Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (6) (2023), 1578 – 1599
DOI : 10.15672/hujms.1118138

Special Issue Paper

An inverse problem of identifying the time-dependent potential and source terms in a two-dimensional parabolic equation

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Abstract

In this article, simultaneous identification of the time-dependent lowest and source terms in a two-dimensional (2D) parabolic equation from knowledge of additional measurements is studied. Existence and uniqueness of the solution is proved by means of the contraction mapping on a small time interval. Since the governing equation is yet ill-posed (very slight errors in the time-average temperature input may cause relatively significant errors in the output potential and source terms), we need to regularize the solution. Therefore, regularization is needed for the retrieval of unknown terms. The 2D problem is discretized using the alternating direction explicit (ADE) method and reshaped as non-linear least-squares optimization of the Tikhonov regularization function. This is numerically solved by means of the MATLAB subroutine *lsqnonlin* tool. Finally, we present a numerical example to demonstrate the accuracy and efficiency of the proposed method. Our numerical results show that the ADE is an efficient and unconditionally stable scheme for reconstructing the potential and source coefficients from minimal data which makes the solution of the inverse problem (IP) unique.

Mathematics Subject Classification (2020). 35R30, 65K10, 35A02

Keywords. existence and uniqueness, contraction mapping, 2D parabolic equation, inverse identification problem, nonlinear optimization, Tikhonov regularization

1. Introduction

Multi-dimensional parabolic equations are mathematical models arise in various processes such as financial market behaviour, seawater desalination, bioheat transfer, fluid dynamics, etc., see [5–8, 18] to mention only a few. Solvability of inverse problems for a multi-dimensional parabolic equation has been attracted attention by many authors. Amongst these inverse problems, great interest is paid to determining the lowest order coefficient in the parabolic equation, especially when this coefficient is dependent only on time. Authors studied the determination of the solely time-dependent diffusion coefficient in a two-dimensional parabolic equation with different boundary and additional

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Received: 18.05.2022; Accepted: 13.01.2023

measurements in [2, 37, 40]. In the references [1, 4, 35], and [36], authors considered the inverse time-dependent lowest term identification problem with various classical (Dirichlet, Neumann, and Robin) and non-classical boundary conditions. On contrary of these references authors investigated the solvability of the inverse problem for both space and time-dependent coefficient in [46]. The authors in [13] investigated the inverse problem of estimating a discontinuous parameter in a quasi-variational inequality involving multi-valued terms. Zeng et al. [49] studied a dynamical system called a differential variational-hemivariational inequality of elliptic type and a nonlinear evolution inclusion problem in a Banach space while the authors in [50] considered a mixed boundary value problem with a nonhomogeneous under very general assumptions on the data. Yamamoto [48] has obtained the Hölder stability estimate for the parabolic equation with the unknown coefficient in a general domain with smooth boundary. In references [38, 39] multidimensional inverse problems for the parabolic equations are also investigated in general domains with smooth boundaries.

In recent years, the authors numerically investigated various inverse problems related to the determination of time-dependent coefficients [20–22, 24–27, 31–34, 43]. The authors in [12] estimated free boundary coming from two new scenarios, aggregation processes and nonlocal diffusion. Snitko [47], theoretically, and Huntul [23], numerically, investigated the inverse problem of determining the time-dependent reaction coefficient in a two-dimensional parabolic problem. Furthermore, Huntul et al. [28, 29] studied numerically the inverse problems for reconstructing the unknown coefficients in a third-order pseudo-parabolic equation from additional and nonlocal integral observations, respectively. Huntul et al. [30] identified the time-dependent potential in a fourth-order pseudo-hyperbolic equation from additional measurement.

In this article, we study the two-dimensional parabolic equation to identify the time-dependent lowest and source function coefficients along with the solution function theoretically, i.e. existence and uniqueness, and numerically, for the first time, in the rectangular domain, using the initial, homogeneous boundary conditions and the additional data as over-specification conditions. The pre-eminent goal of the current work is to undertake the theory and numerical aspect of this problem.

The paper is organized as follows. The proposed inverse problem has been mathematically developed in Section 2. The existence and uniqueness of the solution of the inverse problem is proved in Section 3. Section 4 briefly explains the scheme for solving the direct problem by means of ADE. The description of numerical procedure to solve the minimization of the nonlinear functional has been given in Section 5. The computational outcomes for benchmark test example on the topic are discussed in Section 6. Finally, concluding remarks are revealed in Section 7.

2. Mathematical formulation of the IP

We consider an inverse problem of recovering the time-dependent coefficients $a(t)$ and $b(t)$ in the two-dimensional parabolic equation

$$z_t(x,y,t) - \alpha(t)\left(z_{xx}(x,y,t) + z_{yy}(x,y,t)\right) = a(t)z(x,y,t) + b(t)g(x,y,t) + f(x,y,t), \quad (x,y,t) \in Q_T, \tag{2.1}$$

subject to the initial condition

$$z(x,y,0) = \varphi(x,y), \quad (x,y) \in Q_{xy}, \tag{2.2}$$

the homogeneous boundary conditions

$$z(0,y,t) = z(1,y,t) = 0, \quad (y,t) \in [0,1] \times [0,T], \tag{2.3}$$

$$z(x,0,t) = z(x,1,t) = 0, \quad (x,t) \in [0,1] \times [0,T], \tag{2.4}$$

and over-specification conditions

$$\iint\limits_{Q_{xy}} z(x,y,t)dxdy = E_1(t), \quad t \in [0,T], \tag{2.5}$$

$$z(x_0,y_0,t) + \iint\limits_{Q_{xy}} K(x,y)z(x,y,t)dxdy = E_2(t), \quad t \in [0,T], \tag{2.6}$$

where $Q_T = Q_{xy} \times [0,T]$, Q_{xy} is the rectangle $[0,1] \times [0,1]$, $\alpha(t) \in C[0,T]$ with $\alpha(t) > 0$, $(x_0,y_0) \in (0,1) \times (0,1)$, $K(x,y) \in C(Q_{xy})$, $\varphi(x,y)$, $g(x,y,t)$, $f(x,y,t)$, $E_1(t)$, and $E_2(t)$ are given functions. Physical situations in which the measurement $E_2(t)$ in (2.6) depends on time occur in mass (energy), damage and radioactive decay applications, [10, 11].

3. Existence and uniqueness of the solution of the IP

In this section, first we will give eigenfunctions and eigenvalues of the auxiliary spectral problem and define two useful Banach spaces. Then we will set the existence and uniqueness theorem of the solution of the inverse initial-boundary value problem for the two-dimensional parabolic equation and and prove this theorem by using Banach fixed point theorem.

The spectral problem corresponding to the inverse problem (2.1)–(2.6) is

$$\begin{cases} W_{xx}(x,y) + W_{yy}(x,y) + \mu W(x,y) = 0, & (x,y) \in Q_{xy}, \\ W(0,y) = W(1,y) = 0, & 0 \le y \le 1, \\ W(x,0) = W(x,1) = 0, & 0 \le x \le 1, \end{cases} \tag{3.1}$$

where μ is the separation parameter. The eigenvalues and eigenfunctions of problem (3.1) are

$$\mu_{m,k} = (m\pi)^2 + (k\pi)^2, \quad m,k = 1,2,...,$$

and

$$W_{m,k}(x,y) = 2\sin(m\pi x)\sin(k\pi y), \quad m,k = 1,2,....$$

The problem (3.1) is self-adjoint. It is easy to seen that $\sqrt{2}\sin(m\pi x)$ and $\sqrt{2}\sin(k\pi y)$ are complete orthonormal systems on $[0,1]$. Thus the set of eigenfunctions $W_{m,k}(x,y)$ is complete in $L_2(Q_{xy})$ and forms an orthonormal system on Q_{xy}, see [35].

