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SOME RESULTS ON THE SUBORDINATION PRINCIPLE FOR ANALYTIC FUNCTIONS

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Abstract

The aim of this paper is to introduce the class of the analytic functions called $\mathcal{S}(c)$ and to investigate the various properties of the functions belonging this class. For the functions in this class, some inequalities related to the angular derivative have been obtained.

Key Words: Jack's lemma, Subordination Principle, Schwarz lemma.

Özet

Bu çalışmanın amacı, $\mathcal{S}(c)$ olarak adlandırılan analitik fonksiyonlar sınıfını tanıtmak ve bu sınıfa ait fonksiyonların çeşitli özelliklerini araştırmaktır. Bu sınıftaki fonksiyonlar için, açısal türeve ilişkin bazı eşitsizlikler elde edilmiştir.

Anahtar Kelimeler: Jack's lemma, Subordination Prensibi, Schwarz lemma

1. Introduction

Let g be an analytic function in the unit disc $U = \{z: |z| < 1\}$, $g(0) = 0$ and $g: U \to U$. In accordance with the classical Schwarz Lemma, for any point z in the unit disc U, we have $|g(z)| \leq |z|$ for all $z \in U$ and $|g'(0)| \leq 1$. In addition, if the equality $|g(z)| = |z|$ holds for any $z \neq 0$, or $|g'(0)| = 1$, then g is a rotation; that is $g(z) = ze^{i\theta}$, θ real ([5], p.329). Schwarz lemma has important applications in engineering [16, 17]. In this study, the Shwarz Lemma will be obtained for the following class $\mathcal{S}(c)$ which will be given.
We will use of the following definition and lemma to prove our results [5, 6].

Lemma 1 (Jack's Lemma) *Let $g(z)$ be a non-constant anaytic function in U with $g(0) = 0$. If*

$$|g(z_0)| = \max\{|g(z)|: |z| \leq |z_0|\},$$

then there exists a real number $k \geq 1$ such that

$$\frac{z_0 g'(z_0)}{g(z_0)} = k.$$

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Definition 1 (Subordination Principle) *Let g and h be analytic functions in U. A function is said to be subordinate to h, written as $g(z) \prec h(z)$, if there exists a Schwarz function $\omega(z)$, analytic in U with $\omega(0) = 0$, $|\omega(z)| < 1$ such that $g(z) = h(\omega(z))$.*

Some applications of Jack Lemma and Subordination Principle have been given in [11, 12, 13, 20]. Also, in [13], authors have been studied some first-order differential subordinations.
Let $\mathcal{A}$ denote the class of functions $f(z) = 1 + b_1 z + b_2 z^2 + \ldots$ that are analytic in U. Also, let $\mathcal{S}(c)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ satisfying

$$1 + \alpha z f'(z) \prec 1 + z, \ \ z \in U,$$

where $\alpha \geq \frac{2\sqrt{1+c}}{c}$, $0 < c \leq 1$.
In this paper, we study some of the properties of the clases $\mathcal{S}(c)$. Namely, an upper bound will be obtained for the modulus of the coefficient $b_1 = f'(0)$ for this class. The aim of this paper is to examine some properties of the function $f(z)$ which belongs to the class of $\mathcal{S}(c)$ by employing Jack's Lemma.
Let $f \in \mathcal{S}(c)$ and consider the following function

$$\vartheta(z) = \frac{(f(z))^2 - 1}{c}, \tag{1.1}$$

where we choose the principle branches of the square root.
It is an analytic function in U and $\vartheta(0) = 0$. Now, let us now demonstrate that $|\vartheta(z)| < 1$ in U. From (1.1), we have

$$f(z) = \sqrt{1 + c\vartheta(z)}$$

and let

$$p(z) = 1 + \alpha z f'(z).$$

Then

$$p(z) = 1 + \frac{\alpha c z \vartheta'(z)}{2\sqrt{1 + c\vartheta(z)}}.$$

We assume that there exists a $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |\vartheta(z)| = |\vartheta(z_0)| = 1.$$

