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Some novel analysis of two different Caputo-type fractional-order boundary value problems

Zouaoui Bekri[a], Vedat Suat Erturk[b], Pushpendra Kumar[c], Venkatesan Govindaraj[d]

[a]*Laboratory of Fundamental and Applied Mathematics, University of Oran 1, Ahmed Ben Bella, Es-Senia, Oran-31000, Algeria. Department of Sciences and Technology, Institute of Sciences, Nour Bashir University Center, El-Bayadh-32000, Algeria.*

[b]*Department of Mathematics, Ondokuz Mayis University, Atakum-55200, Samsun, Turkey.*

[c]*Department of Mathematics, National Institute of Technology Puducherry, Karaikal-609609, India.*

[d] *Department of Mathematics, National Institute of Technology Puducherry, Karaikal-609609, India.*

Abstract

Nowadays, several classical order results are being analyzed in the sense of fractional derivatives. In this research work, we discuss two different boundary value problems. In the first half of the paper, we generalize an integer-order boundary value problem into fractional-order and then we demonstrate the existence and uniqueness of the solution subject to the Caputo fractional derivative. First, we recall some results and then justify our main results with the proofs of the given theorems. We conclude our results by presenting an illustrative example. In the other half of the paper, we extend Banach's contraction theorem to prove the existence and uniqueness of the solution to a sequential Caputo fractional-order boundary value problem.

Keywords: Caputo fractional derivative Existence and uniqueness Boundary value problem
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1. Problem-1: Existence of a unique solution for a Caputo-type fractional boundary value problem

Fractional differential equations (FDEs) are the equations that consist of fractional-order values on their differential operators. Currently, fractional differential equations are very useful to describe or define several scientific and engineering problems. Here, we are trying to add some new analysis to the literature on FDEs. In this regard, the following result is taken from one theorem proved in [1], which is also stated without proof in [2] (Theorem 3.3), as well as left as an unsolved problem in [3] (Problem [41.6]).

Email addresses: zouaouibekri@yahoo.fr (Zouaoui Bekri), vserturk@omu.edu.tr (Vedat Suat Erturk), kumarsaraswatpk@gmail.com (Pushpendra Kumar), govindaraj.maths@gmail.com (Venkatesan Govindaraj)

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Theorem 1.1. *([1]) Let us consider $h : [c,d] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous and accepts a uniform Lipschitz condition respectively v and v'*

$$|h(t,v,v') - h(t,w,w')| \leq M|v - w| + N|v' - w'|,$$

for (t,v,v'), $(t,w,w') \in [c,d] \times \mathbb{R}^2$, where $N \geq 0$, $M > 0$ are constants. If

$$M\frac{(d-c)^2}{8} + N\frac{(d-c)}{2} < 1,$$

then the boundary value problem (BVP)

$$v'' = -h(t,v,v'), \quad v(c) = U, \quad v(d) = V,$$

has a unique solution.

Here we try to generalize the above-mentioned result in the sense of a fractional-order Caputo-type BVP (we suggest the readers to see [4] for the fundamental literature along with necessary results and definitions on fractional calculus) at the place of the classic operator v'', that is, we give the proof of the existence of unique solutions to the adopted fractional-order BVP

$$^{C}D_c^{\nu}v(t) = -h(t, v(t), {}^{C}D_c^{\theta}v(t)), \quad c < t < d \tag{1}$$

$$v(c) = U, \qquad v(d) = V. \tag{2}$$

Here, $1 < \nu \leq 2$, $0 < \theta \leq 1$. To date, a number of results related to the existence and uniqueness of the solution to the initial and boundary value problems in fractional-order sense have been studied by many researchers (see, for example, [[5], [6], [7], [8], [9], [10], [11]] and the quotations mentioned therein). At our discretion, our results are the first to report the fractional counterpart of the problem given in Theorem 1.1. In the formulation of this section, we present subsection 1.1 to pose some basic notions from the theory of fractional calculus. We provide subsection 1.2 to simulate our main results where we give important lemma, proposition and a theorem. We solve an example in subsection 1.3. A conclusion completes the research.

1.1. Basic notions

Firstly we present some background materials from the theory of fractional calculus to study the boundary value problems, see [4].

Definition 1.2. *For j^{th} continuously differentiable mapping $v : [0,\infty] \longrightarrow \mathbb{R}$, the Caputo fractional derivative of order ν is specified as*

$$^{C}D_c^{\nu}v(\tau) = \frac{1}{\Gamma(j-\nu)}\int_c^{\tau}(\tau-\varpi)^{j-\nu-1}v^{(j)}(\varpi)d\varpi, \quad j-1 < \nu \leq j, \quad j = [\nu]+1,$$

where $[\nu]$ shows the integer-part of the real number ν.

Definition 1.3. *For a continuous function v, the Riemann-Liouville factional integral of order ν is described by*

$$I_c^{\nu}v(\tau) = \frac{1}{\Gamma(\nu)}\int_c^{\tau}(\tau-\varpi)^{\nu-1}v(\varpi)d\varpi, \quad \nu > 0.$$

Definition 1.4. *For ν, the fractional differential equation $^{c}D_c^{\nu}v(\tau) = 0$ has a general solution is defined in the following form*

$$v(\tau) = m_0 + m_1\tau + m_2\tau^2 + ... + m_{j-1}\tau^{j-1},$$

For some $m_i \in \mathbb{R}$, $i = 0,1,...,j-1 (j = [\nu]+1)$.

