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GENERALIZED FIDUCIAL INFERENCE FOR THE CHEN DISTRIBUTION

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Abstract: The fiducial inference idea was firstly proposed by Fisher [8] as a powerful method in statistical inference. Many authors such as Weeranhandi [24] and Hannig et. al. [12] improved this method from different points of view. Since the Bayesian method has some deficiencies such as assuming a prior distribution when there was little or no information about the parameters, the fiducial inference is used to overcome these adversities. This study deals with the generalized fiducial inference for the shape parameters of the Chen' s two-parameter lifetime distribution with bathtub shape or increasing failure rate [4]. The method based on the inverse of the structural equation which is proposed by Hannig et. al. [12] is used. We propose the generalized fiducial inferences of the parameters with their confidence intervals. Then, these estimations are compared with their maximum likelihood and Bayesian estimations. Simulation results show that the generalized fiducial inference is more applicable than the other methods in terms of the performances of estimators for the shape parameters of the Chen distribution. Finally, a real data example is used to illustrate the theoretical outcomes of these estimation procedures.

Key words: Bayesian inference, Generalized fiducial inference, Interval estimation, Chen distribution, Point estimation.

1. Introduction

The fiducial idea is firstly proposed by Fisher [8] as a powerful method in statistics. It is known that assuming a prior distribution in the case of insufficient information about the parameters causes adversities in Bayesian inference. The main idea of Fisher [8] with the fiducial method was to overcome this deficiency in the Bayesian framework. Then, some deficiencies in the fiducial inference and the philosophical concerns regarding the interpretation of fiducial probability were handled by various authors (See Zabell [28] for more details.). Thus, the idea of fiducial inference was improved by various authors. Recently, Hannig [11, 10] handled generalized fiducial inference. Then, Hannig et al. [12] defined the generalized fiducial inference method based on the inverse of the structural equation. The generalized fiducial inference is actually similar to the likelihood approach. It differs from the likelihood method by switching the role of the parameters and the observed data.

In statistics theory, there are several applications of fiducial inference. For example; Wandler and Hannig [21, 22] considered generalized fiducial inference on the largest mean of a multivariate normal distribution and also inference on the parameters and the extreme quantiles of the generalized Pareto distribution, respectively. Wang et al. [23] handled fiducial inference to construct prediction intervals for an arbitrary probability distribution. Further; O'Reilly and Rueda [17] studied the truncated exponential distribution, Li and Xu [15] studied inference of Birnbaum-Saunders distribution, Yan and Liu [27] studied generalized exponential distribution with the fiducial inference method.

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On the other hand, Chen [4] proposed a two-parameter distribution with bathtub shaped or increasing hazard function. Chen [4] proposed using this model to analyze the lifetime datasets flexibly. It has the following probability density (pdf), distribution (cdf) and failure rate functions

$$f(x;\lambda,\beta)=\lambda\beta x^{\beta-1}e^{x^{\beta}}e^{\lambda(1-e^{x^{\beta}})},\ \ x>0,\ \ \lambda,\beta>0,$$

$$F(x;\lambda,\beta)=1-e^{\lambda(1-e^{x^{\beta}})},$$

and

$$h(x;\lambda,\beta)=\lambda\beta x^{\beta-1}e^{x^{\beta}}.$$

The Chen distribution has a bathtub shape failure rate when $\beta<1$ and also has an increasing failure rate function when $\beta\geq 1$ (see Figure 1).

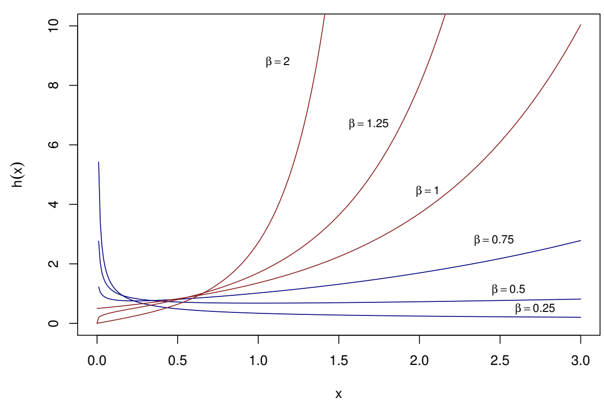


FIGURE 1. Failure rate functions when $\lambda=0.5$

Hence it provides an appropriate conceptual model for some electronic and mechanical products as well as the lifetime of humans. In addition to its positively skewed shape, it also has some other flexible properties (Chen, [4]). Further;

- It leads to the exponential power distribution when $\lambda=1$.
- If $X\sim CH(\lambda,\beta)$, then $Y=(e^{X^{\beta}}-1)\sim Exp(\lambda)$ and $Y=(e^{X^{\beta}}-1)^{\frac{1}{\theta}}\sim Weibull(\lambda,\theta)$.
- It leads to the $Gompertz(1,\lambda)$ distribution when $\beta=1$.

Many authors handled the Chen distribution in terms of statistical inference. For instance; Wu et al. [25] studied the estimation of its shape parameter. Then, Wu [26] studied its parameter estimations under progressive censoring. Sarhan et al. [19] obtained estimations of its parameters. Rastogi and Tripathi [18] handled parameter estimations under hybrid censored data. Ahmed [2] obtained Bayesian estimations and compared them with non-Bayesian estimations under progressive Type-II censoring scheme. Kayal et. al. [14] handled Chen distribution under progressive censoring and Kayal et al. [13] studied inference of its parameters under progressive first-failure censoring.