From Jack's Lemma, we have

$$\vartheta(z_0) = e^{i\theta} \ \text{ and } \ \frac{z_0 \vartheta'(z_0)}{\vartheta(z_0)} = k.$$

Thus, we obtain

$$|p(z_0) - 1| = \left| \frac{\alpha c z_0 \vartheta'(z_0)}{2\sqrt{1 + c\vartheta(z_0)}} \right| = \left| \frac{\alpha c k \vartheta(z_0)}{2\sqrt{1 + c\vartheta(z_0)}} \right|$$

$$\geq \left| \frac{\alpha c k}{2\sqrt{|1 + c e^{i\theta}|}} \right| = \frac{c k \alpha}{2\sqrt{|1 + c\cos\theta + i c \sin\theta|}}$$

$$=\frac{ck\alpha}{2\sqrt[4]{(1+c\cos\theta)^2+c^2\sin^2\theta}}$$
$$=\frac{ck\alpha}{2\sqrt[4]{1+2c\cos\theta+c^2}}$$

and

$$|p(z_0)-1|\geq\frac{ck\alpha}{2\sqrt[4]{1+2c\cos\theta+c^2}}. \tag{1.2}$$

Since the right hand side of (1.2) takes its maximum value for $\cos\theta=1$, we take

$$|p(z_0)-1|\geq\frac{ck\alpha}{2\sqrt[4]{(1+c)^2}}.$$

Also, since $\alpha\geq\frac{2\sqrt{1+c}}{c}$ and $k\geq1$, we obtain

$$|p(z_0)-1|\geq\frac{ck\alpha}{2\sqrt[4]{(1+c)^2}}\geq\frac{ck\frac{2\sqrt{1+c}}{c}}{2\sqrt{1+c}}\geq1$$

This contradicts the $f\in\mathcal{S}(c)$. This implies that there is no point in doing so $z_0\in U$ such that $\max_{|z|\leq|z_0|}|\vartheta(z)|=|\vartheta(z_0)|=1$. Hence, we take $|\vartheta(z)|<1$ in U. From the Schwarz Lemma, we obtain

$$\vartheta(z)=\frac{(f(z))^2-1}{c}$$

$$\vartheta'(z)=\frac{2f(z)f'(z)}{c}$$

$$|\vartheta'(0)|=\left|\frac{2f(0)f'(0)}{c}\right|\leq1$$

and

$$|f'(0)|\leq\frac{c}{2}.$$

As a result, we have the following lemma.

Lemma 2 *If $f\in\mathcal{S}(c)$, then we have*

$$|f'(0)|\leq\frac{c}{2}. \tag{1.3}$$

Now let us consider the following function by taking into account of the zeros, which are different from zero, of the function $f(z)-1$,

$$\Theta(z)=\frac{\vartheta(z)}{\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}.$$

Since $f\in\mathcal{S}(c)$, from Schwarz Lemma, we obtain

$$\Theta(z)=\frac{\vartheta(z)}{\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}=\frac{(f(z))^2-1}{c}\frac{1}{\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}$$
$$=\frac{2(b_1z+b_2z^2+b_3z^3+\ldots)+(b_1z+b_2z^2+b_3z^3+\ldots)^2+\ldots}{c\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}},$$

$$\frac{\Theta(z)}{z}=\frac{2(b_1+b_2z+b_3z^2+\ldots)+z(b_1+b_2z+b_3z^2+\ldots)^2+\ldots}{c\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}},$$

$$|\Theta'(0)| = \frac{2|b_1|}{c\prod_{i=1}^{n}|a_i|} \leq 1$$

and

$$|b_1| \leq \frac{c}{2}\prod_{i=1}^{n}|a_i|.$$

As a result, we get the following lemma.