1.2. Main Results

We give the integral formula for the BVP (1)-(2) in terms of the green function.

Lemma 1.5. *Let us assume that h is a continuous mapping (or function). A mapping $v \in C^1[c,d]$ satisfies* (1)-(2) *if and only if v satisfies the integral equation*

$$v(t) = \frac{(V-U)(t-c)+U(d-c)}{(d-c)} + \int_c^d \mathcal{A}(t,s)h(s,v(s),{}^C D_c^{\theta} v(s))ds$$

where

$$\mathcal{A}(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{(t-c)(d-s)^{\nu-1}}{(d-c)} - (t-s)^{\nu-1}, & c \le s \le t \le d, \\ \frac{(t-c)(d-s)^{\nu-1}}{(d-c)}, & c \le t \le s \le d. \end{cases}$$

Proof. We have

$${}^C D_c^{\nu} v(t) = -h(t),$$

$$I_c^{\nu}\, {}^C D_c^{\nu} v(t) = -I_c^{\nu} h(t) + c_0 + c_1 t, \quad c_0, c_1 \ \text{are constants}.$$

Then,

$$v(t) = -I_c^{\nu} h(t) + c_0 + c_1 t.$$

using the boundary conditions

$$v(c) = U, \ \ i.e \ \ c_0 + cc_1 = U,$$

$$v(d) = V, \ \ i.e \ \ -I_c^{\nu} h(d) + c_0 + c_1 d = V.$$

So, we have a system

$$c_0 + cc_1 = U \longrightarrow (i)$$

$$c_0 + c_1 d = V + I_c^{\nu} h(d) \longrightarrow (ii)$$

i.e

$$(i) \Longrightarrow c_0 = U - cc_1,$$

by putting it in (ii), we get

$$U - cc_1 + c_1 d = V + I_c^{\nu} h(d),$$

$$c_1(d-c) = V - U + I_c^{\nu} h(d),$$

$$c_1 = \frac{1}{(d-c)} \left((V-U) + I_c^{\nu} h(d)\right),$$

and

$$c_0 = U - cc_1 = U - c\left[\frac{1}{(d-c)}((V-U) + I_c^{\nu} h(d))\right],$$

$$c_0 = \frac{U(d-c) - c(V-U)}{(d-c)} - \frac{c}{(d-c)} I_c^{\nu} h(d).$$

By putting the above value in equation $v(t)$, we get

$$v(t) = -I_c^{\nu} h(t) + \frac{U(d-c) - c(V-U)}{(d-c)} - \frac{c}{(d-c)} I_c^{\nu} h(d) + t\left[\frac{1}{(d-c)}((V-U) + I_c^{\nu} h(d))\right],$$

$$v(t) = -I_c^{\nu} h(t) + \frac{(t-c)}{(d-c)} I_c^{\nu} h(d) + \frac{(t-c)(V-U) + U(d-c)}{(d-c)},$$

$$v(t) = -\frac{1}{\Gamma(\nu)} \int_c^t (t-s)^{\nu-1} h(s) ds + \frac{(t-c)}{\Gamma(\nu)(d-c)} \int_c^d (d-s)^{\nu-1} h(s) ds$$

$$+\frac{(t-c)(V-U)+U(d-c)}{(d-c)},$$

$$v(t) = -\frac{1}{\Gamma(\nu)}\int_c^t (t-s)^{\nu-1}h(s)ds + \frac{(t-c)}{\Gamma(\nu)(d-c)}\int_c^t (d-s)^{\nu-1}h(s)ds$$

$$+\frac{(t-c)}{\Gamma(\nu)(d-c)}\int_t^d (d-s)^{\nu-1}h(s)ds + \frac{(V-U)(t-c)+U(d-c)}{(d-c)}.$$

Therefore,

$$v(t) = \frac{(V-U)(t-c)+U(d-c)}{(d-c)} + \frac{1}{\Gamma(\nu)}\int_c^t \left(\frac{(t-c)(d-s)^{\nu-1}}{(d-c)} - (t-s)^{\nu-1}\right)h(s)ds$$

$$+\frac{1}{\Gamma(\nu)}\int_t^d \frac{(t-c)(d-s)^{\nu-1}}{(d-c)}h(s)ds,$$

and the proof is finish. □

Now we establish the following result which is necessary to prove our main results.

Proposition 1.6. *Consider $\mathcal{A}$ be the Green function mentioned in Lemma 1.5. Then,*

$$\int_c^d |\mathcal{A}(t,s)|ds \leq \frac{1}{\Gamma(\nu)}\left(\frac{1}{\nu^{\nu/(\nu-1)}} - \frac{1}{\nu^{(2\nu-1)/(\nu-1)}}\right)(d-c)^{\nu},$$

and

$$\int_c^d \frac{|\mathcal{A}(t,s)|}{\partial t}ds \leq \frac{1}{\Gamma(\nu)}\frac{(d-c)^{\nu}}{\nu}.$$