It should be noted that all cited references are based on classical and Bayesian estimation methods for both complete and censored data sets. The generalized fiducial inference method has never been considered in comparative inference studies based on the Chen distribution. It is known that Newton-Raphson (NR) method can provide unsatisfactory performances explained by the fact that it does not converge in some cases. Since the MLE of β needs some iterative methods such as NR, alternative inference methods for the parameters of the Chen distribution are needed to evaluate. On the other hand, determining a prior distribution in the case of insufficient prior information about the parameters affects the Bayesian inference performance. Consequently, the generalized fiducial method can be worthwhile to overcome these adversities.

In this study, we consider the generalized fiducial inference (GFI) method based on the inverse of the structural equation which is proposed by Hannig et. al. [12]. We obtain the estimation of the unknown parameters of the Chen distribution with the GFI method as an alternative to the maximum likelihood estimation (MLE) and Bayesian estimation methods. We also provide the MLE and Bayesian estimation methods to compare the performances of the estimates and their corresponding confidence intervals. All theoretical outcomes are illustrated with simulation studies and a real-data example.

2. Maximum likelihood estimations (MLE)

The likelihood function of the observed sample from the Chen distribution is given as

$$L(\boldsymbol{x}, \lambda, \beta) = \lambda^n \beta^n e^{(\beta-1)\sum_{i=1}^{n}\log(x_i)} e^{\sum_{i=1}^{n} x_i^\beta} e^{\lambda \sum_{i=1}^{n}\left(1-e^{x_i^\beta}\right)}$$

and the corresponding log-likelihood function is given as

$$\ell(\boldsymbol{x}, \lambda, \beta) = n\log(\lambda) + n\log(\beta) + (\beta-1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n} x_i^\beta + \lambda\sum_{i=1}^{n}\left(1-e^{x_i^\beta}\right).$$

To obtain the MLEs of the parameters, denoted by $\hat{\lambda}$ and $\hat{\beta}$ we should equate the partial derivates of $\ell(\boldsymbol{x}, \lambda, \beta)$ to zero with respect to λ and β respectively. Then we obtain the MLE of λ as given in the following

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n}\left(e^{x_i^{\hat{\beta}}}-1\right)},$$

where $\hat{\beta}$ is the solution of the following non-linear equation

$$\xi(\beta) = \frac{n}{\beta} + \sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n} x_i^\beta \log(x_i) - \frac{n\sum_{i=1}^{n}\left(e^{x_i^\beta} x_i^\beta \log(x_i)\right)}{\sum_{i=1}^{n}\left(e^{x_i^\beta}-1\right)}.$$

Since $\hat{\beta}$ is a fixed point solution of the nonlinear equation $\xi(\beta)$, it can be obtained by using numerical methods such as the Newton-Raphson algorithm. Further, the confidence interval for λ and β can be obtained by the following asymptotic normality when the MLE and its large sample theory exist. That is

$$(\hat{\lambda}, \hat{\beta})^T \longrightarrow N\left((\lambda, \beta)^T, \mathbf{I}_n^{-1}\right),$$

where

$$\mathbf{I}_n^{-1} = -\begin{pmatrix} \frac{\partial^2 \ell}{\partial \lambda^2} & \frac{\partial^2 \ell}{\partial \lambda \partial \beta} \\ \frac{\partial^2 \ell}{\partial \beta \partial \lambda} & \frac{\partial^2 \ell}{\partial \beta^2} \end{pmatrix}^{-1} = \begin{pmatrix} \mathrm{Var}(\hat{\lambda}) & \mathrm{Cov}(\hat{\lambda}, \hat{\beta}) \\ & \mathrm{Var}(\hat{\beta}) \end{pmatrix}.$$

The inverse of the expected Fisher information matrix can be obtained by ***mle.tools*** package [16] in ***R*** [6] software. The presentation of the derivatives is skipped for the sake of simplicity.

Thus, the asymptotic $100(1-\alpha)\%$ confidence intervals (ACI) for λ and β are

$$I^\lambda : \hat{\lambda} \pm z_{1-\alpha/2}\sqrt{\mathrm{Var}(\hat{\lambda})} \quad \text{and} \quad I^\beta : \hat{\beta} \pm z_{1-\alpha/2}\sqrt{\mathrm{Var}(\hat{\beta})},$$

where z_δ denotes $100\delta\%$ percentile of the standard normal distribution.

3. Generalized fiducial inference (GIF)

The main structure of the generalized fiducial inference (GFI) is similar to the likelihood method. It differs from the likelihood method by switching the roles of the data $\boldsymbol{x}$ and the model parameters $\boldsymbol{\theta}$. Let suppose that the data generating equation be

$$\boldsymbol{x} = \boldsymbol{G}(\boldsymbol{U}, \boldsymbol{\theta}),$$

where $\boldsymbol{x}$ is the data, $\boldsymbol{\theta}$ is the parameters, $\boldsymbol{U}$ is a complete known random vector and $\boldsymbol{G}$ is called the structural equation. It is seen from Eq. (3), the distribution of $\boldsymbol{x}$ is determined by using the parameters $\boldsymbol{\theta}$ and random vector $\boldsymbol{U}$. Under some differentiability conditions, Hannig et al. [12] showed that the generalized fiducial distribution for $\boldsymbol{\theta}$ is absolutely continuous with the density

$$f_F(\theta) = \frac{L(x \mid \theta) J(x; \theta)}{\int L\left(x \mid \theta'\right) J\left(x; \theta'\right) \mathrm{d}\theta'},$$