Lemma 3 *Let $f \in \mathcal{S}(c)$ and $a_1, a_2, \ldots, a_n$ be zeros of the function $f(z) - 1$ in U that are different from zero. Then we have*

$$|f'(0)| \leq \frac{c}{2}\prod_{i=1}^{n}|a_i|.$$

There are some important studies about the Schwarz Lemma at the boundary as well. The main source of these studies is Schwarz Lemma at the boundary which is about the below estimation of the modulus of the derivative on some boundary points of the unit disc. The boundary version of Schwarz Lemma is given as follows [14, 19]:

Lemma 4 *If $g(z)$ extends continuously to some boundary point $\xi \in \partial U = \{z: |z| = 1\}$ with $|\xi| = 1$, and if $|g(\xi)| = 1$ and $g'(\xi)$ exists, then*

$$|g'(\xi)| \geq \frac{2}{1+|g'(0)|} \tag{1.4}$$

and

$$|g'(\xi)| \geq 1. \tag{1.5}$$

These inequalities are important in the literature and still continue to be studied among current issues [1, 2, 3, 4, 7, 8, 9, 10, 14, 15].

The following lemma, known as the Julia-Wolff Lemma, is needed in the sequel (see, [18]).

Lemma 5 (Julia-Wolff lemma) *Let g be an analytic function in U, $g(0) = 0$ and $g(U) \subset U$. If, in addition, the function g has an angular limit $g(\xi)$ at $\xi \in \partial U$, $|g(\xi)| = 1$, then the angular derivative $g'(\xi)$ exists and $1 \leq |g'(\xi)| \leq \infty$.*

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{S}(c)$ class. Also, in a class of analytic functions on the unit disc, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained.

Theorem 1 *Let $f \in \mathcal{S}(c)$. Suppose that, for $1 \in \partial U$, f has an angular limit $f(1)$ at the point 1, $f(1) = \sqrt{1+c}$. Then we have*

$$|f'(1)| \geq \frac{c}{2\sqrt{1+c}}. \tag{2.1}$$

Proof. Consider the function

$$\vartheta(z) = \frac{(f(z))^2 - 1}{c}.$$

Also, since $f(1) = \sqrt{1+c}$, we have $|\vartheta(1)| = 1$. Therefore, from (1.5), we obtain

$$1 \leq |\vartheta'(1)| = \left|\frac{2f(1)f'(1)}{c}\right| = \frac{2\sqrt{1+c}}{c}|f'(1)|$$

and

$$|f'(1)| \geq \frac{c}{2\sqrt{1+c}}.$$

Theorem 2 *Under the same assumptions as in Theorem 1, we have*

$$|f'(1)| \geq \frac{1}{\sqrt{1+c}} \frac{c^2}{c+2|f'(0)|}. \tag{2.2}$$

Proof. Let $\vartheta(z)$ function be the same as (1.1). So, from (1.4), we obtain

$$\frac{2}{1+|\vartheta'(0)|} \leq |\vartheta'(1)| = \frac{2\sqrt{1+c}}{c}|f'(1)|.$$

Since

$$|\vartheta'(0)| = \frac{2}{c}|f'(0)|,$$

we take

$$\frac{2}{1+\frac{2}{c}|f'(0)|} \leq \frac{2\sqrt{1+c}}{c}|f'(1)|$$

and

$$|f'(1)| \geq \frac{1}{\sqrt{1+c}} \frac{c^2}{c+2|f'(0)|}.$$

Theorem 3 *Let $f \in S(c)$. Suppose that, for $1 \in \partial U$, f has an angular limit $f(1)$ at the point 1, $f(1) = \sqrt{1+c}$. Then we have*

$$|f'(1)| \geq \frac{c}{2\sqrt{1+c}}\left(1 + \frac{2(c-2|b_1|)^2}{c^2-4|b_1|^2+2c|b_1^2+2b_2|}\right). \tag{2.3}$$

