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Research Article

A constructive approach: From local subgroups to new classes of finite groups

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Abstract

Let G be a finite group and S be a proper subgroup of G. A group G is called an S-(S-quasinormal)-group if every local subgroup of G is either an S-quasinormal subgroup or conjugate to a subgroup of S. The main purpose of this construction is to demonstrate a new way of analyzing the structure of a finite group by the properties and the number of conjugacy classes of its local subgroups.

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1. Introduction

The efforts to investigate the importance of subgroups of prime power order date back to Sylow's theorems. A notion which is closely related is that of local subgroups of a finite group, i.e., the normalizers of non-identity subgroups of prime power order. There are a number of famous normal p-complement theorems, such as the one due to Burnside and the one due to Frobenius. It is always worth noting that Thompson's normal p-complement theorems (cf. [12, 13]) vitalized the field of finite group theory in the 1960s and still exert a powerful influence on the studies of group structure. As is well-known, local subgroups also lie at the heart of the classification of finite simple groups, where Thompson's classification of N-groups [14] is one of the most remarkable contributions.

Indeed, the initial motivation for the present article has its root in the monumental papers of Thompson. In his classification of N-groups, "uniqueness theorems" play a significant role, which virtually state that certain subgroups are contained in a unique maximal subgroup. Without any doubt, its ramifications are worth studying. From our perspective, one may find it intriguing to somehow extend the idea from *maximal subgroups* to *maximal local subgroups*. Note that a proper local subgroup of a group is a *maximal local subgroup* if it is maximal under inclusion among all proper local subgroups. Thus a question can be raised intuitively: What if certain subgroups are contained in unique (conjugacy classes of) maximal local subgroups? This is where the present article originates. We remark that in Definition 1.1, the subgroup S can be a *maximal local subgroup* of G whereas the construction does not restrict it to being so. Namely it can be any proper

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subgroup containing a maximal local subgroup. This makes our results of more general interest.

The present article provides a possible framework of describing finite groups exactly by their local subgroups. It may allow us to see more clearly the structures of different finite groups from the properties and the number of conjugacy classes of their local subgroups. In the present article, we will utilize the S-quasinormal subgroups, which permute with all Sylow subgroups of the group. The topic of conjugacy classes has always been a general topic in group theory (see for example the recent articles [2,3,6,10]). With regard to the number of conjugacy classes of local subgroups, the results in the present article apply to finite groups having exactly one conjugacy class of maximal local subgroup and the present article provides possible approach to exploring finite groups with multiple conjugacy classes of maximal local subgroups.

Recall that a subgroup H of a group G is said to be an S-quasinormal subgroup if for each Sylow subgroup P of G we have $PH = HP$. This concept was first introduced by Kegel [7] and its generalizations have been widely studied by many authors. See for example [1,8,11]. We remark that S-quasinormal subgroups are feasible to control and suitable to be applied to the construction of our approach, which is why we concentrate on this notion in the present article. Now we can give the new definition which plays a central role in this article.

Definition 1.1. Let S be a proper subgroup of a finite group G. Then G is called an S-(S-quasinormal)-group if for every local subgroup L of G one of the following holds:

(1) L is an S-quasinormal subgroup of G;
(2) L is G-conjugate to a subgroup of S.

Remark 1.2. An S-(S-quasinormal)-group G is by definition not a trivial group. See Example 2.2 for examples of S-(S-quasinormal)-groups. One may see that a group can be seen as an S-(S-quasinormal)-group for different S. For instance, as shown in the present article, if a group G is a 1-(S-quasinormal)-group, then it is also an S-(S-quasinormal)-group for an arbitrary proper subgroup S of G.

Section 3 shows that an S-(S-quasinormal)-group has non-trivial Fitting subgroup. In Section 4, 1-(S-quasinormal)-groups are described and a solvability criterion for S-(S-quasinormal)-groups is given. Section 5 consists of an application in investigating the structure of finite groups using the new concept.

Throughout the present article all groups are finite. The standard notation and terminologies follow those in [5]. Let G be a finite group and p a prime. Denote by $\pi(G)$ the set of prime divisors of $|G|$; n_p means the largest power of p dividing an integer n; $[A]B$ denotes a split extension of a normal subgroup A by a complement B; the *Fitting subgroup* of G is denoted by $F(G)$; $Sol(G)$ is the largest solvable normal subgroup of G, which is also called the *solvable radical* of G; $H_G = core_G(H)$ is the largest G-invariant subgroup contained in a subgroup H of G.

2. Preliminaries

Lemma 2.1 ([7]). *Let K be a normal subgroup of G. If U is a subgroup of G, then UK/K is an* S*-quasinormal subgroup of G/K if and only if U is an* S*-quasinormal subgroup of G.*

Example 2.2. The following groups are S-(S-quasinormal)-groups.

(1) The dihedral group of order 8 is an S-(S-quasinormal)-group for any proper subgroup. In particular, it is a 1-(S-quasinormal)-group.
(2) The symmetric group $S_4 = \langle (12), (1234) \rangle$ is an S-(S-quasinormal)-group, where S is a non-normal elementary abelian subgroup of order 4 or isomorphic to the dihedral group of order 8.