Now let us give the following spaces which are Banach spaces:

I:

$$\begin{aligned} B_T \;=\; \Bigg\{ & z(x,y,t) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} z_{m,k}(t)W_{m,k}(x,y) : \; z_{m,k}(t) \in C[0,T], \\ & J_T(z) = \left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} (\mu_{m,k}^{3/2} \left\| z_{m,k}(t) \right\|_{C[0,T]})^2 \right)^{1/2} < +\infty \Bigg\}, \end{aligned}$$

with the norm $\left\| z(x,y,t) \right\|_{B_T} \equiv \left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} (\mu_{m,k}^{3/2} \left\| z_{m,k}(t) \right\|_{C[0,T]})^2 \right)^{1/2}$ which is related with the Fourier coefficients of the function $z(x,y,t)$ by the eigenfunctions $W_{m,k}(x,y) = 2\sin(m\pi x)\sin(k\pi y)$, $m,k = 1,2,....$

II: $E_T = B_T \times C[0,T] \times C[0,T]$ with the norm

$$\left\| v(x,y,t) \right\|_{E_T} = \left\| z(x,y,t) \right\|_{B_T} + \left\| a(t) \right\|_{C[0,T]} + \left\| b(t) \right\|_{C[0,T]},$$

where $v(x,y,t) = \{z(x,y,t), a(t), b(t)\}$.

Before giving the main theorem let us set and prove the following Lemmas:

Lemma 3.1. *Let the assumption*

(A$_1$): $\varphi \in C^{2,2}(Q_{xy})$, $\varphi_{xxx}, \varphi_{xxy}, \varphi_{xyy}, \varphi_{yyy} \in L_2(Q_{xy})$,
$\varphi(x,0) = \varphi(x,1) = \varphi_x(x,0) = \varphi_x(x,1) = \varphi_{yy}(x,0) = \varphi_{yy}(x,1) = 0,\ 0 \le x \le 1,$
$\varphi(0,y) = \varphi(1,y) = \varphi_{xx}(0,y) = \varphi_{xx}(1,y) = 0,\ 0 \le y \le 1;$

be satisfied. Then, the estimates

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left((m\pi)^3 |\varphi_{m,k}|\right)^2 \le \|\varphi_{xxx}(.,.)\|_{L_2(Q_{xy})},$$

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(m^2k\pi^3 |\varphi_{m,k}|\right)^2 \le \|\varphi_{xxy}(.,.)\|_{L_2(Q_{xy})},$$

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left((k\pi)^3 |\varphi_{m,k}|\right)^2 \le \|\varphi_{yyy}(.,.)\|_{L_2(Q_{xy})},$$

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(mk^2\pi^3 |\varphi_{m,k}|\right)^2 \le \|\varphi_{xyy}(.,.)\|_{L_2(Q_{xy})}.$$

are valid.

Proof. Let us show the first one of these estimates is true and the others can be proven analogously. Since

$$\varphi_{m,k} = \iint_{Q_{xy}} \varphi(x,y) W_{m,k}(x,y)\,dxdy,$$

where $W_{m,k}(x,y) = 2\sin(m\pi x)\sin(k\pi y)$, using integration by parts under the assumption (A_1), we get

$$\varphi_{m,k} = \frac{-1}{(m\pi)^3}\iint_{Q_{xy}} \varphi_{xxx}(x,y) 2\cos(m\pi x)\sin(k\pi y)\,dxdy.$$

The term $2\cos(m\pi x)\sin(k\pi y)$ is complete and form an orthonormal system in $L_2(Q_{xy})$, because $\sqrt{2}\cos(m\pi x)$ and $\sqrt{2}\sin(k\pi y)$ are complete orthonormal system on $[0,1]$. Thus from the Bessel's inequality

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left((m\pi)^3 |\varphi_{m,k}|\right)^2 \le \|\varphi_{xxx}(.,.)\|_{L_2(Q_{xy})}.$$

$\square$

Here the notation $C^{2,2}(Q_{xy})$ means that all partial derivatives exist and continuous that include at most two derivatives with respect to the first and second variables.

Analogously, we can get the estimates for $f_{m,k}(t)$ and $g_{m,k}(t)$ as:

Lemma 3.2. *Let the assumption*

(A$_2$): $f(x,y,t) \in C(Q_T)$, $f_x, f_y, f_{xx}, f_{yy}, f_{xy} \in C^{2,2}(Q_{xy})$, $\forall t \in [0,T]$, $f_{xxx}, f_{yyy}, f_{xyy}, f_{xxy} \in L_2(Q_T)$, $f(x,0,t) = f(x,1,t) = f_x(x,0,t) = f_x(x,1,t) = f_{yy}(x,0,t) = f_{yy}(x,1,t) = 0$, $(x,t) \in [0,1]\times[0,T]$, $f(0,y,t) = f(1,y,t) = f_{xx}(0,y,t) = f_{xx}(1,y,t) = 0$, $(y,t) \in [0,1]\times[0,T]$;

be satisfied. Then, the estimates

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left((m\pi)^3 |f_{m,k}(t)|\right)^2 \le \|f_{xxx}(\cdot,\cdot,t)\|_{L_2(Q_{xy})},$$

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(m^2k\pi^3 |f_{m,k}(t)|\right)^2 \le \|f_{xxy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})},$$

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left((k\pi)^3\left|f_{m,k}(t)\right|\right)^2 \le \left\|f_{yyy}(\cdot,\cdot,t)\right\|_{L_2(Q_{xy})},$$

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(mk^2\pi^3\left|f_{m,k}(t)\right|\right)^2 \le \left\|f_{xyy}(\cdot,\cdot,t)\right\|_{L_2(Q_{xy})}.$$

are valid.

Lemma 3.3. *Let the assumption*

(A_3): $g(x,y,t)\in C(Q_T), g_x, g_y, g_{xx}, g_{yy}, g_{xy} \in C^{2,2}\left(Q_{xy}\right), \forall t\in[0,T], g_{xxx}, g_{yyy}, g_{xyy}, g_{xxy} \in L_2\left(Q_T\right), g(x,0,t)=g(x,1,t)=g_x(x,0,t)=g_x(x,1,t)=g_{yy}(x,0,t)=g_{yy}(x,1,t)=0, (x,t)\in[0,1]\times[0,T]$, $g(0,y,t)=g(1,y,t)=g_{xx}(0,y,t)=g_{xx}(1,y,t)=0, (y,t)\in[0,1]\times[0,T]$;

be satisfied. Then, the estimates

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left((m\pi)^3\left|g_{m,k}(t)\right|\right)^2 \le \left\|g_{xxx}(\cdot,\cdot,t)\right\|_{L_2(Q_{xy})},$$

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(m^2k\pi^3\left|g_{m,k}(t)\right|\right)^2 \le \left\|g_{xxy}(\cdot,\cdot,t)\right\|_{L_2(Q_{xy})},$$

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left((k\pi)^3\left|g_{m,k}(t)\right|\right)^2 \le \left\|g_{yyy}(\cdot,\cdot,t)\right\|_{L_2(Q_{xy})},$$

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(mk^2\pi^3\left|g_{m,k}(t)\right|\right)^2 \le \left\|g_{xyy}(\cdot,\cdot,t)\right\|_{L_2(Q_{xy})}.$$

are valid.

Theorem 3.4. *Let the assumptions of Lemma 3.1, Lemma 3.2, Lemma 3.3 and*

(A_4): $E_1(t)\in C^1[0,T],\ \iint\limits_{Q_{xy}}\varphi(x,y)dxdy = E_1(0)$;

(A_5): $E_2(t)\in C^1[0,T],\ \varphi(x_0,y_0)+\iint\limits_{Q_{xy}}K(x,y)\varphi(x,y)dxdy = E_2(0)$,

be satisfied, and $D(t)=E_1(t)g_2(t)-E_2(t)g_1(t)\neq 0,\ \forall t\in[0,T]$, where $g_1(t)=\iint_{Q_{xy}} g(x,y,t)dxdy$, and $g_2(t)=g(x_0,y_0,t)+\iint\limits_{Q_{xy}}K(x,y)g(x,y,t)dxdy$. Then, the inverse problem (2.1)–(2.6) has a unique solution for small T.

