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Research Article

A Polynomial Sequence Generalizing an Integer Sequence Associated with Tribonacci Numbers

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Abstract

Tribonacci polynomial sequence is an extension of Tribonacci numbers. We consider an integer sequence enumerating the number of subsets of S of the set $[n] = \{1, 2, \ldots, n\}$ containing no three consecutive even integers. We define a polynomial sequence generalizing this integer sequence. The polynomial sequence is associated with the Tribonacci polynomials. We find the closed form formula and derive some basic properties of the polynomial sequence.

Keywords: Tribonacci numbers, Tribonacci polynomials, polynomial sequence, consecutive even integers, generating function.

Tribonacci Sayıları ile İlişkili Bir Tamsayı Dizisini Genelleyen Polinom Dizisi

Öz

Tribonacci polinom dizileri Tribonacci sayılarının bir genişlemesidir. $\{1, 2, \ldots, n\}$ kümesinin ardışık üç çift tam sayı içermeyen S alt kümelerinin sayısını veren tam sayı dizisini göz önüne aldık. Bu tamsayı dizisini genelleyen bir polinom dizisi tanımladık. Polinom dizisi Tribonacci polinomları ile ilişkilendirildi. Bu polinom dizisinin kapalı formülünü bulduk ve polinom dizisinin bazı temel özelliklerini elde ettik.

Anahtar Kelimeler: Tribonacci sayıları, Tribonacci polinomları, polinom dizisi, ardışık çift sayılar, üreteç fonksiyon.

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1. Introduction

The Tribonacci polynomials are a polynomial sequence which can be considered as a generalization of the Tribonacci numbers. You can see more about Tribonacci polynomials in [4-9]. The tribonacci polynomial $T_n(x)$ was defined in 1973 by Hoggatt and Bicknell in [3] by the recurrence relation with its initial conditions as follows:

$$T_n(x) = x^2T_{n-1}(x) + xT_{n-2}(x)+T_{n-3}(x), \qquad n \geq 3, \tag{1}$$

$$T_0(x) = 0, \qquad T_1(x) = 1, \qquad T_2(x) = x^2$$

When $x = 1$, we obtain the Tribonacci sequence $(T_n)_{n\geq 0}$. Generating function for Tribonacci polynomial sequence is given in [9] by

$$\sum_{n=0}^{\infty} T_n(x)\, t^n = \frac{t}{1 - x^2t - xt^2 - t^3}, \tag{2}$$

Binet's formula of Tribonacci polynomial is given in [5] by

$$T_n(x) = \frac{\alpha(x)^{n+1}}{(\alpha(x) - \beta(x))(\alpha(x) - \gamma(x))} + \frac{\beta(x)^{n+1}}{(\beta(x) - \alpha(x))(\beta(x) - \gamma(x))} + \frac{\gamma(x)^{n+1}}{(\gamma(x) - \alpha(x)))(\gamma(x) - \beta(x))}, \tag{3}$$

where $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are the distinct roots of $t^3 - x^2t^2 - xt - 1 = 0$ which is the characteristic equation of (1).

Consider the sequence $(a_n)_{n\geq 0}$ counting the number of subsets S of the set [n] = $\{1, 2, \ . \ . \ . \ , n\}$ such that S contains no three consecutive even integers. The sequence $(a_n)_{n\geq 0}$ is studied in detail in [1].

$$a_n = 2a_{n-2} + 4a_{n-4} + 8a_{n-6}, \ \ n \geq 6, \tag{4}$$

$$a_0 = 1, a_1 = 2, \ a_2 = 4, a_3 = 8, a_4 = 16, a_5 = 32.$$

In this paper we first define the polynomial sequence $(a_n(\mathrm{x}))_{n\geq 0}$ using (4) and we obtain some basic properties of the polynomial sequence.