Proof. We determine $\int_c^d |\mathcal{A}(t,s)|ds$, where $\mathcal{A}(t,s) \geq 0$ for all $c \leq t, \quad s \leq d$. Therefore,

$$\int_c^d |\mathcal{A}(t,s)|ds = \frac{1}{\Gamma(\nu)}\left[\int_c^t \left(\frac{(t-c)(d-s)^{\nu-1}}{d-c} - (t-s)^{\nu-1}\right)ds + \int_t^d \frac{(t-c)(d-s)^{\nu-1}}{d-c}ds\right],$$

we calculate the primitives

$$\int_c^d |\mathcal{A}(t,s)|ds = \frac{(t-c)}{\Gamma(\nu)(d-c)}\left[-\frac{1}{\nu}(d-s)^{\nu}\right]_c^t + \frac{1}{\Gamma(\nu)}\left[\frac{1}{\nu}(t-s)^{\nu}\right]_c^t$$

$$+\frac{(t-c)}{\Gamma(\nu)(d-c)}\left[-\frac{1}{\nu}(d-s)^{\nu}\right]_t^d,$$

then,

$$\int_c^d |\mathcal{A}(t,s)|ds = \frac{1}{\Gamma(\nu)}\left[\frac{(t-c)(d-c)^{\nu-1}}{\nu} - \frac{(t-c)^{\nu}}{\nu}\right],$$

implies that

$$\int_c^d \frac{|\mathcal{A}(t,s)|}{\partial t}ds = \frac{1}{\Gamma(\nu)}\left[\frac{(d-c)^{\nu-1}}{\nu} - (t-c)^{\nu-1}\right].$$

Now we define $l : [c,d] \longrightarrow \mathbb{R}$ by

$$l(t) = \frac{(t-c)(d-c)^{\nu-1}}{\nu} - \frac{(t-c)^{\nu}}{\nu},$$

and we also define $l' : [c,d] \longrightarrow \mathbb{R}$ by

$$l'(t) = \frac{(d-c)^{\nu-1}}{\nu} - (t-c)^{\nu-1}.$$

Differentiating the functions l and l', we sharply find that their maximum are achieved at the points

$$t^* = \frac{(d-c)}{\nu^{1/(\nu-1)}} + c,$$

and

$$t_1^* = c.$$

Moreover,

$$l(t^*) = \left(\frac{1}{\nu^{\nu/(\nu-1)}} - \frac{1}{\nu^{(2\nu-1)/(\nu-1)}} \right) (d-c)^{\nu},$$

and

$$l'(t_1^*) = \frac{(d-c)^{\nu-1}}{\nu},$$

that finishes the proof. □

Theorem 1.7. *Let $h : [c,d] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous mapping and agrees to a uniform Lipschitz constraint according to the second variable on $[c,d] \times \mathbb{R}^2$ with Lipschitz constant M, that is,*

$$|h(t, v, v') - h(t, w, w')| \leq M|v - w| + N|v' - w'|,$$

for (t, v, v'), $(t, w, w') \in [c,d] \times \mathbb{R}^2$, where $N \geq 0$, $M > 0$ are constants. If

$$\frac{M}{\Gamma(\nu)} \left(\frac{1}{\nu^{\nu/(\nu-1)}} - \frac{1}{\nu^{(2\nu-1)/(\nu-1)}} \right) (d-c)^{\nu} + \frac{N}{\Gamma(\nu)} \frac{(d-c)^{\nu-1}}{\nu} < 1, \tag{3}$$

then the BVP

$${}^{C}D_c^{\nu} v(t) = -h(t, v(t), {}^{C}D_c^{\theta} v(t)), \quad c < t < d \tag{4}$$

$$v(c) = U, \qquad v(d) = V, \tag{5}$$

exists a unique solution.

Proof. Assume Ω is the Banach space of continuous mappings derived on $[c,d]$ with the norm

$$\|v\| = \max_{t \in [c,d]} \{M|v(t)| + N|v'(t)|\}.$$

By Lemma 1.5, $v \in C^1[c,d]$ is a solution of (4)-(5) if and only if it solves the integral equation

$$v(t) = \frac{(V-U)(t-c) + U(d-c)}{(d-c)} + \int_c^d \mathcal{A}(t,s) h(s, v(s), {}^{c}D_c^{\theta} v(s)) ds.$$

Define the operator $F : \Omega \longrightarrow \Omega$ by

$$Fv(t) = \frac{(V-U)(t-c) + U(d-c)}{(d-c)} + \int_c^d \mathcal{A}(t,s) h(s, v(s), {}^{C}D_c^{\theta} v(s)) ds,$$

for $t \in [c,d]$. Here we will justify that the operator F exists a unique fixed-point. Let $v, w \in \Omega$. Then,

$$\begin{aligned} M|Fv(t) - Fw(t)| &\leq M \int_c^d |\mathcal{A}(t,s)| \left| h(s, v(s), {}^{C}D_c^{\theta} v(s)) - h(s, w(s), {}^{C}D_c^{\theta} w(s)) \right| ds \\ &\leq M \int_c^d |\mathcal{A}(t,s)| \left(M|v(t) - w(t)| + N|v'(t) - w'(t)| \right) ds \\ &\leq M \int_c^d |\mathcal{A}(t,s)| ds \|v - w\| \end{aligned}$$

$$\leq M \frac{1}{\Gamma(\nu)} \left(\frac{1}{\nu^{\nu/(\nu-1)}} - \frac{1}{\nu^{(2\nu-1)/(\nu-1)}} \right) (d-c)^{\nu} \|v - w\|,$$

for $t \in [c, d]$, then similarly

$$N|(Fv)'(t) - (Fw)'(t)| \leq N \int_c^d \frac{|\mathcal{A}(t,s)|}{\partial t} \left| h(s, v(s), {}^C D_c^{\theta} v(s)) - h(s, w(s), {}^C D_c^{\theta} w(s)) \right| ds$$