Proof. For arbitrary $a(t), b(t)\in C[0,T]$, to construct the formal solution of the inverse problem (2.1)–(2.6), we will use the Fourier (Eigenfunction expansion) method. In accordance with this, let us consider

$$z(x,y,t)=\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} z_{m,k}(t)W_{m,k}(x,y), \tag{3.2}$$

is a solution of the inverse problem (2.1)–(2.6), where $W_{m,k}(x,y)$ are the eigenfunctions and $\mu_{m,k}$ are the eigenvalues of the corresponding spectral problem.

Since $z(x,y,t)$ is the formal solution of the inverse problem (2.1)–(2.6), we get the following Cauchy problems with respect to $z_{m,k}(t)$ from the equation (2.1) and initial conditions (2.2),

$$\begin{cases} z'_{m,k}(t)+\mu_{m,k}\alpha(t)z_{m,k}(t)=R_{(m,k)}(t;z,a,b), \\ z_{m,k}(0)=\varphi_{m,k},\ m,k=1,2,..., \end{cases} \tag{3.3}$$

where

$$R_{(m,k)}(t;z,a,b)=a(t)z_{m,k}(t)+b(t)g_{m,k}(t)+f_{m,k}(t),$$

$$z_{m,k}(t)=\iint\limits_{Q_{xy}} z(x,y,t)W_{m,k}(x,y)dxdy,\quad f_{m,k}(t)=\iint\limits_{Q_{xy}} f(x,y,t)W_{m,k}(x,y)dxdy,$$

$$g_{m,k}(t)=\iint\limits_{Q_{xy}} g(x,y,t)W_{m,k}(x,y)dxdy,\quad \varphi_{m,k}=\iint\limits_{Q_{xy}} \varphi(.,.)W_{m,k}(x,y)dxdy.$$

Solving (3.3) we obtain

$$z_{m,k}(t)=\varphi_{m,k}e^{-\int_0^t\mu_{m,k}\alpha(s)ds}+\int_0^t R_{(m,k)}(s;z,a,b)e^{-\int_s^t\mu_{m,k}\alpha(\tau)d\tau}ds. \tag{3.4}$$

Substitute the expression (3.4) into (3.2) to determine $z(x,y,t)$. Then, we get

$$z(x,y,t)=\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left[\varphi_{m,k}e^{-\int_0^t\mu_{m,k}\alpha(s)ds}+\int_0^t R_{(m,k)}(s;z,a,b)e^{-\int_s^t\mu_{m,k}\alpha(\tau)d\tau}ds\right]W_{m,k}(x,y). \tag{3.5}$$

Let us derive the equations of $a(t)$ and $b(t)$. If we integrate the equation (2.1) from $(0,0)$ to $(1,1)$ with respect to x and y, and consider the additional condition (2.5), then we have

$$a(t)E_1(t)+b(t)g_1(t)=E_1'(t)-f_1(t)-\alpha(t)\iint\limits_{Q_{xy}}(z_{xx}+z_{yy})\,dxdy, \tag{3.6}$$

where

$$f_1(t)=\iint\limits_{Q_{xy}} f(x,y,t)dxdy,\quad g_1(t)=\iint\limits_{Q_{xy}} g(x,y,t)dxdy.$$

Since

$$z_{xx}=-\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}z_{m,k}(t)2(m\pi)^2\sin(m\pi x)\sin(k\pi y),$$
$$z_{yy}=-\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}z_{m,k}(t)2(k\pi)^2\sin(m\pi x)\sin(k\pi y),$$

we get

$$\begin{aligned}\iint\limits_{Q_{xy}}(z_{xx}+z_{yy})\,dxdy &= -2\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}z_{m,k}(t)\frac{\mu_{m,k}}{(m\pi)(k\pi)}\left[(-1)^m-1\right]\left[(-1)^k-1\right]\\ &= \begin{cases} 0, & \text{if } m \text{ or } n \text{ is even},\\ -8\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}z_{2m-1,2k-1}(t)\frac{\mu_{2m-1,2k-1}}{(2m-1)(2k-1)\pi^2}, & \text{if } m \text{ and } n \text{are odd}.\end{cases}\end{aligned}$$

Thus from the equation (3.6), we obtain

$$a(t)E_1(t)+b(t)g_1(t)=E_1'(t)-f_1(t)+8\alpha(t)\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}z_{2m-1,2k-1}(t)\frac{\mu_{2m-1,2k-1}}{(2m-1)(2k-1)\pi^2}. \tag{3.7}$$

Similarly, we can obtain

$$a(t)E_2(t)+b(t)g_2(t)=E_2'(t)-f_2(t)+\alpha(t)\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}A_{m,k}\mu_{m,k}z_{m,k}(t), \tag{3.8}$$

where

$$f_2(t) = f(x_0, y_0, t) + \iint\limits_{Q_{xy}} K(x,y) f(x,y,t) dxdy,$$
$$g_2(t) = g(x_0, y_0, t) + \iint\limits_{Q_{xy}} K(x,y) g(x,y,t) dxdy,$$
$$A_{m,k} = W_{m,k}(x_0, y_0) + \iint\limits_{Q_{xy}} K(x,y) W_{m,k}(x,y) dxdy.$$

To derive the equations of $a(t)$ and $b(t)$, let us solve the system of equations (3.7) and (3.8) for $a(t)$ and $b(t)$. Thus we get

$$a(t) = \frac{1}{D(t)}\Bigg[E_1'(t)g_2(t) - E_2'(t)g_1(t) + g_1(t)f_2(t) - g_2(t)f_1(t)$$
$$+\alpha(t)\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(8g_2(t)z_{2m-1,2k-1}(t)\frac{\mu_{2m-1,2k-1}}{(2m-1)(2k-1)\pi^2} - g_1(t)A_{m,k}\mu_{m,k}z_{m,k}(t)\right)\Bigg], \quad (3.9)$$

and

$$b(t) = \frac{1}{D(t)}\Bigg[E_2'(t)E_1(t) - E_1'(t)E_2(t) + E_2(t)f_1(t)$$
$$-E_1(t)f_2(t) + \alpha(t)\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\Bigg(E_1(t)A_{m,k}\mu_{m,k}z_{m,k}(t)$$
$$-8E_2(t)z_{2m-1,2k-1}(t)\frac{\mu_{2m-1,2k-1}}{(2m-1)(2k-1)\pi^2}\Bigg)\Bigg], \quad (3.10)$$

where

$$D(t) = E_1(t)g_2(t) - E_2(t)g_1(t) \neq 0, \quad \forall t \in [0,T].$$

We obtained the system of Volterra integral equations (3.5), (3.9) and (3.10) with respect to $z(x,y,t)$, $a(t)$ and $b(t)$. The inverse problem (2.1)–(2.6) and the system (3.5), (3.9) and (3.10) are equivalent. In other words, solving the system of integral equations (3.5), (3.9) and (3.10) is equivalent to solve the inverse problem (2.1)–(2.6).