2. Main Results

2.1. Recursive Definition of the Polynomial Sequence

Let's define the polinomial sequence $(a_n(x))$ with the help of the recurrence relation (4) as follows:

$$a_n(x) = 2x^4a_{n-2}(x) + \ 4x^2a_{n-4}(x)+8a_{n-6}(x). \tag{5}$$

The first few polynomials are:

$$a_0(x) = 1$$

$$a_1(x) = 2$$

$$a_2(x) = 4$$

$$a_3(x) = 8$$

$$a_4(x) = 16$$

$$a_5(x) = 32$$

$$a_6(x) = 32x^4 + 16x^2 + 8$$

$$a_7(x) = 64x^4 + 32x^2 + 16$$

$$a_8(x) = 64x^8 + 32x^6 + 16x^4 + 64x^2 + 32$$

Notice that $a_n(1) = a_n$.

2.2. Generating Function and the Closed Form Formula of the Polynomial Sequence

Let's try to find generating function $G(x,t)$ of the polynomial sequence $(a_n(x))$ using the formal power series.

$$G(x,t) = \sum_{n=0}^{\infty} a_n(x)t^n$$

To find $G(x,t)$, multiply both sides of the recurrence relation (5) by t^n and sum over the values of n for which the recurrence is valid, namely, over $n \ \geq \ 6$. We get,

$$\sum_{n\geq 6} a_n(x)t^n = \sum_{n\geq 6} 2x^4a_{n-2}(x)t^n + \sum_{n\geq 6} 4x^2a_{n-4}(x)t^n + \sum_{n\geq 6} 8a_{n-6}(x)t^n \tag{6}$$

Then try to relate these sums to the unknown generating function $G(x,t)$. We have

$$\sum_{n\geq 6} a_n(x)t^n$$

$$= G(x,t) - a_0(x) - a_1(x)t - a_2(x)t^2 - a_3(x)t^3 - a_4(x)t^4 - a_5(x)t^5$$

$$= G(x,t) - 1 - 2t - 4t^2 - 8t^3 - 16t^4 - 32t^5,$$

$$\sum_{n\geq 6} 2x^4a_{n-2}(x)t^n = 2x^4t^2 \sum_{n\geq 6} a_{n-2}(x)t^{n-2}$$

$$= 2x^4t^2\,(G(x,t) - a_0(x) - a_1(x)t - a_2(x)t^2 - a_3(x)t^3)$$

$$= 2x^4t^2\,(G(x,t) - 1 - 2t - 4t^2 - 8t^3),$$

$$\sum_{n\geq 6} 4x^2 a_{n-4}(x)t^n = 4x^2t^4 \sum_{n\geq 6} a_{n-4}(x)t^{n-4}$$

$$= 4x^2t^4\,(G(x,t) - 1 - 2t),$$

$$\sum_{n\geq 6} 8a_{n-6}(x)t^n = 8t^6 \sum_{n\geq 6} a_{n-6}(x)t^{n-6}$$

$$= 8t^6 G(x,t).$$

If we write these results on the two sides of (6), we find that

$$G(x,t) - 1 - 2t - 4t^2 - 8t^3 - 16t^4 - 32t^5$$

$$= 2x^4t^2\,(G(x,t) - 1 - 2t - 4t^2 - 8t^3)$$

$$+4x^2t^4\,(G(x,t) - 1 - 2t) + 8t^6G(x,t),$$

$$G(x,t)\frac{1 + 2t + (4 - 2x^4)t^2 + (8 - 4x^4)t^3}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}$$

$$+\frac{(16 - 4x^2 - 8x^4)t^4 + (32 - 8x^2 - 16x^4)t^5}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}. \qquad (7)$$

Substituting $x = 1$, we get the generating function for the integer sequence $(a_n)_{n\geq 0}$.

Theorem 1. *Let $(a_n(x))$ is the polynomial sequence defined by (5). Then we have*

$$a_{2n}(x) = 2^n[T_{n+1}(x^2) + (2 - x^4)T_n(x^2)$$
$$+(4 - x^2 - 2x^4)T_{n-1}(x^2)],$$

$$a_{2n+1}(x) = 2^{n+1}[T_{n+1}(x^2) + (2 - x^4)T_n(x^2)$$

$$+(4 - x^2 - 2x^4)T_{n-1}(x^2),$$

where $T_n(x)$ is the nth Tribonacci polynomial.