$$\leq N \int_c^d \frac{|\mathcal{A}(t,s)|}{\partial t} \left(M|v(t) - w(t)| + N|v'(t) - w'(t)| \right) ds$$

$$\leq N \int_c^d \frac{|\mathcal{A}(t,s)|}{\partial t} ds \|v - w\|$$

$$\leq N \frac{1}{\Gamma(\nu)} \frac{(d-c)^{\nu}}{\nu} \|v - w\|,$$

for $t \in [c, d]$, we have

$$\|Fv - Fw\| \leq \delta \|v - w\|,$$

where

$$\delta := \frac{M}{\Gamma(\nu)} \left(\frac{1}{\nu^{\nu/(\nu-1)}} - \frac{1}{\nu^{(2\nu-1)/(\nu-1)}} \right) (d-c)^{\nu} + \frac{N}{\Gamma(\nu)} \frac{(d-c)^{\nu-1}}{\nu} < 1,$$

here we have applied Proposition 1.6. By Eq. (3), we get that F is a contracting mapping on Ω, according the Banach contraction mapping theorem, we conclude the necessary result i.e. we receive that F exists a unique fixed-point in $C^1[c, d]$. Which implies that the BVP (4)-(5) has a unique solution. □

Remark 1.8. *We notice that when $\nu = 2$ and $\theta = 1$ in Theorem 1.7, in condition* (3)*, we clearly obtain Theorem 1.1.*

1.3. Example

We consider the BVP

$${}^C D_0^{3/2} v(t) = 1 - t^2 - \cos(v(t)) + \sin({}^C D_0^1 v(t)), \quad 0 < t < 1 \tag{6}$$

$$v(0) = 1, \qquad v(1) = 2. \tag{7}$$

Set,

$$\nu = 3/2, \ \theta = 1, \ c = 0 \ \textit{and} \ d = 1$$

Here,

$$h(t, v(t), {}^C D_0^1 v(t)) = 1 - t^2 - \cos(v(t)) + \sin({}^C D_0^1 v(t)),$$

and, therefore,

$$|h(t, v, v') - h(t, w, w')| \leq |\cos(v(t)) - \cos(w(t))| + |\sin({}^C D_0^1 v(t)) - \sin({}^C D_0^1 w(t))|,$$

$$\leq M|v - w| + N|v' - w'|.$$

Moreover, we have

$$\frac{M}{\Gamma(\nu)} \left(\frac{1}{\nu^{\nu/(\nu-1)}} - \frac{1}{\nu^{(2\nu-1)/(\nu-1)}} \right) (b-a)^{\nu} + \frac{N}{\Gamma(\nu)} \frac{(b-a)^{\nu-1}}{\nu} \approx 0.863698 < 1,$$

and therefore (3) is agreed. Now by the applications of Theorem 1.7, we prove that (6)-(7) has a unique solution.

Conclusion

In this section, we have generalized a classical-order boundary value problem into a fractional-order problem. We have demonstrated the existence of the unique solution, subject to the Caputo fractional derivative. We recalled some important results and then proved our main simulations with the support of the given theorems. We have analyzed the correctness of our results by solving an illustrative example.

2. Problem-2: Solvability and uniqueness results for the sequential Caputo fractional double-derivative boundary value problem

In recent years, numerous studies have been produced about the fractional boundary value problems and especially on Caputo fractional derivative type boundary value problems. Different scientific and technological phenomena related to many domains like engineering, biology, physics, chemistry, image processing, signal analysis, and control theory have been studied in the sense of fractional boundary value problems, for the reading, see in ref. [12, 13, 14, 15, 16].

Also, in the last few years, new problems have emerged about sequential fractional derivative boundary value problems related to sequential Riemann-Liouville and Caputo fractional derivatives, see in [17, 18, 19]. We can find some definitions and concepts about the sequential fractional derivative in ref. [20, 21]. Several authors have gone to study these issues with different mathematical methodologies and in a variety of ways. Some of them used Banach, Leray-Schauder, Krasnoselskii fixed point theorems, and the Banach contraction mapping principle. Several researchers worked by applying the Krein-Rutman theorem, fixed point theorem of Darbo-type, and Lyapunov-type inequality, to know more, see the studies [22, 23, 24, 25, 26, 27] and references therein.

In this portion of the study, we consider the following Sequential fractional boundary value problem

$$({}^{C}_{\zeta}D^{\gamma}\ {}^{C}_{\zeta}D^{\delta}\chi)(\tau) + p(\tau)\chi(\tau) = 0, \quad \zeta < \tau < \eta, \tag{8}$$

$$\chi(\zeta) = \chi(\eta) = 0, \tag{9}$$

here $\gamma > 0,\ \ \delta \leq 1,\ 1 < \gamma + \delta \leq 2,\ p \in C[\zeta, \eta],\ \chi$ is a continuous function and ${}^{C}_{\zeta}D$ represents the Caputo fractional derivative. Our focus is on the generalisation of the following result mentioned in [28].