To prove the existence of a unique solution of the system (3.5), (3.9) and (3.10) we need to rewrite this system into operator form and to show that this operator a contraction operator. To this end let us denote $v(x,y,t) = \{z(x,y,t), a(t), b(t)\}$ is a triplet of the functions $z(x,y,t), a(t)$, and $b(t)$ and consider the following operator equation

$$v = \Phi(v), \quad (3.11)$$

where $\Phi(z) \equiv \{\phi_1, \phi_2, \phi_3\}$ and ϕ_1, ϕ_2 and ϕ_3 are connected with the equations (3.5), (3.9) and (3.10), respectively, i.e.

$$\phi_1(v) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\Bigg[\varphi_{m,k}e^{-\int_0^t \mu_{m,k}\alpha(s)ds}$$
$$+\int_0^t R_{(m,k)}(s; z, a, b)e^{-\int_s^t \mu_{m,k}\alpha(\tau)d\tau}ds\Bigg]W_{m,k}(x,y), \quad (3.12)$$

$$\phi_2(v) = \frac{1}{D(t)}\Bigg[E_1'(t)g_2(t) - E_2'(t)g_1(t) + g_1(t)f_2(t) - g_2(t)f_1(t)$$

$$+\alpha(t)\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(8g_2(t)z_{2m-1,2k-1}(t)\frac{\mu_{2m-1,2k-1}}{(2m-1)(2k-1)\pi^2} - g_1(t)A_{m,k}\mu_{m,k}z_{m,k}(t)\right)\Bigg], \quad (3.13)$$

and

$$\phi_3(v) = \frac{1}{D(t)}\Bigg[E_2'(t)E_1(t) - E_1'(t)E_2(t) + E_2(t)f_1(t) - E_1(t)f_2(t)$$

$$+\alpha(t)\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(E_1(t)A_{m,k}\mu_{m,k}z_{m,k}(t) - 8E_2(t)z_{2m-1,2k-1}(t)\frac{\mu_{2m-1,2k-1}}{(2m-1)(2k-1)\pi^2}\right)\Bigg] \quad (3.14)$$

It is easy to see that

$$\mu_{m,k}^{3/2} \le \left[(m\pi)^2 + (k\pi)^2\right][m\pi + k\pi] = (m\pi)^3 + m^2k\pi^3 + mk^2\pi^3 + (k\pi)^3,$$

and

$$\begin{aligned}
|A_{m,k}| &= \left|W_{m,k}(x_0,y_0) + \iint\limits_{Q_{xy}} K(x,y)W_{m,k}(x,y)dxdy\right| \\
&= \left|2\sin(m\pi x_0)\sin(k\pi y_0) + \iint\limits_{Q_{xy}} K(x,y)2\sin(m\pi x)\sin(k\pi y)dxdy\right| \\
&\le 2\left(1 + \iint\limits_{Q_{xy}} |K(x,y)|\, dxdy\right) \le k, \ \ k - \text{constant}.
\end{aligned}$$

Now we can prove that Φ is a contraction operator. Let us carry out this proof in two steps.

I) Let us demonstrate that Φ maps E_T onto E_T continuously. Stated in other words, we require to show $\phi_1(v) \in B_T$ and $\phi_2(v), \phi_3(v) \in C[0,T]$ for arbitrary $v(x,y,t) = \{z(x,y,t), a(t), b(t)\}$ with $z(x,y,t) \in B_T$, $a(t), b(t) \in C[0,T]$.

Let us start with showing that $\phi_1(v) \in B_T$, i.e. we need to verify

$$J_T(\phi_1) = \left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{m,k}^{3/2}\left\|(\phi_1)_{m,k}(t)\right\|_{C[0,T]})^2\right)^{1/2} < +\infty, \quad (3.15)$$

where

$$(\phi_1)_{m,k}(t) = \mathrm{RHS}(z_{m,k}(t)).$$

After some manipulations under the assumptions (A_1)–(A_3), using the estimates in Lemma 3.1, 3.2, and 3.3, we obtain

$$J_T(\phi_1) \le M_1(T) + M_2(T)\|b(t)\|_{C[0,T]} + M_3(T)\|a(t)\|_{C[0,T]}\|z(x,y,t)\|_{B_T}, \quad (3.16)$$

where

$$\begin{aligned}
M_1(T) &= 4\Big(\|\varphi_{xxx}(.,.)\|_{L_2(Q_{xy})} + \|\varphi_{xxy}(.,.)\|_{L_2(Q_{xy})} + \|\varphi_{xyy}(.,.)\|_{L_2(Q_{xy})} \\
&+ \|\varphi_{yyy}(.,.)\|_{L_2(Q_{xy})}\Big) + 4T\Big(\|f_{xxx}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|f_{xxy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} \\
&+ \|f_{xyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|f_{yyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})}\Big), \quad M_2(T) = 4T\Big(\|g_{xxx}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} \\
&+ \|g_{xxy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|g_{xyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|g_{yyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})}\Big), \quad M_3(T) = 4T.
\end{aligned}$$

Since $z(x,y,t) \in B_T$, and $a(t), b(t) \in C[0,T]$, the right hand side of $J_T(\phi_1)$ is finite. Thus $\phi_1(v)$ belongs to the space B_T.

Now let us verify $\phi_2(v), \phi_3(v) \in C[0,T]$. From the equation (3.13) and (3.14) we have

$$\|\phi_2(v)\|_{C[0,T]} \le N_1(T) + N_2(T)\|b(t)\|_{C[0,T]} + N_3(T)\|a(t)\|_{C[0,T]}\|z(x,y,t)\|_{B_T}, \quad (3.17)$$

and

$$\|\phi_3(v)\|_{C[0,T]} \le R_1(T) + R_2(T)\|b(t)\|_{C[0,T]} + R_3(T)\|a(t)\|_{C[0,T]}\|z(x,y,t)\|_{B_T}, \quad (3.18)$$

where

$$\begin{aligned} N_1(T) = \frac{1}{\delta}\Big\{ & \|g_2(t)\|_{C[0,T]}\left(\|E_1'(t)\|_{C[0,T]} + \|f_1(t)\|_{C[0,T]}\right) \\ & + \|g_1(t)\|_{C[0,T]}\left(\|E_2'(t)\|_{C[0,T]} + \|f_2(t)\|_{C[0,T]}\right) + \|\alpha(t)\|_{C[0,T]}\Big(8C_1\|g_2(t)\|_{C[0,T]} \\ & + C_2|A_{m,k}|\,\|g_1(t)\|_{C[0,T]}\Big)\Big[\|\varphi_{xxx}(.,.)\|_{L_2(Q_{xy})} + \|\varphi_{xxy}(.,.)\|_{L_2(Q_{xy})} \\ & + \|\varphi_{xyy}(.,.)\|_{L_2(Q_{xy})} + \|\varphi_{yyy}(.,.)\|_{L_2(Q_{xy})} + T\Big(\|f_{xxx}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} \\ & + \|f_{xxy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|f_{xyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|f_{yyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})}\Big)\Big]\Big\}, \end{aligned}$$

$$\begin{aligned} N_2(T) = \frac{1}{\delta}\Big\{ & T\|\alpha(t)\|_{C[0,T]}\left(8C_1\|g_2(t)\|_{C[0,T]} + C_2|A_{m,k}|\,\|g_1(t)\|_{C[0,T]}\right)\Big[\|g_{xxx}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} \\ & + \|g_{xxy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|g_{xyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|g_{yyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})}\Big]\Big\}, \end{aligned}$$

$$N_3(T) = \frac{T}{\delta}\|\alpha(t)\|_{C[0,T]}\left(8C_1\|g_2(t)\|_{C[0,T]} + C_2|A_{m,k}|\,\|g_1(t)\|_{C[0,T]}\right),$$

and

$$\begin{aligned} R_1(T) = \frac{1}{\delta}\Big\{ & \|E_2(t)\|_{C[0,T]}\left(\|E_1'(t)\|_{C[0,T]} + \|f_1(t)\|_{C[0,T]}\right) + \|E_1(t)\|_{C[0,T]}\Big(\|E_2'(t)\|_{C[0,T]} \\ & + \|f_2(t)\|_{C[0,T]}\Big) + \|\alpha(t)\|_{C[0,T]}\Big(8C_1\|E_2(t)\|_{C[0,T]} \\ & + C_2|A_{m,k}|\,\|E_1(t)\|_{C[0,T]}\Big)\Big[\|\varphi_{xxx}(.,.)\|_{L_2(Q_{xy})} + \|\varphi_{xxy}(.,.)\|_{L_2(Q_{xy})} + \|\varphi_{xyy}(.,.)\|_{L_2(Q_{xy})} \\ & + \|\varphi_{yyy}(.,.)\|_{L_2(Q_{xy})} + T\Big(\|f_{xxx}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|f_{xxy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|f_{xyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} \\ & + \|f_{yyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})}\Big)\Big]\Big\}, \end{aligned}$$

$$\begin{aligned} R_2(T) = \frac{T}{\delta}\Big\{ & \|\alpha(t)\|_{C[0,T]}\left(8C_1\|E_2(t)\|_{C[0,T]} + C_2|A_{m,k}|\,\|E_1(t)\|_{C[0,T]}\right)\Big[\|g_{xxx}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} \\ & + \|g_{xxy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|g_{xyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})} + \|g_{yyy}(\cdot,\cdot,t)\|_{L_2(Q_{xy})}\Big]\Big\}, \end{aligned}$$

$$R_3(T) = \frac{T}{\delta}\|\alpha(t)\|_{C[0,T]}\left(8C_1\|E_2(t)\|_{C[0,T]} + C_2|A_{m,k}|\,\|E_1(t)\|_{C[0,T]}\right),$$

where δ is a constant such that

$$0<\delta\leq \min_{0\leq t\leq T}|D(t)|\,,\quad C_1=\left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\left[(2m-1)+(2k-1)\right]^2\pi^2}\right)^{1/2},$$

$$C_2=\left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(m+k)^2\pi^2}\right)^{1/2}.$$