where $L(x|\theta)$ denotes the joint likelihood function of observed data and

$$J(x, \theta) = \sum\nolimits_{\left(i_1, \ldots, i_p\right)} \left| \det \left(\left(\tfrac{\partial}{\partial u} \boldsymbol{G}(u, \boldsymbol{\theta})\right)^{-1} \tfrac{\partial}{\partial \theta} \boldsymbol{G}(u, \theta) \right) \right|_{u = G^{-1}(x, \theta)} \tag{3.1}$$

where the above sums go $\binom{n}{p}$ over of p-tuples of indexes $i = 1 \leq i_1 < \cdots < i_p \leq n$, $\partial G(u, \theta)/\partial \theta$ and $\partial G(u, \theta)/\partial u$ are respectively $n \times p$ and $n \times n$ Jacobian matrices. For the Chen distribution, we have

$$U_i = F\left(x_i; \lambda, \beta\right), \quad i = 1, \ldots, n, \tag{3.2}$$

where $F(x_i; \lambda, \beta) \equiv 1 - e^{\lambda\left(1 - e^{x^\beta}\right)}$ is the distribution function of the Chen model and U_i denotes the sample from uniform distribution on the range $(0, 1)$. Further, the data generating equation, $x = G(U, \theta)$, can be obtained from Eq. (3.2) and the ith component $x_i = G(U_i, \lambda, \beta)$ can be obtained as

$$x_i = \left[\ln\left(1 - (1/\lambda) \ln(1 - u)\right) \right]^{1/\beta},$$

and we have

$$\left.\frac{\partial G_i}{\partial \lambda}\right|_{u_i = 1 - e^{\lambda\left(1 - e^{x^\beta}\right)}} = \frac{1}{\lambda \beta} \left(e^{-x_i^\beta} - 1\right) x_i^{1-\beta} \qquad \text{and} \qquad \left.\frac{\partial G_i}{\partial \beta}\right|_{u_i = 1 - e^{\lambda\left(1 - e^{x^\beta}\right)}} = -\frac{x_i \ln(x_i)}{\beta}. \tag{3.3}$$

Then, by replacing (3.3) in (3.1) we obtain

$$J(x; \lambda, \beta) = \frac{1}{\lambda \beta^2} \sum_{1 \leq i < j \leq n} \left| g(x_i, x_j, \lambda) \right|,$$

where

$$g(x_i, x_j, \beta) = x_j^{1-\beta} \left(e^{-x_j^\beta} - 1\right) x_i \ln(x_i) - x_i^{1-\beta} \left(e^{-x_i^\beta} - 1\right) x_j \ln(x_j)$$

in that $J(x; \lambda, \beta)$ plays a similar role with the prior distribution in the Bayesian inference and it reveals like a data dependent prior. The joint likelihood function of the observed data is obtained as

$$f_F(\lambda, \beta) \propto \lambda^{n-1} \beta^{n-2} e^{(\beta - 1) \sum_{i=1}^n \log(x_i)} e^{\sum_{i=1}^n x_i^\beta} e^{\lambda \sum_{i=1}^n \left(1 - e^{x_i^\beta}\right)} \sum_{1 \leq i < j \leq n} \left| g(x_i, x_j, \beta) \right|.$$

Thus, the conditional fiducial density function of λ can be obtained in form of the gamma density as

$$f_F(\lambda | \beta, x) = \lambda^{n-1} e^{-\lambda \sum_{i=1}^n \left(e^{x_i^\beta} - 1\right)} \mathrm{GA}\left(n, \sum_{i=1}^n \left(e^{x_i^\beta} - 1\right)\right).$$

Then, the conditional fiducial density function of β is

$$f_F(\beta|\lambda, x) = \beta^{n-2} e^{(\beta-1)\sum_{i=1}^n \log(x_i) + \sum_{i=1}^n x_i^\beta + \lambda \sum_{i=1}^n \left(1 - e^{x_i^\beta}\right)} \sum_{1 \le i < j \le n} |g(x_i, x_j, \beta)| \,.$$

It is clearly seen that estimates of the parameters can be obtained by using the Gibbs algorithm since their conditional densities are obtained. The conditional estimates of λ can be easily generated from the gamma density. However, the density of β in $f_F(\beta|\lambda, x)$ can not be reduced analytically to well known distributions and therefore it is not possible to sample directly by standard methods. The conditional posterior density of β is observed that, it is likely to be the Gaussian distribution.