Theorem 2.1. *Suppose that* $\Psi : [\zeta, \eta] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous function which satisfies a uniform Lipschitz property for the second variable on* $[\zeta, \eta] \times \mathbb{R}$ *and let Lipschitz constant* $\xi > 0$*, that is*

$$\mid \Psi(\tau, \chi) - \Psi(\tau, \varphi) \mid \leq \xi \mid \chi - \varphi \mid,$$

$\forall\ (\tau, \chi), (\tau, \varphi) \in [\zeta, \eta] \times \mathbb{R}$*. If*

$$\eta - \zeta < \left(\frac{\Gamma(\gamma + \delta + 1)(\gamma + \delta)^{\frac{\gamma+\delta+1}{\gamma+\delta-1}}}{\xi \left((\gamma + \delta)^{\frac{\gamma+\delta}{\gamma+\delta-1}} - (\gamma + \delta)^{\frac{1}{\gamma+\delta-1}} \right)} \right)^{\frac{1}{\gamma+\delta}}, \tag{10}$$

so the problem (8)-(9) *admits a unique solution.*

We arrange the sections of the article as follows: In Section 2.1, we remember the result. Section 2.2 contains the singularity result, while at the end a conclusion that summarizes our findings.

2.1. Basic notions

Proposition 2.2. *([4, 29]) The general solution* v *of the following fractional differential equation*

$${}^{C}D^{\nu}_{\zeta}v(\tau) = \Psi(\tau), \ \ \tau \geq \zeta, \ \ 0 < \nu \leq 1,$$

is $v(\tau) = m + ({}_{\zeta}I^{\nu}\Psi)(\tau), \ \ m \in \mathbb{R}.$

2.2. Main Results

We consider now the BVP (8)-(9) with $({}^C_\zeta D^\gamma\ {}^C_\zeta D^\delta \chi)(\tau) = ({}^C_\zeta D^{\gamma+\delta}\chi)(\tau)$ does not take in general (see [30]),

Lemma 2.3. *Let* $0 < \gamma,\ \ \delta > 1$ *be such that* $1 < \gamma + \delta \leq 2$ *and* $p \in C[\zeta, \eta]$ *for some* $\zeta < \eta$*. Then* $\chi \in C[\zeta, \eta]$ *is a solution of the fractional boundary value problem* (8)-(9)*, if and only if, it satisfies the integral equation*

$$\chi(\tau) = \int_\zeta^\eta \hbar(\tau, s)\ p(s)\ \chi(s) ds, \tag{11}$$

where

$$\hbar(\tau, s) = \frac{1}{\Gamma(\gamma+\delta)} \begin{cases} \frac{(\eta-s)^{\gamma+\delta-1}(\tau-\zeta)^\delta}{(\eta-\zeta)^\delta} - (\tau - s)^{\gamma+\delta-1}, & \zeta \leq s \leq \tau \leq \eta, \\ \frac{(\eta-s)^{\gamma+\delta-1}(\tau-\zeta)^\delta}{(\eta-\zeta)^\delta}, & \zeta \leq \tau \leq s \leq \eta. \end{cases}$$

Proof. The guide is merely an iterative application of Proposition 2.2 with the uses of boundary conditions. We have

$$({}^C_\zeta D^\gamma\ {}^C_\zeta D^\delta \chi)(\tau) = -p(\tau)\chi(\tau),$$

$$({}^C_\zeta I^\gamma\ {}^C_\zeta D^\gamma\ {}^C_\zeta D^\delta \chi)(\tau) = -{}^C_\zeta I^\gamma p(\tau)\chi(\tau).$$

So,

$$({}^C_\zeta D^\delta \chi)(\tau) = -{}^C_\zeta I^\gamma p(\tau)\chi(\tau) + m_0 + m_1\tau, \quad m_0, m_1 \ \textit{are constants},$$

we repeat the integration

$$({}^C_\zeta I^\delta\ {}^C_\zeta D^\delta \chi)(\tau) = -{}^C_\zeta I^\gamma p(\tau)\chi(\tau) + m_0 + m_1\tau.$$

We get

$$\chi(\tau) = -{}^C_\zeta I^\gamma\ {}^C_\zeta I^\delta p(\tau)\chi(\tau) + m_0 + m_1(\tau - \zeta)^\delta$$

i.e.

$$\chi(\tau) = -{}^C_\zeta I^{\gamma+\delta} p(\tau)\chi(\tau) + m_0 + m_1(\tau - \zeta)^\delta,$$

using the conditions (9)

$$\chi(\zeta) = 0 \Longrightarrow m_0 = 0,$$

and

$$\chi(\eta) = 0 \Longrightarrow -{}^C_\zeta I^{\gamma+\delta} p(\eta)\chi(\eta) + m_1(\eta - \zeta)^\delta = 0$$

i.e.

$$m_1 = \frac{1}{(\eta - \zeta)^\delta}\ {}^C_\zeta I^{\gamma+\delta} p(\eta)\chi(\eta).$$

So the solution is given by

$$\chi(\tau) = -{}^C_\zeta I^{\gamma+\delta} p(\tau)\chi(\tau) + \frac{(\tau - \zeta)^\delta}{(\eta - \zeta)^\delta}\ {}^C_\zeta I^{\gamma+\delta} p(\eta)\chi(\eta).$$