The right hand sides of (3.17) and (3.18) are bounded. Thus, $\phi_2(v)$ and $\phi_3(v)$ are continuous in other words $\phi_2(v)$ and $\phi_3(v)$ belong to the space $C[0,T]$. Since $\phi_1(v)\in B_T$ and $\phi_2(v),\phi_3(v)\in C[0,T]$, we can conclude that Φ maps E_T onto itself continuously.

II) In this step our aim is to show Φ is contraction mapping operator. Assume that let v_1 and v_2 be any two elements of E_T. We know that

$$\|\Phi(v_1)-\Phi(v_2)\|_{E_T}=\|\phi_1(v_1)-\phi_1(v_2)\|_{B_T}+\|\phi_2(v_1)-\phi_2(v_2)\|_{C[0,T]}+\|\phi_3(v_1)-\phi_3(v_2)\|_{C[0,T]}\,,$$

where

$$v_i=\left\{z^i(x,y,t),a^i(t),b^i(t)\right\},\quad i=1,2.$$

Now consider the following differences

$$\phi_1(v_1)-\phi_1(v_2)=\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\int_0^t\Big[R_{(m,k)}(s;z^1,a^1,b^1)$$
$$-R_{(m,k)}(s;z^2,a^2,b^2)\Big]e^{-\int_s^t\mu_{m,k}\alpha(\tau)d\tau}dsW_{m,k}(x,y),\tag{3.19}$$

$$\phi_2(v_1)-\phi_2(v_2)=\frac{\alpha(t)}{D(t)}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{8g_2(t)\mu_{2m-1,2k-1}}{(2m-1)(2k-1)\pi^2}\left(z^1_{2m-1,2k-1}(t)-z^2_{2m-1,2k-1}(t)\right)\right.$$
$$\left.-g_1(t)A_{m,k}\mu_{m,k}\left(z^1_{m,k}(t)-z^2_{m,k}(t)\right)\right)\tag{3.20}$$

and

$$\phi_3(v_1)-\phi_3(v_2)=\frac{\alpha(t)}{D(t)}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\left(E_1(t)A_{m,k}\mu_{m,k}\left(z^1_{m,k}(t)-z^2_{m,k}(t)\right)\right.$$
$$\left.-\frac{8E_2(t)\mu_{2m-1,2k-1}}{(2m-1)(2k-1)\pi^2}\left(z^1_{2m-1,2k-1}(t)-z^2_{2m-1,2k-1}(t)\right)\right).\tag{3.21}$$

As in the estimates (3.16)–(3.18), we can obtain from the last equations

$$\|\phi_1(v_1)-\phi_1(v_2)\|_{B_T}\leq M_2(T)\left\|b^1-b^2\right\|_{C[0,T]}$$
$$+M_3(T)\left\|z^2\right\|_{B_T}\left\|a^1-a^2\right\|_{C[0,T]}+M_3(T)\left\|a^1\right\|_{C[0,T]}\left\|z^1-z^2\right\|_{B_T},\tag{3.22}$$

$$\|\phi_2(v_1)-\phi_2(v_2)\|_{C[0,T]}\leq N_2(T)\left\|b^1-b^2\right\|_{C[0,T]}$$
$$+N_3(T)\left\|z^2\right\|_{B_T}\left\|a^1-a^2\right\|_{C[0,T]}+N_3(T)\left\|a^1\right\|_{C[0,T]}\left\|z^1-z^2\right\|_{B_T},\tag{3.23}$$

and

$$\|\phi_3(v_1)-\phi_3(v_2)\|_{C[0,T]}\leq R_2(T)\left\|b^1-b^2\right\|_{C[0,T]}$$
$$+R_3(T)\left\|z^2\right\|_{B_T}\left\|a^1-a^2\right\|_{C[0,T]}+R_3(T)\left\|a^1\right\|_{C[0,T]}\left\|z^1-z^2\right\|_{B_T}.\tag{3.24}$$

From the these inequalities it follows that

$$\|\Phi(v_1)-\Phi(v_2)\|_{E_T}\leq C(T,a^1,z^2)\,\|v_1-v_2\|_{E_T}\,,\tag{3.25}$$

where

$$C(T,a^1,z^2)=\max\Big\{M_2(T)+N_2(T)+R_2(T), M_3(T)\left\|z^2\right\|_{B_T}+N_3(T)\left\|z^2\right\|_{B_T}$$
$$+R_3(T)\left\|z^2\right\|_{B_T}, M_3(T)\left\|a^1\right\|_{C[0,T]}+N_3(T)\left\|a^1\right\|_{C[0,T]}+R_3(T)\left\|a^1\right\|_{C[0,T]}\Big\}.$$

For sufficiently small T such as $C(T,a^1,z^2)$ tends to zero, i.e. $0<C(T,a^1,z^2)<1$ for sufficiently small T. This implies that the operator Φ is contraction operator.

Thus from the first and second steps the operator Φ is contraction mapping operator and maps E_T onto itself continuously. Then in regard to Banach fixed point theorem the solution of the operator $v=\Phi(v)$ exists and unique. □

4. Numerical solution of the Direct problem

We consider in this section, the direct initial boundary value problem given by equations (2.1)–(2.4), where $a(t)$, $b(t)$, $\alpha(t)$, $g(x,y,t)$, $\varphi(.,.)$ and $f(x,y,t)$ are known and the solution $z(x,y,t)$ is to be determined. We subdivide Q_T into M_1, M_2 and N subintervals of equal step lengths Δx, Δy and Δt, where $\Delta x=1/M_1$, $\Delta y=1/M_2$, and $\Delta t=T/N$, respectively. At the node (i,j,n), we denote $z_{i,j}^n:=z(x_i,y_j,t_n)$, where $x_i=i\Delta x$, $y_j=j\Delta y$, $t_n=n\Delta t$, $a^n:=a(t_n)$, $b^n:=b(t_n)$, $\alpha^n:=\alpha(t_n)$, $g_{i,j}^n:=g(x_i,y_j,t_n)$ and $f_{i,j}^n:=f(x_i,y_j,t_n)$ for $i=\overline{0,M_1}$, $j=\overline{0,M_2}$, $n=\overline{0,N}$.

4.1. Alternating direction explicit (ADE) scheme

Based on the method described in [3, 9, 45], in this section an unconditionally stable numerical procedure for solving nonlinear a two-dimensional parabolic equation (2.1) with initial and boundary conditions (2.2)–(2.4) will be described.