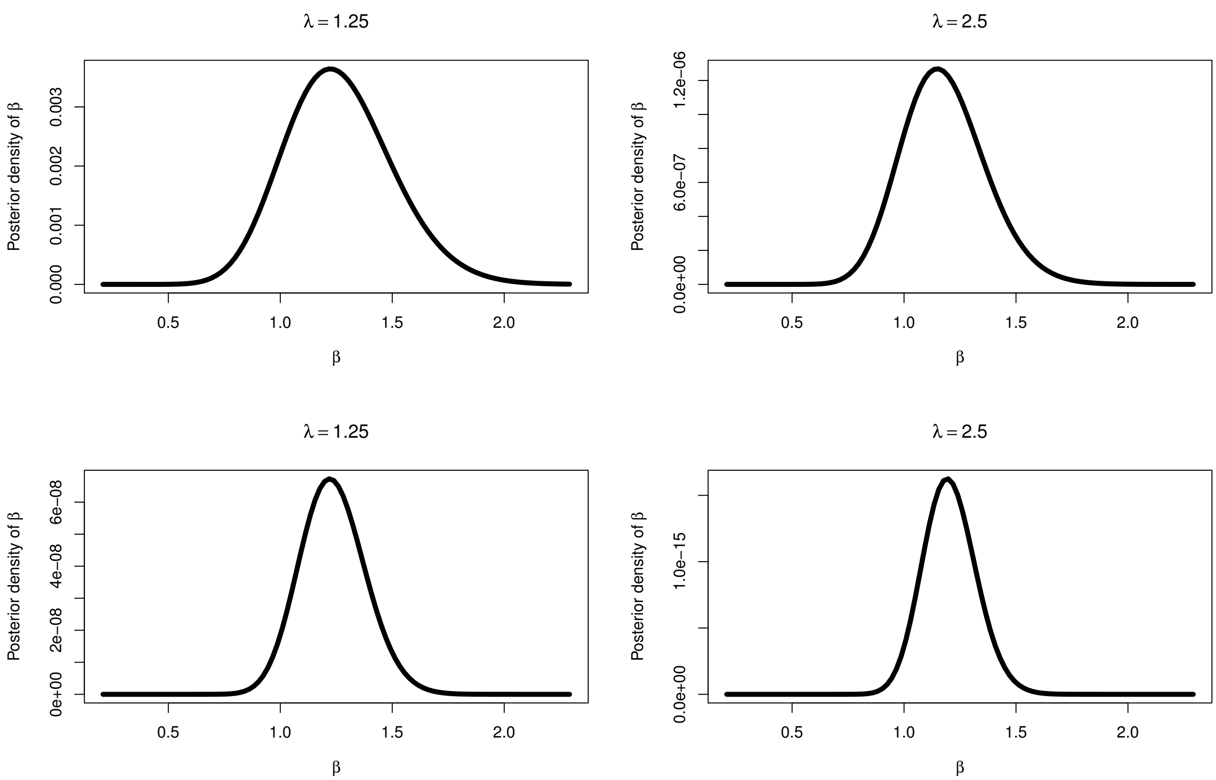


FIGURE 2. The posterior fiducial density of β for $n = 20$ (up) and $n = 50$ (down)

In this case, we propose to use the Metropolis-Hasting (M-H) sampling in Gibbs algorithm with normal proposal distribution (see Figure 2) as suggested by Tierney [20]. The Gibbs algorithm with the M-H sampling for the fiducial inference of the Chen distribution can be given as follows:

- **Step 1:** Start by using the initial values of $\lambda^{(0)}$ and $\beta^{(0)}$.
- **Step 2:** Set $t = 1$.
- **Step 3:** Generate $\lambda^{(t)}$ from $\text{GA}\left(n, \sum_{i=1}^n \left(e^{x_i^\beta} - 1\right)\right)$.
- **Step 4:** Draw a candidate $\beta^{(t)}$ from $f_F(\beta|\lambda, x)$ by using Metropolis-Hastings methods with normal proposal.
- **Step 5:** Repeat 2-4, M times.

It is known that a Markov chain algorithm naturally generates autocorrelated samples (Zhang, [29]) and so we should use a thinning operation to reduce the produced autocorrelation. We discard the first B_0 values as burn-in period and take every $L-$th (as an integer) observation from the remaining $(M-B_0)$ variates as an independent and identically distributed (i.i.d.) observation in thinning procedure. Thus, we obtain $M^{'}=(M-B_0)/L$ i.i.d. observations. If we denote the thinning procedure applied observations as $\alpha_i^{(t^{'})}$ and $\lambda_i^{(t^{'})}$ for $i=(B_0+L, B_0+2L, B_0+3L, \cdots, M)$, we obtain the fiducial inferences of the parameters as

$$\hat{\lambda}_F = \frac{1}{M'}\sum_{i=B_0+L}^{M}\lambda_i^{(t^{'})} \qquad \text{and} \qquad \hat{\beta}_F = \frac{1}{M'}\sum_{i=B_0+L}^{M}\beta_i^{(t^{'})}.$$

Then, the $100(1-\alpha)\%$ the fiducial cofidence intervals are

$$I_F^{\hat{\lambda}} \cong \left[\hat{\lambda}_{\alpha/2}, \hat{\lambda}_{1-\alpha/2}\right] \quad \text{and} \quad I_F^{\hat{\beta}} \cong \left[\hat{\beta}_{\alpha/2}, \hat{\beta}_{1-\alpha/2}\right],$$

where $\hat{\lambda}_\alpha$ and $\hat{\beta}_\alpha$ are the $100\alpha\%$th quantile of the $\lambda_i^{(t^{'})}$ and $\beta_i^{(t^{'})}$.

4. Bayesian inference

This section deals with the Bayesian inference to provide comparative estimates for the fiducial and maximum likelihood inference of the parameters. Since $J(x;\lambda,\beta)$ plays a similar role with the prior distribution in the Bayesian context and similarity of the mathematical structure of the fiducial inference process, the Bayesian inference method is handled as an alternative inference procedure. For this purpose, we first assume that the unknown parameters λ and β follow independent gamma priors such that $\pi(\lambda)\sim \mathrm{GA}(a_1,b_1)$ and $\pi(\beta)\sim \mathrm{GA}(a_2,b_2)$ with density functions are given as in the following

$$\pi(\lambda)\propto \lambda^{a_1-1}e^{-\lambda b_1} \qquad \text{and} \qquad \pi(\beta)\propto \beta^{a_2-1}e^{-\beta b_2},$$

where hyper parameters a_i and b_i, $(i=1,2)$ are assumed as non-negative and known.