Proof. If $A(x,t)$ is the generating function for even terms of the polinomial sequence $(a_n(x))_{n\geq 0}$ then it is clear that $A(x,t) = \frac{1}{2}(G(x,t) + G(x,-t))$. From (7) we get,

$$A(x,t) = \frac{1 + (2 - x^4)2t^2 + (4 - x^2 - 2x^4)4t^4}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6},$$

$$A(x,t) = \frac{1}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}$$

$$+(2 - x^4)\frac{2t^2}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}$$

$$+(4 - x^2 - 2x^4)\frac{4t^4}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}. \qquad (8)$$

Let's write the generating function of the Tribonacci polynomial sequence with initial conditions $T_0(x) = 0,\ T_1(x) = 1, T_2(x) = x^2$ which is given in (2):

$$t(x,z) = \frac{z}{1 - x^2z - xz^2 - z^3}.$$

Let's indicate the correspondence between a sequence and its generating function with a double-sided arrow as follows:

$$\langle 1, x^2, x^4 + x, \dots \rangle \ \leftrightarrow\ \frac{1}{1 - x^2z - xz^2 - z^3} \qquad (9)$$

$$\langle 0, 1, x^2, x^4 + x, \dots \rangle \ \leftrightarrow\ \frac{z}{1 - x^2z - xz^2 - z^3} \qquad (10)$$

$$\langle 0, 0, 1, x^2, x^4 + x, \dots \rangle \ \leftrightarrow\ \frac{z^2}{1 - x^2z - xz^2 - z^3} \qquad (11)$$

If we right-shift the polynomial sequence in (9) by adding respectively one and two leading zeros, we obtain the polynomial sequences (10) and (11). Hence (9), (10) and (11) are respectively generating functions of the polynomial sequences $(T_{n+1}(x))$, $(T_n(x))$ and $(T_{n-1}(x))$.

Substituting x^2 for x and writing $z = 2t^2$ into (9), (10) and (11). Together with these and using (8) we get the coefficients of t^{2n} which gives the exact formula for the polynomial sequence $(a_{2n}(x))$,

$$a_{2n}(x) = 2^n[T_{n+1}(x^2) + (2 - x^4)T_n(x^2)$$
$$+(4 - x^2 - 2x^4)T_{n-1}(x^2)]$$

where $T_n(x)$ is the Tribonacci polynomial sequence defined by (1).

If $B(x,t)$ is the generating function for odd terms of the polinomial sequence then it is clear that $B(x,t) = \frac{1}{2}(G(x,t) - G(x,-t))$. Using (7) we get,

$$B(x,t) = \frac{2t + (8 - 4x^4)t^3 + (32 - 8x^2 - 16x^4)t^5}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}$$

$$B(x,t) = t[\frac{2}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}$$

$$+(4 - 2x^4)\frac{2t^2}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}$$

$$+(8 - 2x^2 - 4x^4)\ \frac{4t^4}{1 - 2x^4t^2 - 4x^2t^4 - 8t^6}] \qquad (12)$$

Substituting (9), (10) and (11) into the equation (12) we get the coefficients of t^{2n+1} which gives the general term for the polynomial subsequence $(a_{2n+1}(x))$

$$a_{2n+1}(x) = 2^n[2T_{n+1}(x^2) + (4 - 2x^4)T_n(x^2)$$

$$+(8 - 2x^2 - 4x^4)T_{n-1}(x^2)]\,,$$

$$a_{2n+1}(x) = 2^{n+1}[T_{n+1}(x^2) + (2 - x^4)T_n(x^2)$$

$$+(4 - x^2 - 2x^4)T_{n-1}(x^2)].$$

where $T_n(x)$ is the Tribonacci polynomial defined by (1).

The proof is completed.