Proof. Let $\vartheta(z)$ be the same as in the proof of Theorem 1 and $m(z) = z$. By the maximum principle, for each $z \in U$, we have the inequality $|\vartheta(z)| \leq |m(z)|$. So,

$$\begin{aligned} d(z) &= \frac{\vartheta(z)}{m(z)} = \frac{1}{z}\left(\frac{(f(z))^2-1}{c}\right) \\ &= \frac{\left(1+b_1z+b_2z^2+\dots\right)^2-1}{cz} \\ &= \frac{2(b_1+b_2z+b_3z^2+\dots)+z\left(b_1+b_2z+b_3z^2+\dots\right)^2+\dots}{c} \end{aligned}$$

is an analytic function in U and $|d(z)| \leq 1$ for $z \in U$. In particular, we have

$$|d(0)| = \frac{2|b_1|}{c} \leq 1 \tag{2.4}$$

and

$$|d'(0)| = \frac{2}{c}|b_1^2 + 2b_2|.$$

Furthermore, with simple calculations, we take

$$\frac{\vartheta'(1)}{\vartheta(1)} = |\vartheta'(1)| \geq |m'(1)| = \frac{m'(1)}{m(1)}.$$

The auxiliary function

$$w(z) = \frac{d(z)-d(0)}{1-\overline{d(0)}d(z)}$$

is analytic in U, $w(0) = 0$, $|w(z)| < 1$ for $|z| < 1$ and $|w(1)| = 1$ for $1 \in \partial U$. From (1.4), we obtain

$$\frac{2}{1+|w'(0)|} \leq |w'(1)| = \frac{1-|d(0)|^2}{\left|1-\overline{d(0)}d(1)\right|^2}|d'(1)|$$

$$\leq \frac{1+|d(0)|}{1-|d(0)|}\{|\vartheta'(1)|-|m'(1)|\}$$
$$=\frac{c+2|b_1|}{c-2|b_1|}\left(\frac{2\sqrt{1+c}}{c}|f'(1)|-1\right).$$

Since

$$w'(z)=\frac{1-|d(0)|^2}{\left(1-\overline{d(0)}d(z)\right)^2}d'(z)$$

and

$$|w'(0)|=\frac{|d'(0)|}{1-|d(0)|^2}=\frac{\frac{2}{c}|b_1^2+2b_2|}{1-\left(\frac{2|b_1|}{c}\right)^2}=\frac{2c|b_1^2+2b_2|}{c^2-4|b_1|^2},$$

we obtain

$$\frac{2}{1+\frac{2c|b_1^2+2b_2|}{c^2-4|b_1|^2}}\leq\frac{c+2|b_1|}{c-2|b_1|}\left(\frac{2\sqrt{1+c}}{c}|f'(1)|-1\right),$$

$$\frac{2(c^2-4|b_1|^2)}{c^2-4|b_1|^2+2c|b_1^2+2b_2|}\frac{c-2|b_1|}{c+2|b_1|}\leq\frac{2\sqrt{1+c}}{c}|f'(1)|-1$$

and

$$|f'(1)|\geq\frac{c}{2\sqrt{1+c}}\left(1+\frac{2(c-2|b_1|)^2}{c^2-4|b_1|^2+2c|b_1^2+2b_2|}\right).$$

Theorem 4 *Let $f\in\mathcal{S}(c)$ and $a_1,a_2,\ldots,a_n$ be zeros of the function $f(z)-1$ in U that are different from zero. Suppose that, for $1\in\partial U$, f has an angular limit $f(1)$ at the point 1, $f(1)=\sqrt{1+c}$. Then we have*

$$|f'(1)|\geq\frac{c}{2\sqrt{1+c}}\left(1+\prod_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}\right. \tag{2.5}$$
$$\left.+\frac{2\left(c\prod_{i=1}^{n}|a_i|-2|b_1|\right)^2}{\left(c\prod_{i=1}^{n}|a_i|\right)^2-4|b_1|^2+c\prod_{i=1}^{n}|a_i|\left|2b_2+b_1^2+2b_1\prod_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}\right).$$