Therefore,

$$\chi(\tau) = \frac{1}{\Gamma(\gamma+\delta)} \int_\zeta^\tau \left(\frac{(\eta - s)^{\gamma+\delta-1}(\tau - \zeta)^\delta}{(\eta - \zeta)^\delta} - (\tau - s)^{\gamma+\delta-1} \right) p(s)\chi(s) ds$$
$$+ \frac{1}{\Gamma(\gamma+\delta)} \int_\tau^\eta \frac{(\eta - s)^{\gamma+\delta-1}(\tau - \zeta)^\delta}{(\eta - \zeta)^\delta} p(s)\chi(s) ds,$$

and the proof is finish. □

Now, we start with defining the first part of the function h by

$$h(\tau, s) = \frac{(\eta - s)^{\gamma+\delta-1}(\tau - \zeta)^{\delta}}{(\eta - \zeta)^{\delta}} - (\tau - s)^{\gamma+\delta-1},\ \zeta \le s \le \tau \le \eta,$$

for all $\tau \in (\zeta, \eta)$. Then, $h(\tau, s) < 0$ is equivalent to the following formula

$$\frac{(\eta - s)^{\gamma+\delta-1}(\tau - \zeta)^{\delta}}{(\eta - \zeta)^{\delta}} - (\tau - s)^{\gamma+\delta-1} < 0,$$

which is equivalent to

$$\left(\frac{\tau - \zeta}{\eta - \zeta}\right)^{\delta} < \left(\frac{\tau - s}{\eta - s}\right)^{\gamma+\delta-1}.$$

We put some arrangements on the previous inequality, we get

$$\left(\frac{\tau - \zeta}{\eta - \zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} \eta - \tau < s\left[\left(\frac{\tau - \zeta}{\eta - \zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} - 1\right].$$

Since

$$\left(\frac{\tau - \zeta}{\eta - \zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} - 1 < 0,$$

finally we find

$$s < \frac{\left(\frac{\tau-\zeta}{\eta-\zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} \eta - \tau}{\left(\frac{\tau-\zeta}{\eta-\zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} - 1}.$$

We define a function Φ by

$$\Phi(\tau) = \frac{\left(\frac{\tau-\zeta}{\eta-\zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} \eta - \tau}{\left(\frac{\tau-\zeta}{\eta-\zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} - 1}, \quad \tau \in [\zeta, \eta)$$

$$\Phi(\eta) = \lim_{\tau \longrightarrow \eta} \frac{\left(\frac{\tau-\zeta}{\eta-\zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} \eta - \tau}{\left(\frac{\tau-\zeta}{\eta-\zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} - 1} = \eta - \frac{(\gamma + \delta - 1)}{\delta}(\eta - \zeta).$$

Now we show it $\zeta < \Phi(\tau) < \tau$ on (ζ, η), we have

$$\begin{aligned}
\zeta < \frac{\left(\frac{\tau-\zeta}{\eta-\zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} \eta - \tau}{\left(\frac{\tau-\zeta}{\eta-\zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} - 1} &\Longleftrightarrow \left(\frac{\tau - \zeta}{\eta - \zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} \zeta - \zeta < \left(\frac{\tau - \zeta}{\eta - \zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} \eta - \tau \\
&\Longleftrightarrow (\tau - \zeta) > \left(\frac{\tau - \zeta}{\eta - \zeta}\right)^{\frac{\delta}{\gamma+\delta-1}} (\eta - \zeta) \\
&\Longleftrightarrow 1 > \left(\frac{\tau - \zeta}{\eta - \zeta}\right)^{\frac{1-\gamma}{\gamma+\delta-1}} \\
&\Longleftrightarrow \tau < \eta.
\end{aligned}$$

Now we establish the following result which is necessary to prove our main results.

Proposition 2.4. *Consider $\hbar$ be the Green function mentioned in Lemma 2.3 . Then,*

$$\int_{\zeta}^{\eta} |\hbar(\tau,s)|ds = \frac{1}{\Gamma(\gamma+\delta)}\bigg(-2(\tau-\Phi(\tau))^{\gamma+\delta} + 2\frac{(\tau-\zeta)^{\delta}}{(\eta-\zeta)^{\delta}}(\eta-\Phi(\tau))^{\gamma+\delta}$$

$$+(\tau-\zeta)^{\gamma+\delta} - (\tau-\zeta)^{\delta}(\eta-\zeta)^{\gamma}\bigg). \tag{12}$$

Proof. In general, we have to prove that $\hbar(\tau,s) < 0$ for every $\tau \in (\zeta,\eta)$ in which $\zeta \le s \le \Phi(\tau)$. Therefore, to get on the quantity value

$$\max_{\tau\in[\zeta,\eta]} \int_{\zeta}^{\eta} |\hbar(\tau,s)|ds, \tag{13}$$

we need to do the following

$$\int_{\zeta}^{\eta} |\hbar(\tau,s)|ds = \frac{1}{\Gamma(\gamma+\delta)}\bigg\{\int_{\zeta}^{\Phi(\tau)}\bigg[(\tau-s)^{\gamma+\delta-1} - \frac{(\eta-s)^{\gamma+\delta-1}(\tau-\zeta)^{\delta}}{(\eta-\zeta)^{\delta}}\bigg]ds$$

$$+\int_{\Phi(\tau)}^{\tau}\bigg[\frac{(\eta-s)^{\gamma+\delta-1}(\tau-\zeta)^{\delta}}{(\eta-\zeta)^{\delta}} - (\tau-s)^{\gamma+\delta-1}\bigg]ds$$

$$+\int_{\tau}^{\eta}\frac{(\eta-s)^{\gamma+\delta-1}(\tau-\zeta)^{\delta}}{(\eta-\zeta)^{\delta}}\,ds\bigg\}.$$

