beta = 0.9, \alpha = 2,$ and $\vartheta = 1$

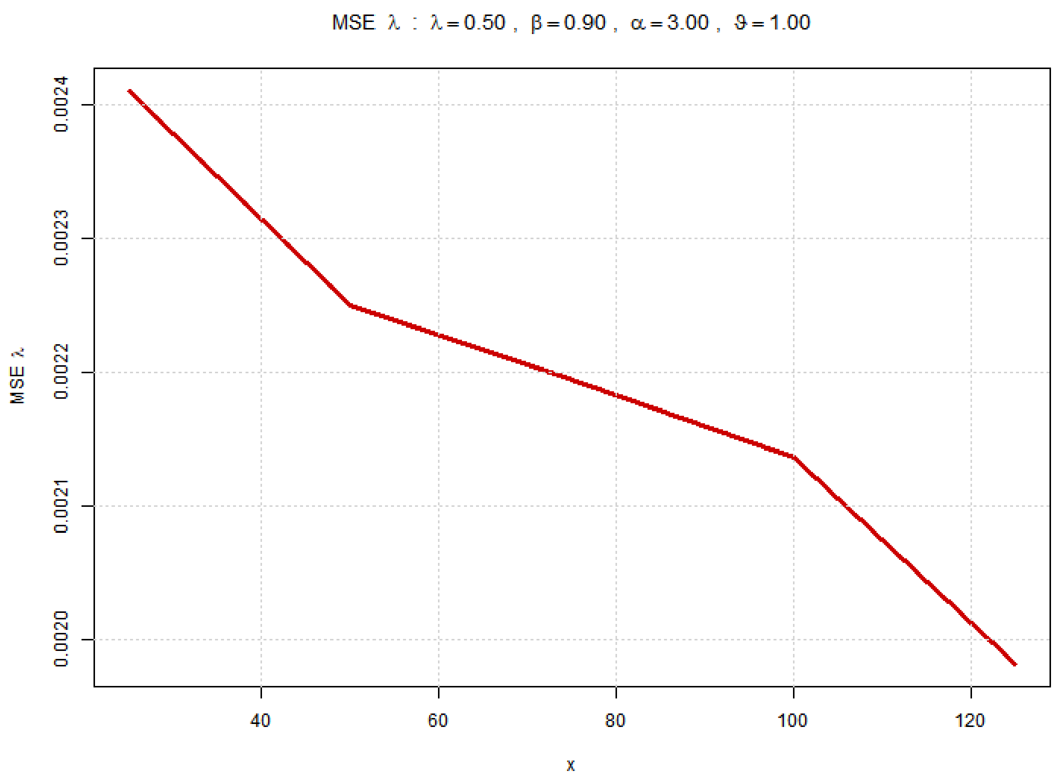

Fig. 14. MSE Lambda for $\lambda = 0.5, \beta = 0.9, \alpha = 3,$ and $\vartheta = 1$