Notice that,

$$a_{2n}(1) = a_{2n} = 2^n T_{n+2},$$

$$a_{2n+1}(1) = a_{2n+1} = 2^{n+1} T_{n+2}.$$

2.3. The Sum of the First n Terms of the Polynomial Sequence

In [4] the sum of the Tribonacci polynomials is obtained as

$$\sum_{k=0}^{n} T_k\,(x) = \frac{T_{n+2}(x) + (1 - x^2)T_{n+1}(x) + T_n(x) - 1}{x^2 + x}.$$

Theorem 2. *Let $(a_n(x))$ is the polynomial sequence defined by (5) and $T_n(x)$ is the nth Tribonacci polynomial. Then for $n \geq 1$ we have*

$$\sum_{k=0}^{2n} a_k\,(x) = \frac{2^{n+1}A_n(x) + 30x^4 + 12x^2 - 63}{2x^4 + 4x^2 + 7}$$

$$\sum_{k=0}^{2n+1} a_k\,(x) = \frac{2^{n+1}B_n(x) + 30x^4 + 12x^2 - 63}{2x^4 + 4x^2 + 7}$$

where

$$A_n(x) = [(-22x^4 - 8x^2 + 51)T_{n+1}(x^2)$$

$$+(22x^8 + 8x^6 - 55x^4 + 4x^2 + 28)T_n(x^2)$$

$$+(4x^8 + 18x^6 - 16x^4 - 37x^2 + 35)T_{n-1}(x^2)],$$

$$B_n(x) = [(-20x^4 - 4x^2 + 58)T_{n+1}(x^2)$$

$$+(20x^8 + 4x^6 - 58x^4 + 12x^2 + 42)T_n(x^2)$$

$$+(8x^6 - 26x^4 - 28x^2 + 63)T_{n-1}(x^2)].$$

Proof. Let $(S_n(x))_{n\geq 0}$ be the sum of first n terms of the polynomial sequence $(a_n(x))$:

$$S_n(x) = \sum_{k=0}^{n} a_k\,(x)$$

Using recurrence relation (5) and its initial conditions we have

$$a_n(x) = 2x^4 a_{n-2}(x) + 4x^2 a_{n-4}(x) + 8a_{n-6}(x),$$

$$a_0(x) = 1,\ a_1(x) = 2,\ a_2(x) = 4,\ a_3(x) = 8,$$
$$a_4(x) = 16,\ a_5(x) = 32.$$

For $n > 5$, we can write the following equalities:

$$a_6(x) = 2x^4 a_4(x) + 4x^2 a_2(x) + 8a_0(x)$$

$$a_7(x) = 2x^4 a_5(x) + 4x^2 a_3(x) + 8a_1(x)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_n(x) = 2x^4 a_{n-2}(x) + 4x^2 a_{n-4}(x) + 8a_{n-6}(x)$$

Adding all these equations term by term and substituting initial values we have

$$S_n(x) = \frac{(2x^4 + 4x^2 + 8)[a_n(x) + a_{n-1}(x)]}{2x^4 + 4x^2 + 7}$$

$$+\frac{(4x^2 + 8)\,[a_{n-2}(x) + a_{n-3}(x)]}{2x^4 + 4x^2 + 7}$$

$$+\frac{8\,[a_{n-4}(x) + a_{n-5}(x)] + 30x^4 + 12x^2 - 63}{2x^4 + 4x^2 + 7}.$$

Let's write $S_{2n}(x)$:

$$S_{2n}(x) = \frac{(2x^4 + 4x^2 + 8)[a_{2n}(x) + a_{2n-1}(x)]}{2x^4 + 4x^2 + 7}$$

$$+\frac{(4x^2 + 8)\,[a_{2n-2}(x) + a_{2n-3}(x)]}{2x^4 + 4x^2 + 7}$$

$$+\frac{8\,[a_{2n-4}(x) + a_{2n-5}(x)]}{2x^4 + 4x^2 + 7}$$

$$+\frac{30x^4 + 12x^2 - 63}{2x^4 + 4x^2 + 7}\,. \qquad (13)$$

Using (13) and Theorem 1 we have

$$S_{2n}(x) = \frac{2^{n+1}A_n(x) + 30x^4 + 12x^2 - 63}{2x^4 + 4x^2 + 7},$$

where

$$A_n(x) = [(x^4 + 2x^2 + 4)T_{n+1}(x^2)$$

$$+(-x^8 - 2x^6 - x^4 + 7x^2 + 14)T_n(x^2)$$

$$+(-3x^8 - 8x^6 - 10x^4 + 11x^2 + 31)T_{n-1}(x^2)$$

$$+(-2x^8-8x^6-22x^4+4x^2+47)T_{n-2}(x^2)$$

$$+(-2x^6-8x^4+x^2+14)T_{n-3}(x^2)$$

$$+(-2x^4-x^2+4)T_{n-4}(x^2).$$

Since $T_n(x)=x^2T_{n-1}(x)+xT_{n-2}(x)+T_{n-3}(x)$, using this fact we have

$$\sum_{k=0}^{2n} a_k(x)=\frac{2^{n+1}A_n(x)+30x^4+12x^2-63}{2x^4+4x^2+7}, \qquad (14)$$