We calculate the primitives

$$\int_{\zeta}^{\eta} |\hbar(\tau,s)|ds = \frac{1}{\Gamma(\gamma+\delta)}\bigg(-2(\tau-\Phi(\tau))^{\gamma+\delta} + 2\frac{(\tau-\zeta)^{\delta}}{(\eta-\zeta)^{\delta}}(\eta-\Phi(\tau))^{\gamma+\delta}$$

$$+(\tau-\zeta)^{\gamma+\delta} - (\tau-\zeta)^{\delta}(\eta-\zeta)^{\gamma}\bigg).$$

The proof is complete. □

It is easy to notice that the original function from the previous equality has a maximum on (ζ,η), but we couldn't get it in an analytical form. To define we put

$$\Sigma(\gamma+\delta,\zeta,\eta) = \max_{\tau\in[\zeta,\eta]}\frac{1}{\Gamma(\gamma+\delta)}\bigg(-2(\tau-\Phi(\tau))^{\gamma+\delta} + 2\frac{(\tau-\zeta)^{\delta}(\eta-\Phi(\tau))^{\gamma+\delta}}{(\eta-\zeta)^{\delta}}$$

$$+(\tau-\zeta)^{\gamma+\delta} - (\tau-\zeta)^{\delta}(\eta-\zeta)^{\gamma}\bigg).$$

Then we get the following value which checks the exact true value (10) generated by formula (13).

Theorem 2.5. *Suppose that $\Psi : [\zeta,\eta]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function which verify a uniform Lipschitz property for the second variable on $[\zeta,\eta]\times\mathbb{R}$ and let Lipschitz constant be $\xi > 0$, that is*

$$|\Psi(\tau,\chi) - \Psi(\tau,\varphi)| \le \xi\,|\chi-\varphi|,$$

$\forall\ (\tau,\chi),(\tau,\varphi) \in [\zeta,\eta]\times\mathbb{R}$. *If*

$$\Sigma(\gamma+\delta,\zeta,\eta) < \frac{1}{\xi}, \tag{14}$$

therefore the problem (8)-(9) *admits a unique solution.*

Proof. Assume that Υ is a Banach space contains continuous functions defined on $[\zeta, \eta]$ provided by the norm

$$\|\chi\| = \max_{\tau\in[\zeta,\eta]} |\chi(\tau)|.$$

By Lemma 2.3 , $\chi \in C[\zeta, \eta]$ is a solution of (8)-(9) iff it satisfies the integral equation (11)

Define the operator $\Lambda : \Upsilon \longrightarrow \Upsilon$ by

$$\Lambda\chi(\tau) = \int_\zeta^\eta \hbar(\tau, s)\ p(s)\,\chi(s)ds,$$

is equivalence to

$$\Lambda\chi(\tau) = \int_\zeta^\eta \hbar(\tau, s)\ \varpi(s,\chi(s))ds,$$

for $\tau \in [\zeta, \eta]$. We prove that the operator Λ admits a unique point. Suppose that $\chi, \varphi \in \Upsilon$. Then

$$\begin{aligned}|\Lambda\chi(\tau) - \Lambda\varphi(\tau)| &\le \int_\zeta^\eta \hbar(\tau, s)|\varpi(s,\chi(s)) - \varpi(s, \varphi(s))|ds\\ &\le \int_\zeta^\eta \hbar(\tau, s)(\xi|\chi(s) - \varphi(s)|)ds\\ &\le \xi \int_\zeta^\eta \hbar(\tau, s)ds\|\chi - \varphi\|\\ &\le \xi\Sigma(\gamma + \delta, \zeta, \eta)\|\chi - \varphi\|.\end{aligned}$$

In the view of the assumption (10) and by using Proposition 2.4, we have (14). We can conclude that Λ is a contracting mapping on Υ and under the light of the Banach contraction mapping theorem, we get the targeted result i.e. we get that Λ admits a unique fixed point in $C[\zeta, \eta]$, this implies that the problem (8)-(9) admits a unique solution. The proof is finished. □

Corollary 2.6. *Let $\Psi : [\zeta, \eta] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function which verify a uniform Lipschitz property for the second variable on $[\zeta, \eta] \times \mathbb{R}$ and let Lipschitz constant $\xi > 0$, i.e*

$$| \Psi(\tau,\chi) - \Psi(\tau, \varphi) |\le \xi \mid \chi - \varphi \mid,$$

$\forall\ (\tau,\chi), (\tau, \varphi) \in [\zeta, \eta] \times \mathbb{R}$. *If*

$$\eta - \zeta < \frac{2\sqrt{2}}{\sqrt{\xi}}, \tag{15}$$

therefore the system

$$\chi''(\tau) = -\Psi(\tau,\chi(\tau)), \quad \zeta < \tau < \eta$$
$$\chi(\zeta) = c, \quad \chi(\eta) = d, \quad c, d \in \mathbb{R}$$

admits a unique solution.

Proof. Let $0 < \gamma = 1$, $\delta = 1 \le 1$, and $\gamma + \delta = 2$. Then, it is very easy to achieve our proof with the following result

$$\Sigma(2, \zeta, \eta) = \frac{1}{\Gamma(2)} \max_{\tau\in[\zeta,\eta]} (\tau - \zeta)(\eta - \tau) = \frac{(\eta - \zeta)^2}{8}.$$

Therefore, the inequality in (14) is represented by the following

$$\frac{(\eta - \zeta)^2}{8} < \frac{1}{\xi},$$

which is equivalent to the formula (15). We completed the proof. □

Conclusion

In this second half of the study, we have extended Banach's contraction theorem by applying it to the boundary value problem of double-sequential fractional order. We have demonstrated the existence of the unique solution subject to the Caputo fractional derivative of double-sequential fractional order. We have proposed some important concepts and then proved them by employing some basic theories from the literature. We have established a valid analysis of our results along with the novel proofs.