The joint posterior density function of data, λ and β can be obtained by using the observed sample and the prior distributions for the parameters as in the following

$$\mathcal{L}(X,\lambda,\beta)=\mathcal{L}(X|\lambda,\beta)\pi(\lambda)\pi(\beta)$$

and the joint posterior density of λ and β given data is obtained by

$$\mathcal{L}(\lambda,\beta|X)=\frac{\mathcal{L}(X|\lambda,\beta)\pi(\lambda)\pi(\beta)}{\int_0^\infty\int_0^\infty \mathcal{L}(X|\lambda,\beta)\pi(\lambda)\pi(\beta)d\lambda d\beta}$$

and for the Chen distribution we have the following joint posterior density of the parameters

$$\mathcal{L}(\lambda,\beta|X)\propto \lambda^{n+a_1-1}\beta^{n+a_2-1}e^{-\beta\left(b_2-\sum_{i=1}^n\log(x_i)\right)}e^{\sum_{i=1}^n x_i^\beta}e^{-\lambda\left(b_1+\sum_{i=1}^n\left(e^{x_i^\beta}-1\right)\right)}.$$

Then, it is easily seen that the conditional posterior density functions of λ and β, denoted by $f_B(\lambda|\beta,x)$ and $f_B(\beta|\lambda,x)$, are obtained as

$$f_B(\lambda|\beta,x)\propto \lambda^{n+a_1-1}e^{-\lambda\left(b_1+\sum_{i=1}^n\left(e^{x_i^\beta}-1\right)\right)} \propto \mathrm{GA}(n+a_1,\sum_{i=1}^n\left(e^{x_i^\beta}-1\right)+b_1)$$

and

$$f_B(\beta|\lambda,x)\propto \beta^{n+a_2-1}e^{\beta\left(\sum_{i=1}^n\log(x_i)-b_2\right)+\sum_{i=1}^n x_i^\beta+\lambda\sum_{i=1}^n\left(1-e^{x_i^\beta}\right)}.$$

As in the fiducial process, we can easily generate samples of λ from the gamma density and Metropolis-Hasting with normal proposal is needed distribution for β. Thus, the point estimates of the parameters, $\hat{\lambda}_B$ and $\hat{\beta}_B$, can be obtained using the Gibbs sampling algorithm which was described in the fiducial process. Finally, the highest posterior density (HPD) $100(1-\gamma)\%$ credible intervals for the Bayesian estimates proposed by Chen and Shao [3] can be constructed as

$$I_B^{\hat{\lambda}} \cong \left(\hat{\lambda}_{B[\frac{\gamma}{2}(M-B_0)]}, \hat{\lambda}_{B[(1-\frac{\gamma}{2})(M-B_0)]}\right) \qquad \text{and} \qquad I_B^{\hat{\beta}} \cong \left(\hat{\beta}_{B[\frac{\gamma}{2}(M-B_0)]}, \hat{\beta}_{B[(1-\frac{\gamma}{2})(M-B_0)]}\right)$$

where $[\frac{\gamma}{2}(M-B_0)]$ and $[(1-\frac{\gamma}{2})(M-B_0)]$ are the smallest integers less than or equal to $\frac{\gamma}{2}(M-B_0)$ and $(1-\frac{\gamma}{2})(M-B_0)$, respectively.

5. Simulation studies

In this section, we perform some simulation studies to evaluate the performances of the generalized fiducial (GFI), ML and Bayesian estimators for the shape parameters of the Chen distribution. We consider three different combinations of the parameters (λ, β) as $(1.25, 1.25)$, $(0.50, 0.75)$ and $(2.00, 1.00)$. Small, moderate and larger sample sizes are considered as 10, 25 and 50, respectively. We run Markov chain with 3500 iteration, the first 500 values are discarded as Burn-in period then every third observation are taken in thinning procedure to generate uncorrelated and independent Markov chains. We replicate each chain 1000 times. The estimations are evaluated with their biases and mean squared errors (MSE). Further, we provide 95% approximate confidence intervals of the estimations and evaluated them according to their average lengths (AL) and coverage probabilities (CP). In the Bayesian estimations, we use the small and non-negative hyper-parameters as $a_1 = a_2 = b_1 = b_2 = 0.0001$ suggested by Congdon ([5], page 69) which are almost like Jeffrey's priors but they are proper, inversely. The biases and the MSEs of the estimates are reported in Table 1 and 2 then the corresponding credible intervals with their ALs and CPs are given in Table 3 and 4.

TABLE 1. The performances of estimations for β based on GFI, MLE and Bayesian methods

$\hat{\beta}$			Bias			MSE		
λ	β	n	GFI	MLE	BYS	GFI	MLE	BYS
1.25	1.25	10	0.02782	0.23687	0.08564	0.15144	0.22183	0.17905
1.25	1.25	25	0.01232	0.09554	0.04193	0.05587	0.06492	0.05984
1.25	1.25	50	0.00273	0.04103	0.01636	0.02553	0.02742	0.02650
0.50	0.75	10	0.02556	0.11549	0.02966	0.03893	0.05390	0.04403
0.50	0.75	25	0.01128	0.04732	0.01714	0.01466	0.01632	0.01521
0.50	0.75	50	0.00267	0.02032	0.00608	0.00666	0.00702	0.00682
2.00	1.00	10	0.02283	0.18994	0.06587	0.10344	0.14840	0.11809
2.00	1.00	25	0.00790	0.07682	0.03285	0.03770	0.04354	0.03988
2.00	1.00	50	0.00290	0.03294	0.01268	0.01755	0.01881	0.01809