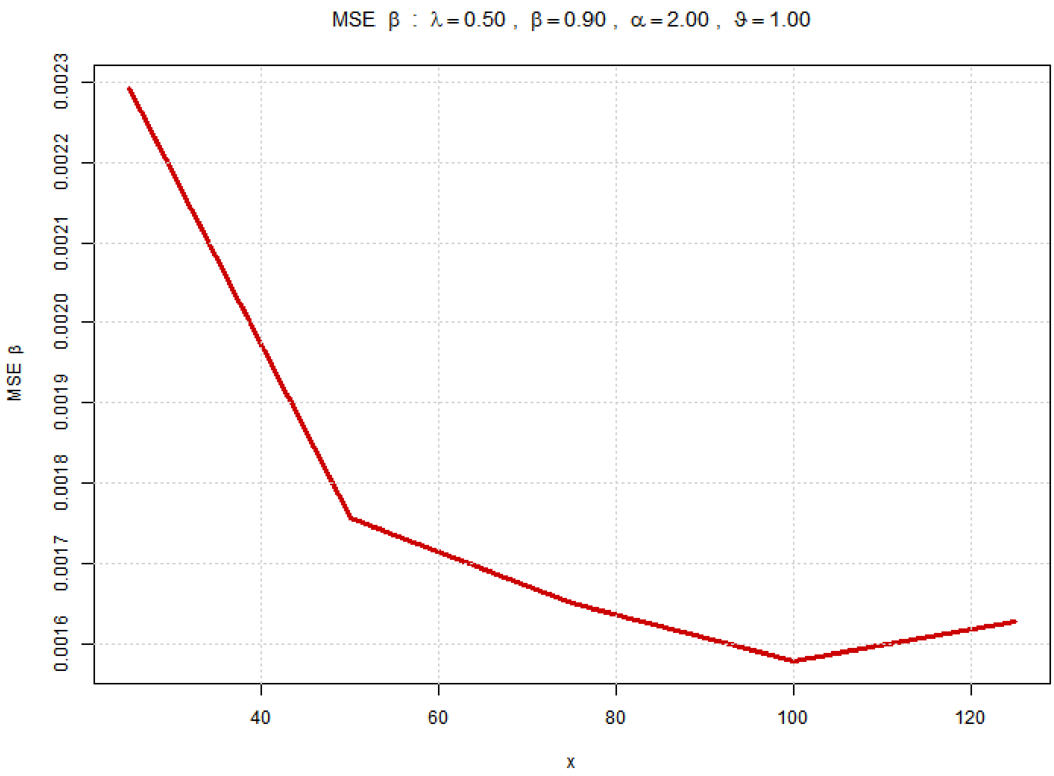

Fig. 15. MSE Beta for $\lambda = 0.5, \beta = 0.9, \alpha = 2,$ and $\vartheta = 1$

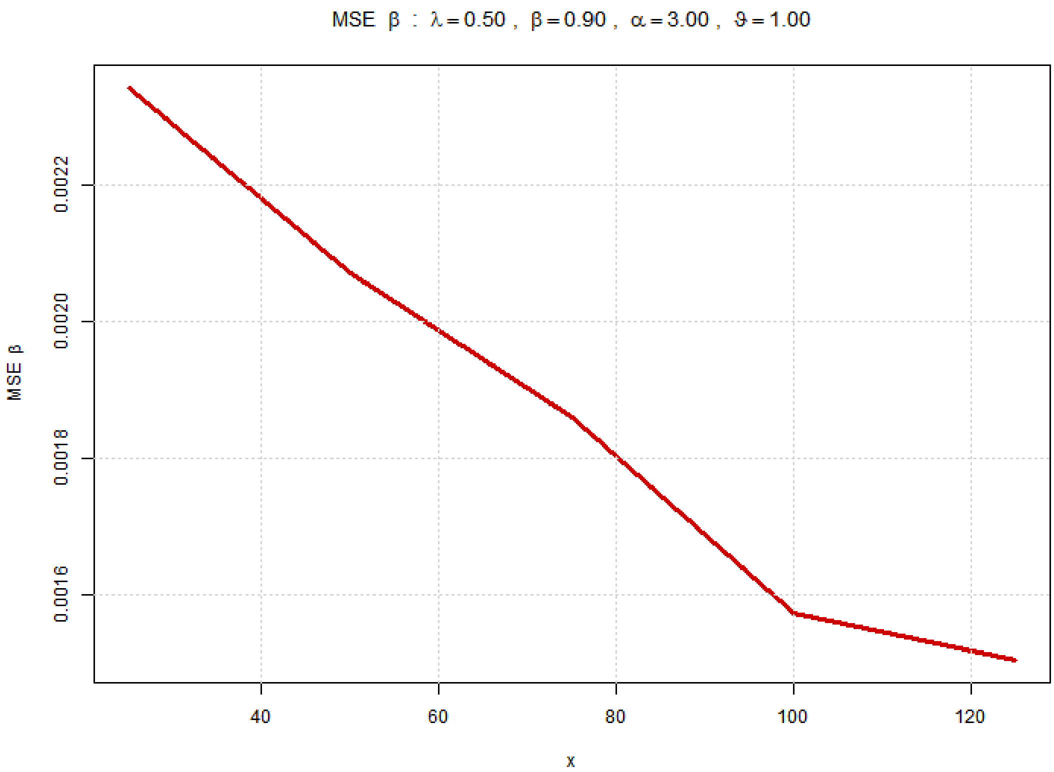

Fig. 16. MSE Beta for $\lambda = 0.5, \beta = 0.9, \alpha = 3,$ and $\vartheta = 1$