Proof. Let $\vartheta(z)$ be as in (1.1) and $a_1,a_2,\ldots,a_n$ be zeros of the function $f(z)-1$ in U that are different from zero. Also, consider the function

$$B(z)=z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}.$$

Here, $B(z)$ is analytic in U and $|B(z)|<1$ for $|z|<1$. By the maximum principle for each $z\in U$, we have

$$|\vartheta(z)|\leq|B(z)|.$$

Consider the function

$$t(z)=\frac{\vartheta(z)}{B(z)}=\left(\frac{(f(z))^2-1}{c}\right)\frac{1}{z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}$$
$$=\frac{2(b_1+b_2z+b_3z^2+\ldots)+z(b_1+b_2z+b_3z^2+\ldots)^2+\ldots}{c}\frac{1}{\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}.$$

$t(z)$ is analytic in U and $|t(z)|<1$ for $|z|<1$. In particular, we have

$$|t(0)| = \frac{2|b_1|}{c\prod_{i=1}^{n}|a_i|}$$

and

$$|t'(0)| = \frac{\left|2b_2+b_1^2+2b_1\prod_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{c\prod_{i=1}^{n}|a_i|}.$$

The auxiliary function

$$r(z) = \frac{t(z)-t(0)}{1-\overline{t(0)}t(z)}$$

is analytic in U, $|r(z)| < 1$ for $|z| < 1$ and $r(0) = 0$. For $1 \in \partial U$ and $f(1) = \sqrt{1+c}$, we take $|r(1)| = 1$.
From (1.4), we obtain

$$\frac{2}{1+|r'(0)|} \leq |r'(1)| = \frac{1-|t(0)|^2}{\left|1-\overline{t(0)}t(1)\right|}|t'(1)|$$
$$\leq \frac{1+|t(0)|}{1-|t(0)|}(|\vartheta'(1)| - |B'(1)|).$$

It can be seen that

$$|r'(0)| = \frac{|t'(0)|}{1-|t(0)|^2}$$

and

$$|r'(0)| = \frac{\frac{\left|2b_2+b_1^2+2b_1\prod_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{c\prod_{i=1}^{n}|a_i|}}{1-\left(\frac{2|b_1|}{c\prod_{i=1}^{n}|a_i|}\right)^2} = c\prod_{i=1}^{n}|a_i|\frac{\left|2b_2+b_1^2+2b_1\prod_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{\left(c\prod_{i=1}^{n}|a_i|\right)^2-4|b_1|^2}.$$

Also,we have

$$|B'(1)| = 1+\prod_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}.$$

Therefore, we obtain

$$\frac{2}{1+c\prod_{i=1}^{n}|a_i|\frac{\left|2b_2+b_1^2+2b_1\prod_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}{(c\prod_{i=1}^{n}|a_i|)^2-4|b_1|^2}}$$
$$\leq \frac{c\prod_{i=1}^{n}|a_i|+2|b_1|}{c\prod_{i=1}^{n}|a_i|-2|b_1|}\left(\frac{2\sqrt{1+c}|f'(1)|}{c}-1-\prod_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}\right),$$
$$\frac{2((c\prod_{i=1}^{n}|a_i|)^2-4|b_1|^2)}{(c\prod_{i=1}^{n}|a_i|)^2-4|b_1|^2+c\prod_{i=1}^{n}|a_i|\left|2b_2+b_1^2+2b_1\prod_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}$$
$$\leq \frac{c\prod_{i=1}^{n}|a_i|+2|b_1|}{c\prod_{i=1}^{n}|a_i|-2|b_1|}\left(\frac{2\sqrt{1+c}|f'(1)|}{c}-1-\prod_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}\right),$$
$$\frac{2(c\prod_{i=1}^{n}|a_i|-2|b_1|)^2}{(c\prod_{i=1}^{n}|a_i|)^2-4|b_1|^2+c\prod_{i=1}^{n}|a_i|\left|2b_2+b_1^2+2b_1\prod_{i=1}^{n}\frac{1-|a_i|^2}{a_i}\right|}$$
$$\leq \frac{2\sqrt{1+c}|f'(1)|}{c}-1-\prod_{i=1}^{n}\frac{1-|a_i|^2}{|1-a_i|^2}$$

and so, we get inequality (2.5).