TABLE 2. The performances of estimations for λ based on GFI, MLE and Bayesian methods

$\hat{\lambda}$				Bias			MSE	
λ	β	n	GFI	MLE	BYS	GFI	MLE	BYS
1.25	1.25	10	0.16140	0.33639	0.23991	2.14111	2.36967	2.26649
1.25	1.25	25	0.05823	0.10341	0.07452	1.10221	1.14054	1.11397
1.25	1.25	50	0.01813	0.03750	0.02475	0.73433	0.74718	0.73594
0.50	0.75	10	0.06494	0.03236	0.05080	0.75997	0.74430	0.74981
0.50	0.75	25	0.02384	0.00724	0.01654	0.45178	0.44779	0.44844
0.50	0.75	50	0.00997	0.00145	0.00619	0.31458	0.31361	0.31244
2.00	1.00	10	0.39813	0.85419	0.64314	4.67404	5.32950	5.25041
2.00	1.00	25	0.13142	0.25914	0.18812	2.16615	2.27428	2.21826
2.00	1.00	50	0.04035	0.09662	0.06515	1.38913	1.42470	1.39976

TABLE 3. The performances of aproximate confidence intervals for β based on GFI, MLE and Bayesian methods

$\hat{\beta}$				AL			CP	
λ	β	n	GFI	MLE	BYS	GFI	MLE	BYS
1.25	1.25	10	1.54294	1.62707	1.59565	94.60	96.30	95.20
1.25	1.25	25	0.90351	0.91553	0.90552	95.30	95.20	95.40
1.25	1.25	50	0.60660	0.61492	0.60934	93.70	94.40	94.00
0.50	0.75	10	0.79377	0.80036	0.79757	94.70	93.70	95.50
0.50	0.75	25	0.45399	0.45716	0.45258	95.10	94.10	95.10
0.50	0.75	50	0.30712	0.30958	0.30627	93.20	93.90	93.40
2.00	1.00	10	1.25740	1.33258	1.29795	94.50	95.90	96.00
2.00	1.00	25	0.74608	0.75799	0.74957	95.50	95.60	95.70
2.00	1.00	50	0.50429	0.51124	0.50675	93.40	94.20	93.80

TABLE 4. The performances of aproximate confidence intervals for λ based on GFI, MLE and Bayesian methods

$\hat{\lambda}$				AL			CP	
λ	β	n	GFI	MLE	BYS	GFI	MLE	BYS
1.25	1.25	10	2.14111	2.36967	2.26649	96.90	95.90	95.60
1.25	1.25	25	1.10221	1.14054	1.11397	96.20	96.20	95.50
1.25	1.25	50	0.73433	0.74718	0.73594	95.10	95.30	94.60
0.50	0.75	10	0.75997	0.74430	0.74981	96.30	92.10	95.20
0.50	0.75	25	0.45178	0.44779	0.44844	94.80	94.00	94.30
0.50	0.75	50	0.31458	0.31361	0.31244	94.90	94.00	95.50
2.00	1.00	10	4.67404	5.32950	5.25041	95.70	96.40	95.60
2.00	1.00	25	2.16615	2.27428	2.21826	95.80	97.50	95.60
2.00	1.00	50	1.38913	1.42470	1.39976	95.00	96.80	95.10

We observe satisfying consistency in the performances of the estimators. The biases, MSEs and the ALs of the confidence intervals decrease parallel to increasing sample sizes in all sets of the parameters. In whole cases, the GFI estimates have smaller biases, MSEs and ALs even in the small samples that are powerful side of the Bayesian estimation method. The differences between the performances of the proposed estimators are decreasing with the increasing sample sizes. In the whole case, the CPs are pretty close to their actual value 0.95. Many various values of the parameters are performed but only a few of them are reported here.

6. Numerical example

In this section, a real-life data is illustrated to compare different estimation procedures studied in this study. The data set which is given by Hand et al. [9] is handled. This data represents the survival period of 45 patients treated with chemotherapy. This data set is also fitted for the Chen distribution by Kayal et al. [13]. The data is given below as

1 63 105 129 182 216 250 262 301 301 342 354 356 358 380 383 383
388 394 408 460 489 499 523 524 535 562 569 675 676 748 778 786
797 955 968 1000 1245 1271 1420 1551 1694 2363 2754 2950

We divided data points by 3000 to simplify computations. Then, we fit this dataset with the Chen distribution by using the MLE, GFI and Bayesian inference methods.

We also evaluate the convergence of the Markov chains. We perform Markov chains 100 500 times and we discard the first 500 values as burn-in period and we count in every tenth variate as independent and uncorrelated samples. Thus, we obtain 10 000 uncorrelated and independent samples. In the Bayesian procedure, we use very small non-negative values of the hyper-parameters, i.e. $a_1 = a_2 = b_1 = b_2 = 0.0001$, as suggested by Congdon ([5], page 69) which are almost like Jeffrey's priors but they are proper, inversely.

The parameter estimates of the parameters and their corresponding confidence intervals are obtained as given in Tables 5-6, respectively.

Table 5. Estimations, K-S test values and p- values for the real data example

	MLE	GFI	Bayes
λ	3.1979	3.0759	3.1290
β	0.9804	0.9444	0.9571
K-S	0.1778 (0.4756)	0.1556 (0.6476)	0.2222 (0.2165)

Table 6. Confidence intervals with their lengths for the real data example

	ACI	FCI	BCI
λ	(1.9157,4.4803)	(1.9810,4.4835)	(1.9921, 4.5752)
	2.5646	2.5025	2.5831
β	(0.7348,1.2261)	(0.7108,1.2014)	(0.7212,1.2079)
	0.4914	0.4906	0.4867

The Kolmogorov–Smirnov (KS) test statistics and the associated p-values for all inference procedures are obtained as bigger than 0.05. Therefore, we can not reject the null hypothesis that this data set comes from the Chen distribution. Also, the estimated density and the emprical cdf plots support this observation as seen in Figure 3.