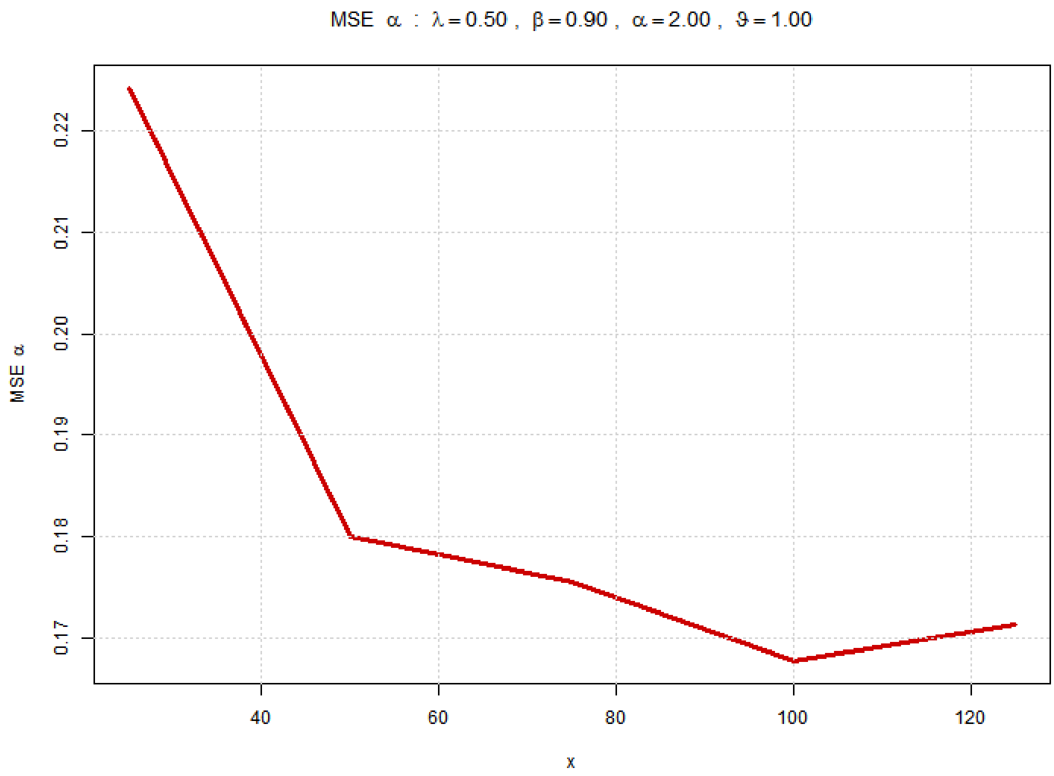

Fig. 17. MSE Alpha for $\lambda = 0.5, \beta = 0.9, \alpha = 2,$ and $\vartheta = 1$

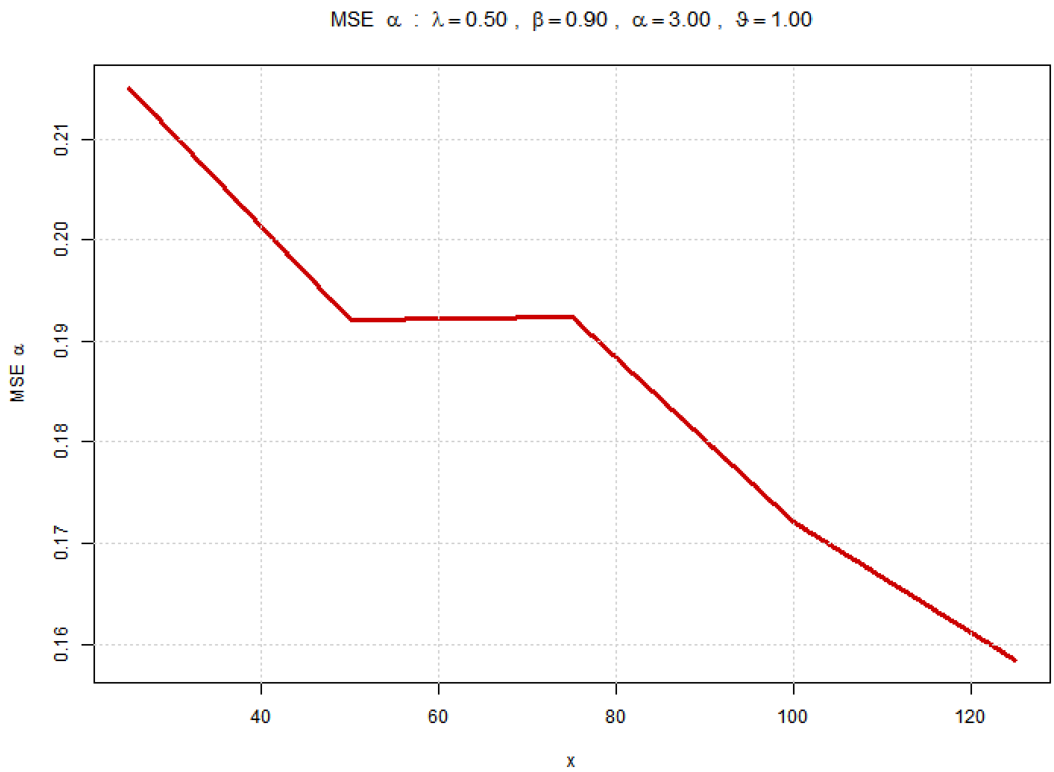

Fig. 18. MSE Alpha for $\lambda = 0.5, \beta = 0.9, \alpha = 3,$ and $\vartheta = 1$