Let $u_{i,j}^n$ and $v_{i,j}^n$ be the solutions of the following equations which are multilevel finite difference discretization of equation (2.1):

$$\frac{\tilde{u}_{i,j}^{n+1}-\tilde{u}_{i,j}^n}{\Delta t}-\alpha^n\left(\frac{u_{i+1,j}^n-u_{i,j}^n-u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{(\Delta x)^2}+\frac{u_{i,j+1}^n-u_{i,j}^n-u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{(\Delta y)^2}\right)$$
$$=a^n\left(\frac{u_{i,j}^n+u_{i,j}^{n+1}}{2}\right)+b^ng_{i,j}^n+f_{i,j}^n,\quad i=\overline{1,M_1-1},\quad j=\overline{1,M_2-1},\quad n=\overline{0,N},\tag{4.1}$$

$$\frac{v_{i,j}^{n+1}-v_{i,j}^n}{\Delta t}-\alpha^n\left(\frac{v_{i+1,j}^{n+1}-v_{i,j}^{n+1}-v_{i,j}^n+v_{i-1,j}^n}{(\Delta x)^2}+\frac{v_{i,j+1}^{n+1}-v_{i,j}^{n+1}-v_{i,j}^n+v_{i,j-1}^n}{(\Delta y)^2}\right)$$
$$=a^n\left(\frac{v_{i,j}^{n+1}+v_{i,j}^n}{2}\right)+b^ng_{i,j}^n+f_{i,j}^n,\quad i=\overline{M_1-1,1},\quad j=\overline{M_2-1,1},\quad n=\overline{0,N}.\tag{4.2}$$

Rearranging the terms in (4.1) and (4.2), we obtain the explicit calculations of $u_{i,j}^{n+1}$ and $v_{i,j}^{n+1}$ as follows:

$$u_{i,j}^{n+1}=A^nu_{i,j}^n+B^n(u_{i+1,j}^n+u_{i-1,j}^{n+1})+C^n(u_{i,j+1}^n+u_{i,j-1}^{n+1})+D^n\left(b^ng_{i,j}^n+f_{i,j}^n\right),$$
$$i=\overline{1,M_1-1},\quad j=\overline{1,M_2-1},\quad n=\overline{0,N},\tag{4.3}$$

$$v_{i,j}^{n+1}=A^nv_{i,j}^n+B^n(v_{i+1,j}^{n+1}+v_{i-1,j}^n)+C^n(v_{i,j+1}^{n+1}+v_{i,j-1}^n)+D^n\left(b^ng_{i,j}^n+f_{i,j}^n\right),$$
$$i=\overline{M_1-1,1},\quad j=\overline{M_2-1,1},\quad n=\overline{0,N},\tag{4.4}$$

where

$$A^n=\frac{1-\lambda^n}{1+\lambda^n},\quad B^n=\frac{(\Delta t)\alpha^n}{(\Delta x)^2(1+\lambda^n)},\quad C^n=\frac{(\Delta t)\alpha^n}{(\Delta y)^2(1+\lambda^n)},$$
$$D^n=\frac{\Delta t}{1+\lambda^n},\quad \lambda^n=\Delta t\left(\frac{\alpha^n}{(\Delta x)^2}+\frac{\alpha^n}{(\Delta y)^2}-\frac{a^n}{2}\right).$$

The initial (2.2) and homogeneous boundary conditions (2.3) and (2.4) are given as

$$u^0_{i,j}=v^0_{i,j}=\varphi(x_i,y_j),\quad i=\overline{0,M_1},\quad j=\overline{0,M_2}, \tag{4.5}$$

$$u^n_{0,j}=v^n_{0,j}=0,\quad u^n_{M_1,j}=v^n_{M_1,j}=0,\quad j=\overline{0,M_2},\quad n=\overline{1,N}, \tag{4.6}$$

$$u^n_{i,0}=v^n_{i,0}=0,\quad u^n_{i,M_2}=v^n_{i,M_2}=0,\quad i=\overline{0,M_1},\quad n=\overline{1,N}. \tag{4.7}$$

From (4.3), $u^{n+1}_{i,j}$ can be computed explicitly. In this case, calculations proceed from the grid point close to the boundaries $x=0$ and $y=0$, as i,j increasing. The needed values such as $u^{n+1}_{i-1,j}$, $u^{n+1}_{i,j-1}$, $u^n_{i,j}$, $u^n_{i+1,j}$ and $u^n_{i,j+1}$ will be known from initial and boundary conditions (4.5)–(4.7). Similarly, $v^{n+1}_{i,j}$ can be calculated explicitly from (4.4) beginnig at the boundaries $x=1$ and $y=1$ and marching in a ssquence of decreasing i and j, i.e. $i=M_1-1,M_1-2,...,1$, $j=M_2-1,M_2-2,...,1$. These values are then substituted into the simple arithmetic mean approximation

$$z^{n+1}_{i,j}=\frac{u^{n+1}_{i,j}+v^{n+1}_{i,j}}{2}. \tag{4.8}$$

The double integral in (2.5) and (2.6) is approximated using the trapezoidal rule [15,17], as follows:

$$\int_0^1\int_0^1 z(x,y,t)dxdy=\frac{1}{4M_1M_2}\Bigg[z(0,0,t_n)+z(1,0,t_n)+z(0,1,t_n)+z(1,1,t_n)$$
$$+2\sum_{i=1}^{M_1-1}z(x_i,0,t_n)+2\sum_{i=1}^{M_1-1}z(x_i,1,t_n)+2\sum_{j=1}^{M_2-1}z(0,y_j,t_n)+2\sum_{j=1}^{M_2-1}z(1,y_j,t_n)$$
$$+4\sum_{j=1}^{M_2-1}\sum_{i=1}^{M_1-1}z(x_i,y_j,t_n)\Bigg],\quad n=\overline{1,N}, \tag{4.9}$$

$$\int_0^1\int_0^1 K(x,y)z(x,y,t)dxdy=\frac{1}{4M_1M_2}\Bigg[K(0,0)z(0,0,t_n)+K(1,0)z(1,0,t_n)$$
$$+K(0,1)z(0,1,t_n)+K(1,1)z(1,1,t_n)+2\sum_{i=1}^{M_1-1}K(x_i,0)z(x_i,0,t_n)$$
$$+2\sum_{i=1}^{M_1-1}K(x_i,1)z(x_i,1,t_n)+2\sum_{j=1}^{M_2-1}K(0,y_j)z(0,y_j,t_n)+2\sum_{j=1}^{M_2-1}K(1,y_j)z(1,y_j,t_n)$$
$$+4\sum_{j=1}^{M_2-1}\sum_{i=1}^{M_1-1}K(x_i,y_j)z(x_i,y_j,t_n)\Bigg],\quad n=\overline{1,N}. \tag{4.10}$$

5. Numerical solution of the IP

In this section, our goal is to obtain simultaneously stable reconstructions for the coefficients $a(t)$ and $b(t)$ and the temperature $z(x,y,t)$, satisfying equations (2.1)–(2.6). The inverse problem can be formulated as a nonlinear least-squares minimization of the least-squares objective function given as follows.

$$F(a,b) = \left\| \iint\limits_{Q_{xy}} z(x,y,t)dxdy - E_1(t) \right\|^2 + \left\| z(x_0,y_0,t) + \iint\limits_{Q_{xy}} K(x,y)z(x,y,t)dxdy - E_2(t) \right\|^2, \tag{5.1}$$

or, in discretizations form

$$F(\underline{a},\underline{b}) = \sum_{n=1}^{N} \left[\iint\limits_{Q_{xy}} z(x,y,t_n)dxdy - E_1(t_n) \right]^2 + \sum_{n=1}^{N} \left[z(x_0,y_0,t_n) + \iint\limits_{Q_{xy}} K(x,y)z(x,y,t_n)dxdy - E_2(t_n) \right]^2, \tag{5.2}$$

where $z(x,y,t)$ solves (2.1)–(2.4) for given $a(t)$ and $b(t)$, respectively. The minimization of the objective function (5.2) is performed using the MATLAB toolbox routine *lsqnonlin*, which does not require supplying by the user the gradient of the objective function, [41]. This subroutine attempts to minimize a sum of squares, which starts from initial guesses, based on the physical constraints $a(t)$ and $b(t)$. Thus, the lower and upper bounds for the coefficients $a(t)$ and $b(t)$ are -10^2 and 10^2, respectively. These bounds allow a wide search range for the unknown. Moreover, within *lsqnonlin*, we apply the interior-reflective Newton approach based Trust Region Reflective algorithm [14].