Theorem 5 *Let $f \in \mathcal{S}(c)$, $f(z)-1$ has no zeros in U except $z=0$ and $b_1>0$. Suppose that, for $1\in \partial U$, f has an angular limit $f(1)$ at the point 1, $f(1)=\sqrt{1+c}$. Then we have*

$$|f''(1)| \geq \frac{c}{2\sqrt{1+c}}\left(1-\frac{2\ln^2\left(\frac{2b_1}{c}\right)b_1}{2\ln\left(\frac{2b_1}{c}\right)b_1-|b_1^2+2b_2|}\right). \tag{2.6}$$

Proof. Let $b_1>0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function $f(z)-1$ has no zeros in U except $z=0$, we denote by $\ln d(z)$ the analytic branch of the logarithm normed by the condition

$$\ln d(0)=\ln\left(\frac{2b_1}{c}\right)<0.$$

The auxiliary function

$$\phi(z)=\frac{\ln d(z)-\ln d(0)}{\ln d(z)+\ln d(0)}$$

is analytic in the unit disc U, $|\phi(z)|<1$, $\phi(0)=0$ and $|\phi(1)|=1$ for $1\in\partial U$.
From (1.4), we obtain

$$\frac{2}{1+|\phi'(0)|}\leq|\phi'(1)|=\frac{|2\ln d(0)|}{|\ln d(1)+\ln d(0)|^2}\left|\frac{d'(1)}{d(1)}\right|$$
$$=\frac{-2\ln d(0)}{\ln^2 d(0)+\arg^2 d(1)}\{|\vartheta'(1)|-1\}.$$

Replacing $\arg^2 d(1)$ by zero, then

$$\frac{1}{1-\frac{1}{2\ln\left(\frac{2b_1}{c}\right)}\frac{|b_1^2+2b_2|}{b_1}}\leq\frac{-1}{\ln\left(\frac{2b_1}{c}\right)}\left\{\frac{2\sqrt{1+c}}{c}|f'(1)|-1\right\}$$

and

$$1-\frac{2\ln^2\left(\frac{2b_1}{c}\right)b_1}{2\ln\left(\frac{2b_1}{c}\right)|b_1|-|b_1^2+2b_2|}\leq\frac{2\sqrt{1+c}}{c}|f'(1)|.$$

Thus, we obtain the inequality (2.6).

Theorem 6 *Let $f\in\mathcal{S}(c)$, $f(z)-1$ has no zeros in U except $z=0$ and $b_1>0$. Then we have*

$$|b_1^2+2b_2|\leq 2\left|b_1\ln\left(\frac{2b_1}{c}\right)\right|. \tag{2.7}$$

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 5. Here, $\phi(z)$ is analytic in the unit disc U, $|\phi(z)|<1$, $\phi(0)=0$. Therefore, the function $\phi(z)$ satisfies the assumptions of the Schwarz Lemma. Thus, we obtain

$$1\geq|\phi'(0)|=\frac{|2\ln d(0)|}{|\ln d(0)+\ln d(0)|^2}\left|\frac{d'(0)}{d(0)}\right|=\frac{-1}{2\ln d(0)}\left|\frac{d'(0)}{d(0)}\right|$$
$$=\frac{1}{2\ln\left(\frac{2b_1}{c}\right)}\frac{|b_1^2+2b_2|}{b_1}$$

and

$$|b_1^2+2b_2|\leq 2\left|b_1\ln\left(\frac{2b_1}{c}\right)\right|.$$

Conflicts of interest

The authors declare that there are no potential conflicts of interest relevant to this article.

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