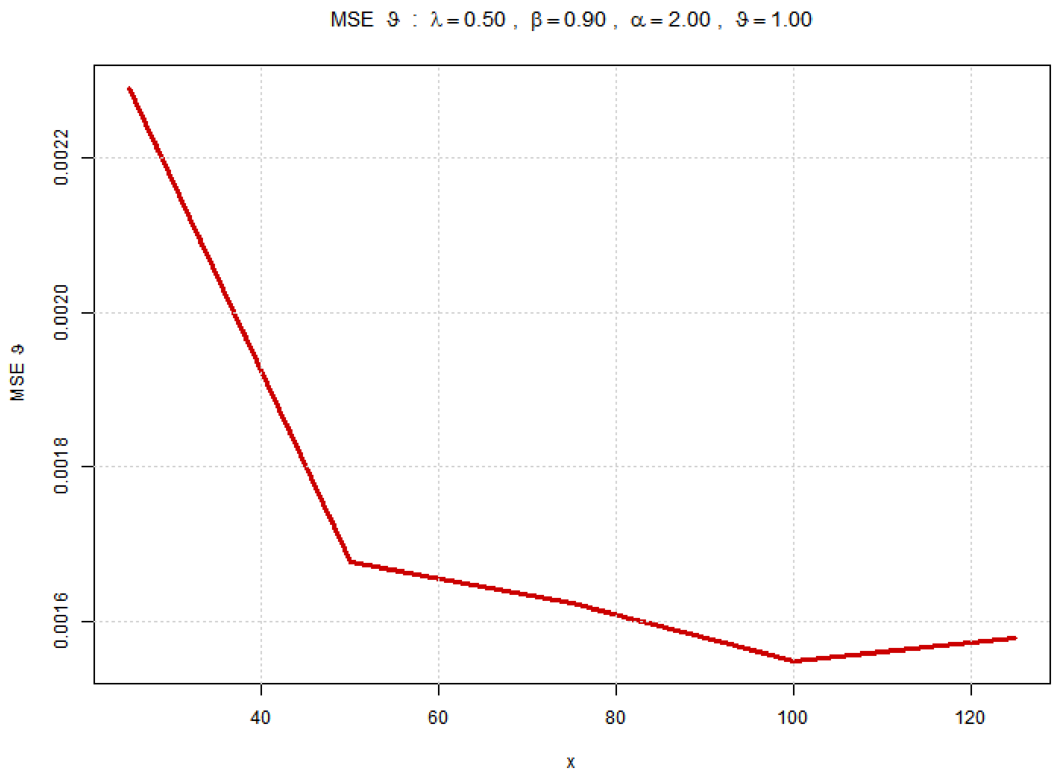

Fig. 19. MSE Vartheta for $\lambda = 0.5, \beta = 0.9, \alpha = 2,$ and $\vartheta = 1$