6. Numerical results and discussion

In this section, we present numerical results for the terms $a(t)$ and $b(t)$ together with the temperature $z(x,y,t)$, in the case of exact and noisy data (2.1)–(2.6). We employ the root mean square errors (RMSE), in order to assess th accuracy of the numerical results, defined as follows.

$$\text{RMSE}(a) = \left[\frac{T}{N} \sum_{n=1}^{N} \left(a^{Numerical}(t_n) - a^{Exact}(t_n) \right)^2 \right]^{1/2}, \tag{6.1}$$

$$\text{RMSE}(b) = \left[\frac{T}{N} \sum_{n=1}^{N} \left(b^{Numerical}(t_n) - b^{Exact}(t_n) \right)^2 \right]^{1/2}. \tag{6.2}$$

For simplicity, we take $T = 1$.

The inverse problem given by (2.1)–(2.6) is solved subject to both exact and noisy measurements (2.5) and (2.6). The noisy data are numerically formulated as follows:

$$E_1^{\epsilon 1}(t_n) = E_1(t_n) + \epsilon 1_n, \quad E_2^{\epsilon 2}(t_n) = E_2(t_n) + \epsilon 2_n, \quad n = \overline{1,N}, \tag{6.3}$$

where $\epsilon 1_n$ and $\epsilon 2_n$ are random variables generated from a Gaussian normal distribution with mean zero and standard deviations σ_1 and σ_2 given by

$$\sigma_1 = \max_{t\in[0,T]} |E_1(t)| \times p, \quad \sigma_2 = \max_{t\in[0,T]} |E_2(t)| \times p, \tag{6.4}$$

where p represents the percentage of noise.

Let us investigate the problem proposed in equations (2.1)–(2.6) with unknown coefficients $a(t)$ and $b(t)$, with the input data:

$$\begin{aligned}\varphi(x,y) = -(-1+x)^5x^5(-1+y)^5y^5,\ z(0,y,t) = z(1,y,t) = 0,\ \alpha(t) = \frac{1+t}{200},\\ z(x,0,t) = z(x,1,t) = 0,\ g(x,y,t) = x^3y^3(-1+x)^3(-1+y)^3e^t,\\ f(x,y,t) = \frac{e^t}{20}(-1+x)^3x^3(-1+y)^3y^3(-20(1+t) - 20(-1+x)^2x^2(-1+y)^2y^2\\ +20t(-1+x)^2x^2(-1+y)^2y^2 + (1+t)(2(-1+y)^2y^2 - 9x(-1+y)^2y^2\\ -2x^3(2-9y+9y^2) + x^4(2-9y+9y^2) + x^2(2-9y+18y^2-18y^3+9y^4))),\end{aligned} \tag{6.5}$$

$$E_1(t) = \int_0^1\int_0^1 z(x,y,t)dxdy = -\frac{e^t}{7683984}, \tag{6.6}$$

$$E_2(t) = z(x_0,y_0,t) + \int_0^1\int_0^1 K(x,y)z(x,y,t)dxdy = -\frac{545785e^t}{503577575424}, \tag{6.7}$$

where

$$x_0 = 0.5, \quad y_0 = 0.5, \quad K(x,y) = 1,$$

and

$$D(t) = E_1(t)g_2(t) - E_2(t)g_1(t) = \frac{3401e^{2t}}{201431030169600} \neq 0,\ \forall t \in [0,1], \tag{6.8}$$

where

$$\begin{aligned}g_1(t) = \int_0^1\int_0^1 g(x,y,t)dxdy = \frac{e^t}{19600},\\ g_2(t) = g(0.5,0.5,t) + \int_0^1\int_0^1 K(x,y)g(x,y,t)dxdy = \frac{1481e^t}{5017600}.\end{aligned}$$

We observe that the conditions of Theorem 1 is fulfilled and thus, the uniqueness condition of the solution is guaranteed. In fact, it can easily be checked by direct substitution that the analytical solution is given by

$$z(x,y,t) = -x^5y^5(x-1)^5(y-1)^5e^t, \quad (x,y,t) \in \overline{Q_T}, \tag{6.9}$$

and

$$a(t) = t, \quad b(t) = 1+t, \quad t \in [0,1]. \tag{6.10}$$

First, we assess the accuracy of the direct problem (2.1)–(2.4) with the input data (6.5) when $a(t)$ and $b(t)$ are known and given by (6.10). The numerical results for the interior temperature $z(x,y,t)$ have been obtained in excellent agreement with the analytical solution (6.9) and therefore they are not presented. Apart from the interior temperature, other output of interest is the data (2.5) and (2.6), which analytically is given by (6.6) and (6.7). Figure 1 shows that the analytical and numerical solutions (6.9) obtained with $M_1 = M_2 = 10$ and with various numbers of time steps $N \in \{10, 20, 40\}$ are in very good agreement. Also, the RMSE defined by

$$\mathrm{RMSE}(E_1) = \left[\frac{1}{N}\sum_{n=1}^{N}\left(E_1^{numerical}(t_n) - E_1^{exact}(t_n)\right)^2\right]^{1/2}, \tag{6.11}$$

$$\mathrm{RMSE}(E_2) = \left[\frac{1}{N}\sum_{n=1}^{N}\left(E_2^{numerical}(t_n) - E_2^{exact}(t_n)\right)^2\right]^{1/2}, \tag{6.12}$$

indicated in Table 1, shows more clearly their decreases as the grid size becomes smaller.