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References

[1] W.G. Kelley and A.C. Peterson, theory of differential equations, Springer, 2010.
[2] P.B. Bailey, L.F.Shampine and P.E. Waltman, Nonlinear two-point boundaryvalue problem,Academic Press, 1968.
[3] R.P. Agarwal and Donal O'Regan, An Introduction to Ordinary Differential Equations, Springer-Verlag, 2008.
[4] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and applications of fractional differential equations Elsevier, 2006.
[5] C.F.Li, X.N.Luo and Y. Zhou, Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations, Computers and Mathematics with Applications, 59(3),1363–1375, 2010.
[6] S. Zhang, Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives, Nonlinear Analysis, 71, 2087–2093, 2009.
[7] T. Trif, Existence of solutions to initial value problems for nonlinear fractional differential equations on the semi-axis, Fractional Calculus and Applied Analysis 16 (3), 595-612, 2013.
[8] Y. Cui, Uniqueness of solution for boundary value problems for fractional differential equations, Applied Mathematics Letters, 51, 48-54, 2016.
[9] Z. Bekri, V.S. Erturk, & P. Kumar, On the existence and uniqueness of a nonlinear q-difference boundary value problem of fractional order. International Journal of Modeling, Simulation, and Scientific Computing, 13(01), 2250011, (2022).
[10] V.S. Erturk, A. Ali, K. Shah, P. Kumar, & T. Abdeljawad, Existence and stability results for nonlocal boundary value problems of fractional order. Boundary Value Problems, 2022(1), 1–15.
[11] P. Kumar, V. Govindaraj, Z.A. Khan, Some novel mathematical results on the existence and uniqueness of generalized Caputo-type initial value problems with delay. AIMS Mathematics, 7(6), 10483–10494.
[12] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon, 1993.
[13] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[14] J. Sabatier, O.P. Agrawal, J.A.T. Machado, Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht, 2007.
[15] V.S. Erturk, A. Ahmadkhanlu, P. Kumar, & V. Govindaraj, Some novel mathematical analysis on a corneal shape model by using Caputo fractional derivative. Optik, 261, 169086,(2022).
[16] V.S. Erturk, A.K. Alomari, P. Kumar, & M. Murillo-Arcila, Analytic Solution for the Strongly Nonlinear Multi-Order Fractional Version of a BVP Occurring in Chemical Reactor Theory. Discrete Dynamics in Nature and Society, 2022.
[17] M. Klimek, Sequential fractional differential equations with Hadamard derivative, Commun. Nonl Sci. Numer. Simul. 16 (2011) 4689-4697.
[18] D. Baleanu, O.G. Mustafa, R.P. Agarwal, On Lp-solutions for a class of sequential fractional differential equations, Appl.Math.Comput. 218 (2011) 2074–2081.
[19] C. Bai, Impulsive periodic boundary value problems for fractional differential equation involving Riemann–Liouville sequential fractional derivative, J. Math. Anal. Appl. 384 (2011), 211–231.
[20] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley and Sons, New York, 1993.
[21] A.S. Vatsala, B. Sambandham, Sequential Caputo versus Nonsequential Caputo Fractional Initial and Boundary Value Problems, Int J of Diff Equ, V 15, Number 2, pp. 531-546 (2020).
[22] B. Ahmed, J.J. Nieto, Sequential fractional differential equations with three-point boundary conditions, Comput and Math with Appl 64 (2012) 3046-3052.
[23] N. Phuangthong, S.K. Ntouyas, J. Tariboon and K. Nonlaopon, Nonlocal Sequential Boundary Value Problems for Hilfer Type Fractional Integro-Differential Equations and Inclusions, Mathematics 2021, 9, 615.
[24] J. Tariboon, A. Cuntavepanit, S.K. Ntouyas and W. Nithiarayaphaks, Separated Boundary Value Problems of Sequential Caputo and Hadamard Fractional Differential Equations, Hindawi J of Funct Spac, V 2018, Art ID 6974046, 8 p.
[25] A. Tudorache and R. Luca, Positive Solutions of a Fractional Boundary Value Problem with Sequential Derivatives, Symmetry 2021, 13, 1489.

[26] Z. Baitiche, K. Guerbati, M. Benchohra, J. Henderson, Boundary Value Problems for Hybrid Caputo Sequential Fractional Differential Equations, Communi on Appl Nonl Analy, Vol 27(2020), N 4, 1–16.
[27] R.A.C. Ferreira, Note on a uniqueness result for a two-point fractional boundary value problem, Applied Mathematics Letters 90 (2019) 75-78.
[28] B. Ahmad, Sharp estimates for the unique solution of two-point fractional-order boundary value problems, Appl. Math. Lett. 65(2017) 77–82.
[29] Y. Zhou, Yong, Basic theory of fractional differential equations, World Scientific, Publishing Co: Pte. Ltd, 2014.
[30] K. Diethelm, The analysis of fractional differential equations, Lecture Notes in Mathematics, 2004, Springer, Berlin, 2010.