Proof. (1) It follows from the fact that the Sylow subgroup of this group is exactly the whole group.

(2) Every non-normal elementary abelian subgroup of order 4 is the normalizer of some subgroup of order 2 in S_4. But such subgroups are not S-quasinormal as they do not permute with the Sylow 3-subgroups of S_4. Since other local subgroups of G are S-quasinormal, by Definition 1.1 the group S_4 is an S-(S-quasinormal)-group for a subgroup S of S_4 containing a non-normal elementary abelian subgroup of order 4. □

Lemma 2.3. *Let G be an S-*(S-quasinormal)*-group. If N is a normal subgroup of G with $N \subseteq S$, then G/N is an S/N-*(S-quasinormal)*-group.*

Proof. It is clear that S/N is a proper subgroup of G/N. Let H/N be any non-identity subgroup of G/N of prime power order. It follows that $|H/N| = p^n$ for some prime p and $n \geqslant 1$. Denote by P a Sylow p-subgroup of H. Then Frattini argument gives that $N_{G/N}(H/N) = N_G(P)N/N$. If $N_G(P)$ is an S-quasinormal subgroup of G, then by Lemma 2.1, $N_G(P)N/N$ is an S-quasinormal subgroup of G/N and so $N_{G/N}(H/N)$ is an S-quasinormal subgroup of G/N. Now suppose that $N_G(P)$ is not an S-quasinormal subgroup of G. Then $N_G(P)$ is conjugate to a subgroup of S. Let $N_G(P)^x \leqslant S$ for some $x \in G$. Noting $N \leqslant S$, we have $N_{G/N}(H/N)^{xN} = (N_G(P)N/N)^{xN} = N_G(P)^x N/N \leqslant S/N$. Thus G/N is an S/N-(S-quasinormal)-group. □

Lemma 2.4. *Let G be an S-*(S-quasinormal)*-group. If N is a non-solvable minimal normal subgroup of G, then*

(1) $N \nsubseteq S$;
(2) $G = NS$.

Proof. (1) Assume that the statement is false and let $N \leqslant S$. Denote by P a non-identity Sylow subgroup of N. Since N is a characteristically simple group, $N_G(P)$ is not an S-quasinormal subgroup of G. By the definition of S-(S-quasinormal)-groups, $N_G(P)$ is a G-conjugate of some subgroup of S, i.e., $N_G(P)^x \leqslant S$ for some $x \in G$. It follows from Frattini argument that $G = N_G(P)N = N_G(P)^x N \leqslant S$ and therefore $G = S < G$, a contradiction.

(2) Assume that the subgroup NS is a proper subgroup of G. Then G is an NS-(S-quasinormal)-group. Now (1) gives $N \nsubseteq NS$, which is absurd. □

3. Two solvable subgroups of an *S*-(S-quasinormal)-group

The next theorem implies that, for an S-(S-quasinormal)-group, all normal subgroups contained in S are solvable.

Theorem 3.1. *Let G be an S-*(S-quasinormal)*-group. Then S_G is solvable.*

Proof. We write $T = S_G$ and denote $N = Sol(T)$ to be the solvable radical of T. It follows from Lemma 2.3 that G/N is an S/N-(S-quasinormal)-group. If $N \neq 1$, then an induction indicates that T/N is solvable and that T is hence solvable. Now we may let $N = 1$ and $T > 1$. Consequently, there is a non-solvable minimal normal subgroup M of G in $T \leqslant S$. This is in contradiction to Lemma 2.4 and entails that $T = 1$. Thus the lemma follows. □

Fitting subgroups are always applied as an important part of group-theoretic proofs. We prove in the following theorem that any S-(S-quasinormal)-group has non-identity Fitting subgroup.

Theorem 3.2. *If G is an S-*(S-quasinormal)*-group, then the Fitting subgroup $F(G) > 1$. In particular, G is not a non-abelian simple group.*

Proof. Assume that the theorem is false and let G be a counterexample of the smallest order. Suppose that S is a proper subgroup of G and that G is an S-(S-quasinormal)-group with $F(G) = 1$. Let N be a minimal normal subgroup of G. Since $F(N) \leqslant F(G) = 1$, the minimal normal subgroup N of G is not solvable. Now Lemma 2.4 gives that $N \not\subseteq S$ and $G = NS$. The proof is divided into the following steps.

(i) G is a non-abelian simple group.

Let $S_0 = S \cap N < N$. We first prove that N is an S_0-(S-quasinormal)-group. Assume that $P > 1$ is a subgroup of N of prime power order. As N is a characteristically simple group, $N_G(P)$ is not an S-quasinormal subgroup of G. By the definition of S-(S-quasinormal)-groups, $N_G(P)$ is conjugate to a subgroup of S. Suppose that $N_G(P)^g \leqslant S$ for some $g \in G$. Recall that $G = NS$. It follows that there exist $n \in N$ and $x \in S$ such that $g = nx$. We now have $N_N(P) = N \cap N_G(P) \leqslant N \cap S^{g^{-1}} = N \cap S^{n^{-1}} = (N \cap S)^{n^{-1}} = S_0^{n^{-1}}$. This proves that N is an S_0-(S-quasinormal)-group.