Further, the convergence of the Markov chains is evaluated with trace (Figures 4-5), density (Figures 6-7) and running mean (ergodic average) (Figures 8-9) plots. A trace plot is a plot of the parameter values in each iteration of the Markov chain against the iteration number. It is expected to observe that the Markov chain disperses around its center with a similar variation. In our example, trace plots of the Markov chains provide expectations and fluctuate around their centers with similar variations. Further, the posterior density plots of λ and β via GFI and Bayesian methods obtained almost symmetrical and in the shapes of unimodal. The trace, density and

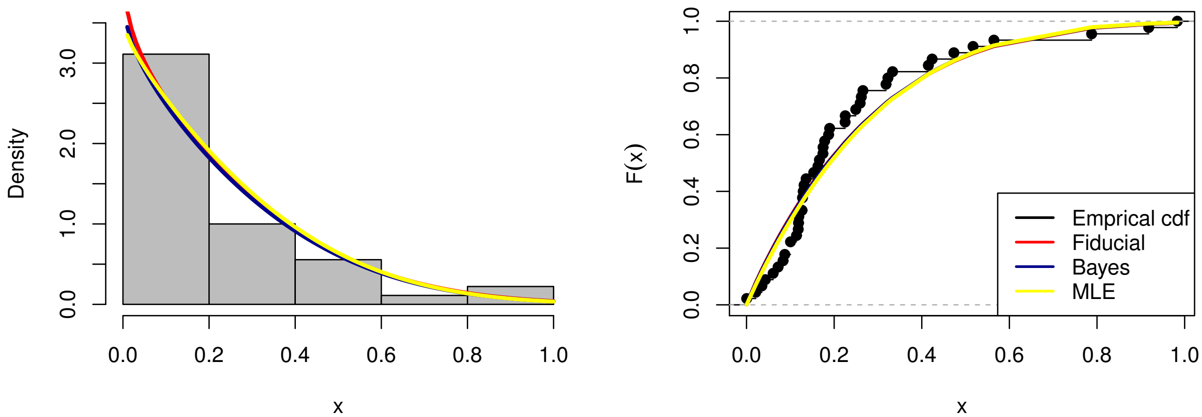


FIGURE 3. Estimated density and empirical cdf for real-data example fitted by the Chen distribution

running mean plots can be drawn using by the **traplot**, **denplot** and **rmeanplot** functions in library **mcmcplots** (Curtis et al., [7]) in **R** [6] software.

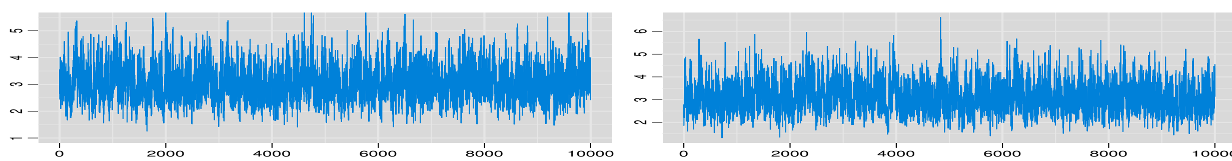
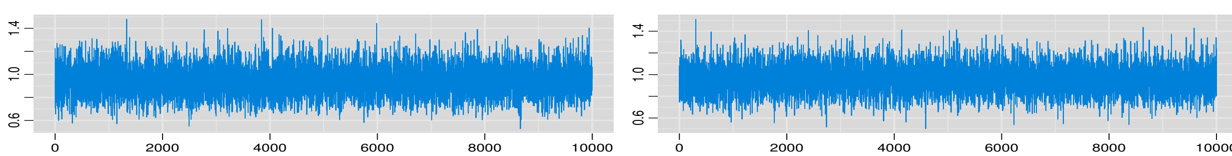


FIGURE 4. Trace plots for λ via GFI (on left) and the Bayesian (on right) methods



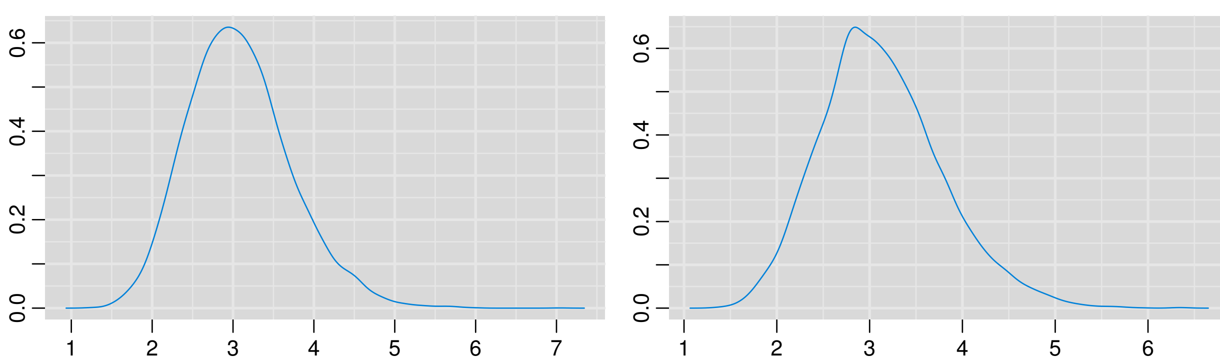


FIGURE 6. Density plot for the posterior distribution of λ via GFI (on left) and the Bayesian (on right) methods