where

$$A_n(x)=[(-22x^4-8x^2+51)T_{n+1}(x^2)$$

$$+(22x^8+8x^6-55x^4+4x^2+28)T_n(x^2)$$

$$+(4x^8+18x^6-16x^4-37x^2+35)T_{n-1}(x^2)].$$

Let's find the following summation formula for $n\geq 1$,

$$\sum_{k=0}^{2n+1} a_k(x).$$

$$\sum_{k=0}^{2n+1} a_k(x)=\sum_{k=0}^{2n} a_k(x)+a_{2n+1}(x)$$

From Theorem 1 and (14) we have

$$\sum_{k=0}^{2n+1} a_k(x)=\frac{2^{n+1}B_n(x)+30x^4+12x^2-63}{2x^4+4x^2+7}, \qquad (15)$$

where

$$B_n(x)=[(-20x^4-4x^2+58)T_{n+1}(x^2)$$

$$+(20x^8+4x^6-58x^4+12x^2+42)T_n(x^2)$$

$$+(8x^6-26x^4-28x^2+63)T_{n-1}(x^2)].$$

The proof is completed.

Corollary 1. *Let $(a_n(x))$ is the polynomial sequence defined by (5) and $T_n(x)$ is the nth Tribonacci polynomial. For $n\geq 1$ we have*

$$\sum_{k=0}^{n} a_{2k}(x)=\frac{2^{n+1}B_n(x)+30x^4+12x^2-63}{3(2x^4+4x^2+7)},$$

$$\sum_{k=0}^{n} a_{2k+1}(x)=\frac{2^{n+2}B_n(x)+60x^4+24x^2-126}{3(2x^4+4x^2+7)},$$

where

$$B_n(x)=[(-20x^4-4x^2+58)T_{n+1}(x^2)$$

$$+(20x^8+4x^6-58x^4+12x^2+42)T_n(x^2)$$

$$+(8x^6-26x^4-28x^2+63)T_{n-1}(x^2)].$$

Proof. As a consequence of Theorem 1, for every $n\geq 0$ we have

$$a_{2n+1}(x)=2a_{2n}(x).$$

Hence, it is an immediate consequence of Theorem 2 and this fact.

2.4. Limit of the Ratio of Consecutive Terns of the Polynomial Sequence

The limit of the ratio of consecutive terms of the Tribonacci numbers is given in [2] as follows:

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\alpha \qquad (16)$$

where

$$\alpha=\frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}.$$

Corollary 2. *Let $(a_n(x))$ is the polynomial sequence defined by (5) and $\alpha(x)$ is the real root of characteristic equation of (1). Then we have*

$$\lim_{n\to\infty}\frac{a_{2n+1}(x)}{a_{2n}(x)}=2, \qquad (17)$$

$$\lim_{n\to\infty}\frac{a_{2n}(x)}{a_{2n-1}(x)}=\alpha(x^2). \qquad (18)$$

Proof. (17) is an immediate consequence of Theorem 1. We can easily obtain (18) using Theorem 1 and the Binet's formula of Tribonacci polynomial given in (3).

Taking $x=1$, we have

$$\lim_{n\to\infty}\frac{a_{2n+1}}{a_{2n}}=2, \qquad (19)$$

$$\lim_{n\to\infty}\frac{a_{2n}}{a_{2n-1}}=\alpha. \qquad (20)$$

You can also see [1] for the limits given in (19) and (20).

3. Conclusions

In this paper, we define a polynomial sequence $(a_n(x))$ which is a generalization of the integer sequence (a_n) given in [1]. The polynomial sequence is associated with the Tribonacci polynomials and we get some properties of the polynomial sequence.

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