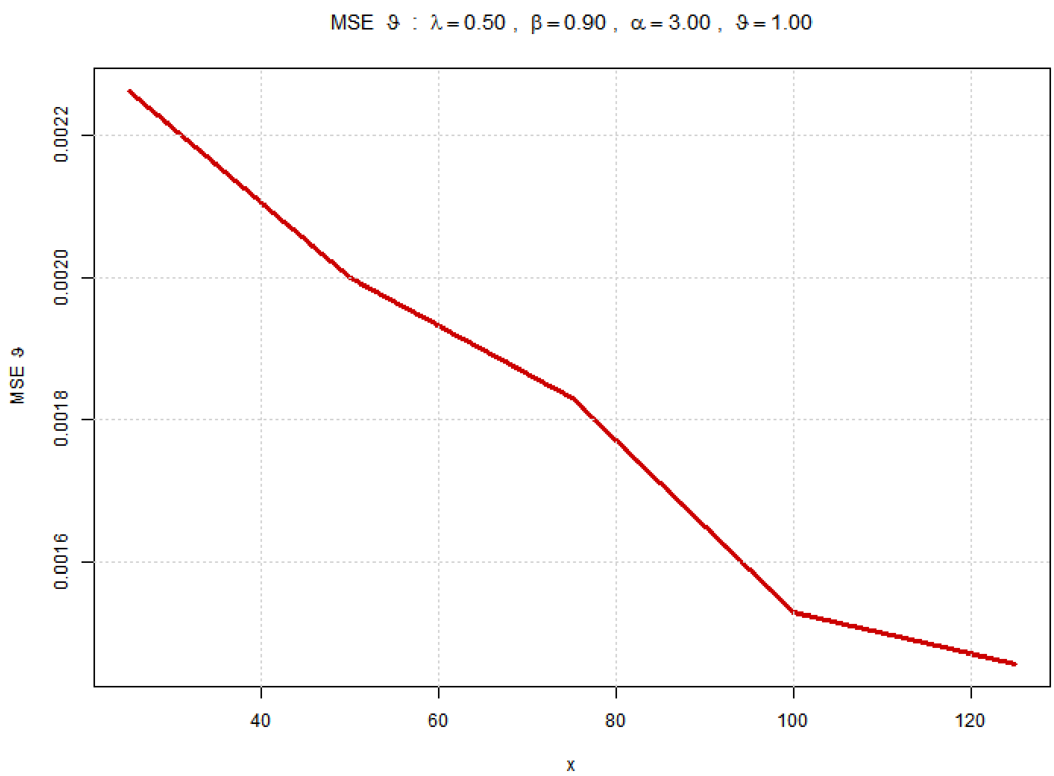

Fig. 20. MSE Vartheta for $\lambda = 0.5, \beta = 0.9, \alpha = 3,$ and $\vartheta = 1$

Figure [5-20] respectively show as the sample size grows, the average bias for the parameter estimators fluctuates upward and downward. The MSE for the parameter estimators, on the other hand, revealed a declining pattern as the sample size increased.

4. Applications

In this section, we provide application to real data sets to demonstration the applicability of the NE-G family. The real-world data set was collected on the breaking stress of carbon fibres with a length of 50 mm (GPa). The data has already been used by [16] and [17]. The following is the data set:

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2,0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66,1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5,1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84,2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62,1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

Table 1. MLEs and Goodness-of-fit measures for real data set

Model	MLE	$-\ell$	W	A	Ks	P-value
NE-U	λ=1.3844747	85.97571	0.08519182	0.518155	0.088342	0.6816
	β= 0.3061201					
	α= 2.6127508					
	ϑ=21.1906288					
Nak-Wei	λ=1.6103057	86.19047	0.09335068	0.5278914	0.10898	0.4133
	β= 1.1390558					
	α=0.2809454					
	ϑ=0.8761429					
GIK-Exp	λ= 4.0944733	89.55029	0.1992844	1.069601	0.12743	0.2342
	β=0.8866774					
	θ=6.6609265					
	α=0.5132424					
Exp.-Wei	λ= 0.5379012	91.99278	0.1818074	0.9721329	0.15906	0.07089
	β=1.8126305					
	α=0.3763266					
	ϑ= 1.8069314					
GOG-Exp	λ= 2.9664896	86.58091	0.1057304	0.6069994	0.094865	0.5925
	β=1.5056397					
	α=0.6131095					
OGG-Wei	λ= 1.9006879	159.4275	0.18397049	0.6766763	0.62739	2.2e-16
	β= 0.4242168					
	α=1.5138327					
	ϑ= 0.7524188					
Exp.	α= 2.9664896	132.9944	0.2462793	1.33359	0.35813	8.883e-08
Bet-Exp	λ=7.6786602	91.4858	0.2527494	1.369781	0.13384	0.1878
	β=7.2017403					
	α=0.2758384					

A summary of the fitted information criteria and MLEs is shown in Table 1. With the minimum of Cramér-von Misses (W), Anderson Darling (A), and Kolmogorov-Smirnov test (Ks) and the maximum value of log-likelihood, the proposed Nakagami exponential-uniform distribution has been sorted. As you can see, all the criteria pointed to the NE-U as the best model. Notably, NE-U's P-Value is higher than that of every other distribution.

5. Conclusion

This study proposes the appropriate NE-G family of distribution for any parent continuous distribution G. Quantile functions, moments, moments-generating functions, incomplete moments, entropy, and order statistics are some of the statistical and mathematical aspects of the new generator studied. The maximum likelihood estimates of model parameters are derived. We finally fitted the proposed NE-U model, among others, to real-life data and found that the Nakagami Exponential Uniform distribution outperformed its competitors. We anticipate that this generalisation will lead to other statistical applications.

Author Contributions

This work was equally contributed by all the writers. The authors read and approved the last version of the paper.

Conflicts of Interest

There are no conflicts of interest declared by the authors.

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