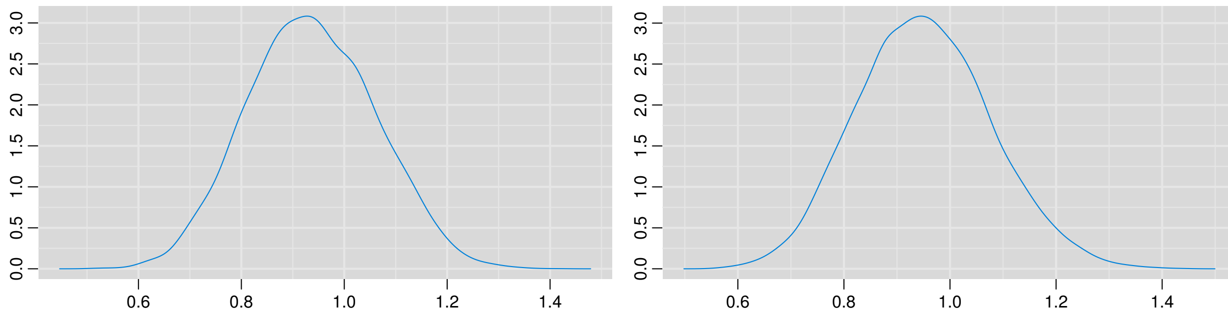


FIGURE 7. Density plot for the posterior distribution of β via GFI (on left) and the Bayesian (on right) methods

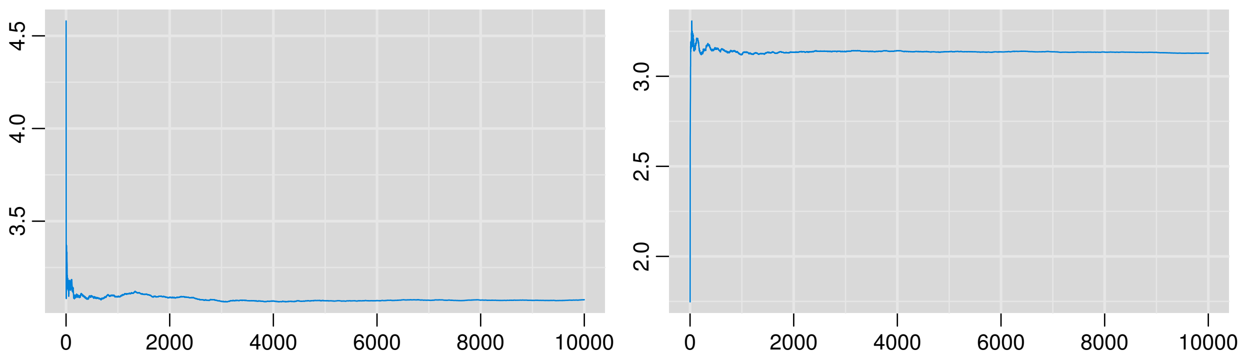


FIGURE 8. Running mean plot for λ via GFI (on left) and the Bayesian (on right) methods

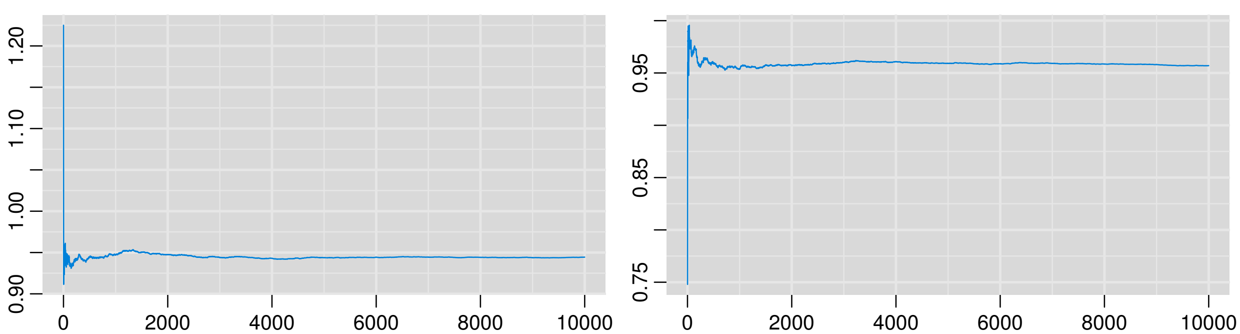


FIGURE 9. Running mean plot for β via GFI (on left) and the Bayesian (on right) methods

7. Conclusions

On the basis of this study, the generalized fiducial inference is considered for the parameter estimates of the Chen distribution. Further, the MLE and Bayesian procedures are handled as alternative methods to this inference method. The performances of the simulation schemes show that the GFI method has superiority in parameter estimations of the Chan distribution over the classical and Bayesian estimation methods. The GFI method provides better results than MLE and Bayesian methods in most cases even in the case of small, moderate or large sample sizes. The theoretical findings are also evaluated on a real data example. Additionally, the convergence of the Markov chains generated in the GFI and Bayesian procedures are provided. These observations are supported by the graphical methods. Consequently, the generalized fiducial inference method based on the inverse of the structural equation which is proposed by Hannig et. al. [12] should be proposed as a more efficient estimator for the parameter estimation of the Chen distribution.

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