The choice of G entails that the minimal normal subgroup N is G itself. Otherwise $F(G) \geqslant F(N) > 1$. Now we have deduced that G is a non-abelian simple group.

(ii) Every local subgroup of G is conjugate to a subgroup of S.

Since by (i) G is a non-abelian simple group, the local subgroups of G are proper subgroups. The simplicity of G also forces that S is not an S-quasinormal subgroup. Now (ii) follows immediately from the definition of S-(S-quasinormal)-groups.

(iii) A contradiction.

Assume that $P_1 \in \mathrm{Syl}_p(G)$ for a prime p dividing $|G : S|$. Then it is obvious that $N_G(P_1)$ is not conjugate to any subgroup of S, contradicting (ii). □

4. Solvable *S*-(S-quasinormal)-groups

This section considers solvable S-(S-quasinormal)-groups. We first show that 1-(S-quasinormal)-groups are solvable, and, in particular, nilpotent.

Theorem 4.1. *If G is a* 1-(S-quasinormal)*-group, then G is a nilpotent group.*

Proof. Since G is a 1-(S-quasinormal)-group, every local subgroup of G is S-quasinormal. Let p be a prime divisor of $|G|$ and let P be an arbitrary Sylow p-subgroup of G. Then $N_G(P)P^* = P^*N_G(P)$ for any Sylow p-subgroup P^* of G. It follows that $N_G(P)P^*$ is subgroup of G. Now the order of $N_G(P)$ forces that $N_G(P) \cap P^* = P^*$ and hence $P^* \leqslant N_G(P)$, otherwise the number $|N_G(P)P^*|_p$ would exceed the largest p-power dividing $|G|$. Then P^* is also a Sylow p-subgroup of $N_G(P)$ as $|P| = |P^*|$. But P is normal in $N_G(P)$ and therefore $N_G(P)$ has unique Sylow p-subgroup. This indicates that $P = P^*$, i.e., Sylow subgroup P is normal in G. Thus G is nilpotent. □

When the structure of S changes, an S-(S-quasinormal)-group is not necessarily nilpotent (cf. Example 2.2(2)). We now provide an equivalent condition for an S-(S-quasinormal)-group to be solvable.

Theorem 4.2. *An S-*(S-quasinormal)*-group G is solvable if and only if S is solvable.*

Proof. The necessity is clear. It follows from Theorem 3.2 that $F(G) \neq 1$ and therefore G has a minimal normal subgroup U that is elementary abelian. If $SU < G$, then G is an SU-(S-quasinormal)-group, where SU is solvable. Since Lemma 2.3 implies that G/U is an SU/U-(S-quasinormal)-group, we can obtain from induction that G/U is solvable, and hence G is solvable. Thus we may let $G = [U]S$. The solvability of G in this situation is obvious as both G/U and U are solvable. □

5. An application

Although Definition 1.1 does not concern the structure of S, the following theorem indicates that there are no finite groups all of whose non-S-quasinormal local subgroups are contained in a single conjugacy class of non-abelian simple subgroups.

Theorem 5.1. *If G is an S-(S-quasinormal)-group, then S is not a non-abelian simple group.*

Proof. Suppose that the theorem is false. Let G be a counterexample of the smallest order. Then G is an S-(S-quasinormal)-group with S a non-abelian simple subgroup of G. In view of Theorem 3.2, $F(G) \neq 1$ and we can identify a minimal normal subgroup P of G that is elementary abelian. Suppose that PS is a proper subgroup of G. Then Lemma 2.3 indicates that G/P is a PS/P-(S-quasinormal)-group. However, $PS/P \cong S$ is a non-abelian simple group, in contradiction to the minimal choice of G. Now we may let $G = [P]S$.

We claim that G is a Frobenius group with Frobenius kernel P. Suppose that a non-identity element $x \in P$ satisfies $C_G(x) > P$. Then $C_G(x) \cap S > 1$, and hence there exists an element $y \in C_G(x) \cap S$ of prime order. Write $Y = \langle y \rangle$. Consequently, $x \in N_G(Y)$ and it follows that $P \cap N_G(Y) > 1$. If $N_G(Y)^g \leqslant S$ for some $g \in G$, then $P \cap S > 1$, a contradiction. Thus $N_G(Y)$ is an S-quasinormal subgroup of G, which is again contradicting the simplicity of S as $N_G(P) \cap S > 1$. We now have that $C_G(x) \subseteq P$ for all $1 \neq x \in P$ and thus G is a Frobenius group with Frobenius kernel P and Frobenius complement S.

The structure of Frobenius groups has been indicated in [4, p. 86], which yields that the odd-order Sylow subgroups of S are cyclic and that its Sylow 2-subgroups are either cyclic or generalized quaternion. An immediate contradiction is that S is metacyclic provided that the Sylow 2-subgroups of S are cyclic. Assume otherwise that S has a generalized quaternion Sylow 2-subgroup. Then the Brauer-Suzuki theorem asserts that the order of the center of S is 2, contradicting the simplicity of S. This final contradiction completes the proof. □

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