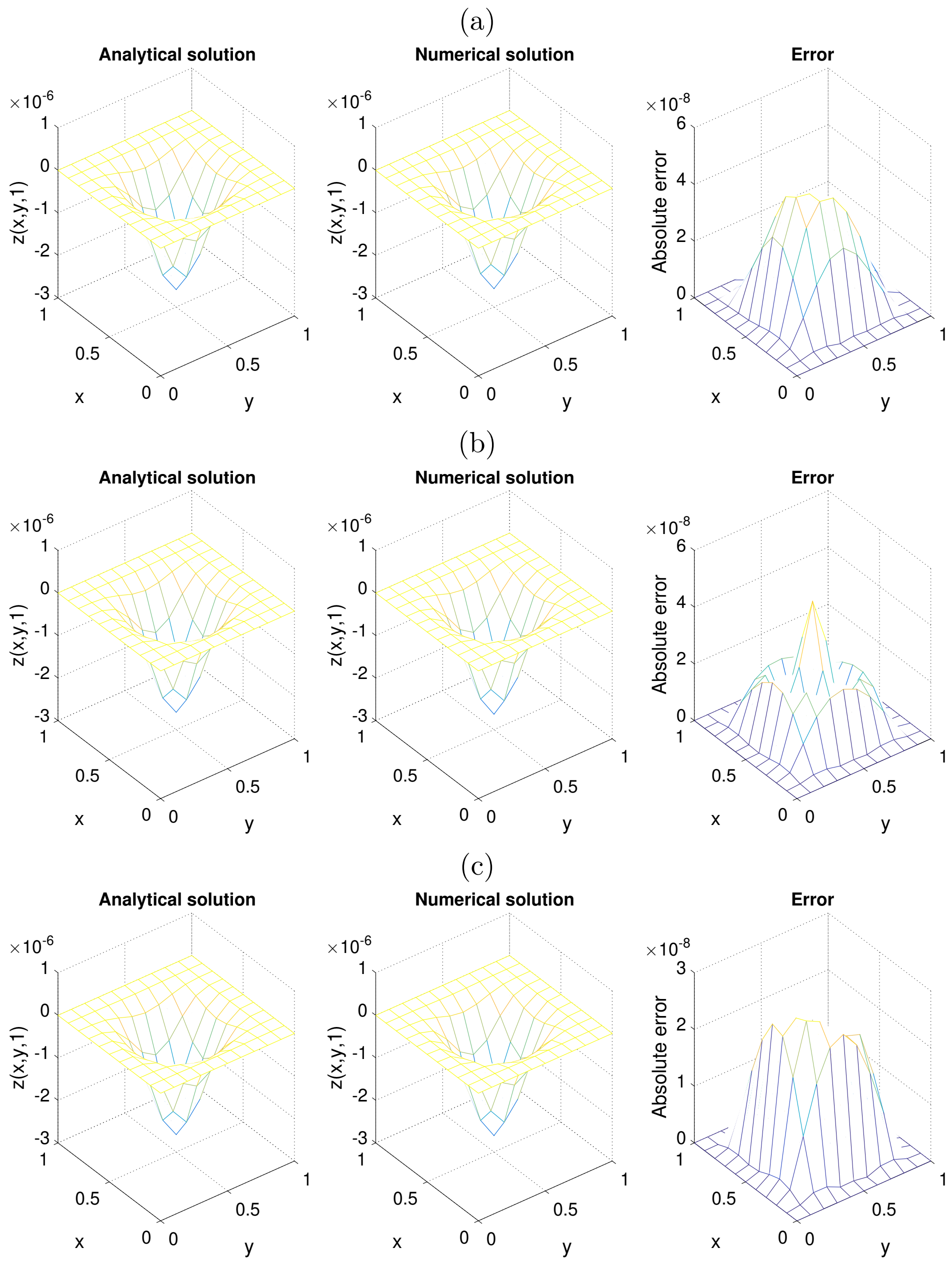

Figure 1. The analytical (6.9) and approximate solutions for the temperature $z(x, y, 1)$, with absolute errors for $\Delta x = \Delta y = \frac{1}{10}$ and with: (a) $\Delta t = \frac{1}{10}$, (b) $\Delta t = \frac{1}{20}$ and (c) $\Delta t = \frac{1}{40}$, for the direct problem.

Table 1. The RMSE values ((6.11) and (6.12)) for $E_1(t)$ and $E_2(t)$, with $M_1 = M_2 = 10$ and with various $N \in \{10, 20, 40\}$, for direct problem.

$M_1 = M_2 = 10$	RMSE(E_1)	RMSE(E_2)
$N = 10$	5.4607E-8	2.2271E-8
$N = 20$	3.0359E-9	6.6868E-9
$N = 40$	1.9196E-9	2.7236E-9

Now, we investigate the inverse problem. We fix $M_1 = M_2 = 10$ and $N = 40$ and start the investigation for reconstructing the unknown coefficients $a(t)$ and $b(t)$ and the temperature $z(x, y, t)$ in absence of noise in the measured data (6.3). We take the initial guesses for the vectors **a** and **b** as follows:

$$a^0(t_n) = a(0) = 0, \quad b^0(t_n) = b(0) = 1, \quad n = \overline{1, N}. \tag{6.13}$$

The objective function (5.2), as a function of the number of iterations, is plotted in Figure 2(a). From this figure, it can be seen that a fast monotonic decreasing convergence is achieved in about 19 iterations to reach a very low prescribed tolerance of $O(10^{-21})$. The exact (6.10) and approximate solutions to the functions $a(t)$ and $b(t)$ are portrayed in Figures 2(b) and 2(c). It is observed that the numerical outcomes are accurate with RMSE(a) =0.0110 and RMSE(b) =1.1928E-4. No regularization was found necessary to penalise the nonlinear least-squares objective functional (5.2) for $p = 0$ noise. Nevertheless, for higher amounts of noise the instability in retrieving the coefficients $a(t)$ and $b(t)$ will become apparent and regularization would need to be employed.

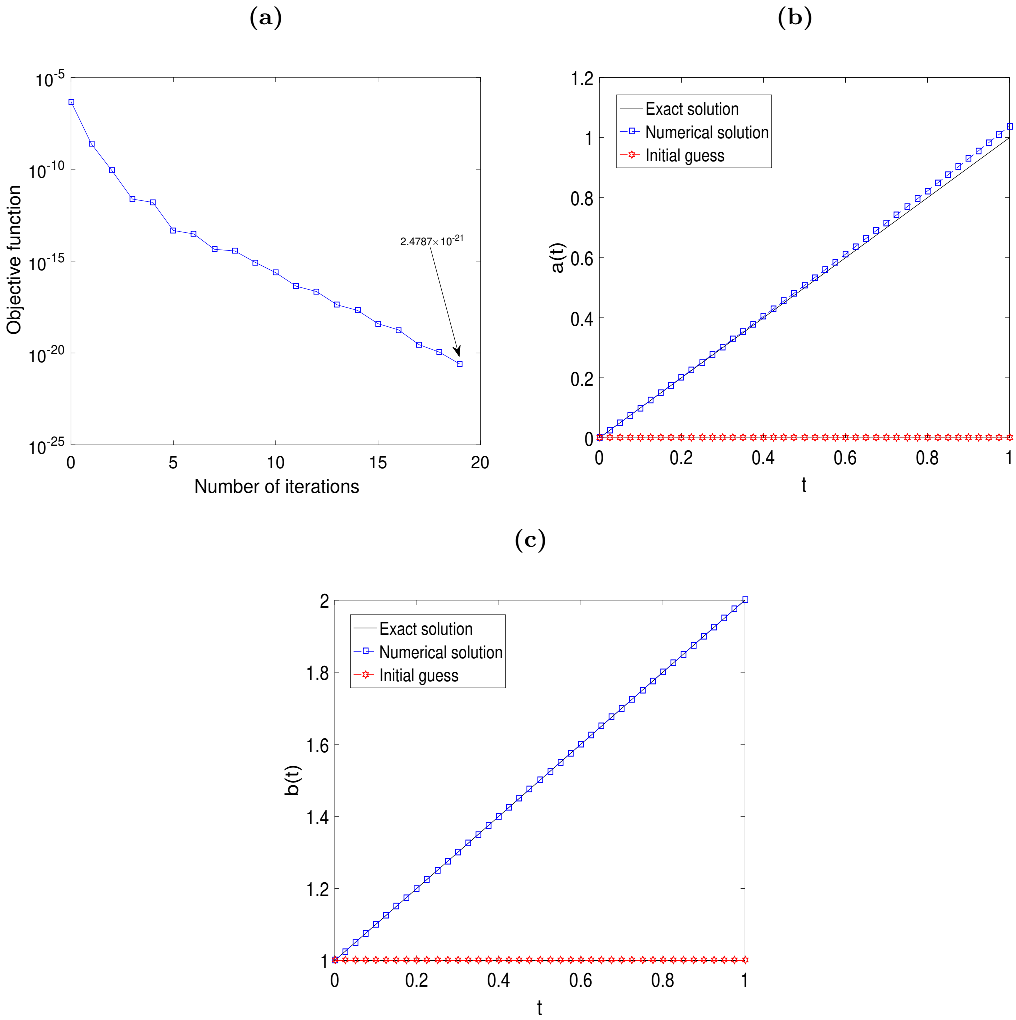

Figure 2. (a) The objective function F (5.2) versus no. of iterations, and the approximate and analytical exact curves (6.10) for: (b) the potential $a(t)$ and (c) the source $b(t)$, in absence of noise and regularization.

Next, we associate 1%, 3% noise with the simulated data (2.5) and (2.6), as in equation (6.4). It is significant to note that the inverse problem is not well posed therefore, we anticipate that the cost function needs to be regularized for the sake of stability and accuracy in results. Figures 3 and 4 show visuals of the reconstructed terms $a(t)$ and $b(t)$. From Figures 3(a), 3(c) and 4(a), 4(c) it is clear that, as expected, we obtain inaccurate and unstable solutions with RMSE$(a) = 0.3209$ and RMSE$(b) = 0.0086$ for $p = 1\%$, and RMSE$(a) = 0.5520$ and RMSE$(b) = 0.0100$ for $p = 3\%$, respectively, as the problem is noise sensitive and ill-posed. Hence, regularization process is crucial for stable solutions. For this, we penalise the objective function F (5.1) by adding $\beta\left(\|a(t)\|^2 + \|b(t)\|^2\right)$ to it, where $\beta > 0$ is the Tikhonov's regularization parameter to be chosen. Then, in discretised form of Tikhonov functional recasts as

$$F_\beta(\underline{a}, \underline{b}) = F(\underline{a}, \underline{b}) + \beta\left(\sum_{n=1}^{N}[a^n]^2 + \sum_{n=1}^{N}[b^n]^2\right). \tag{6.14}$$

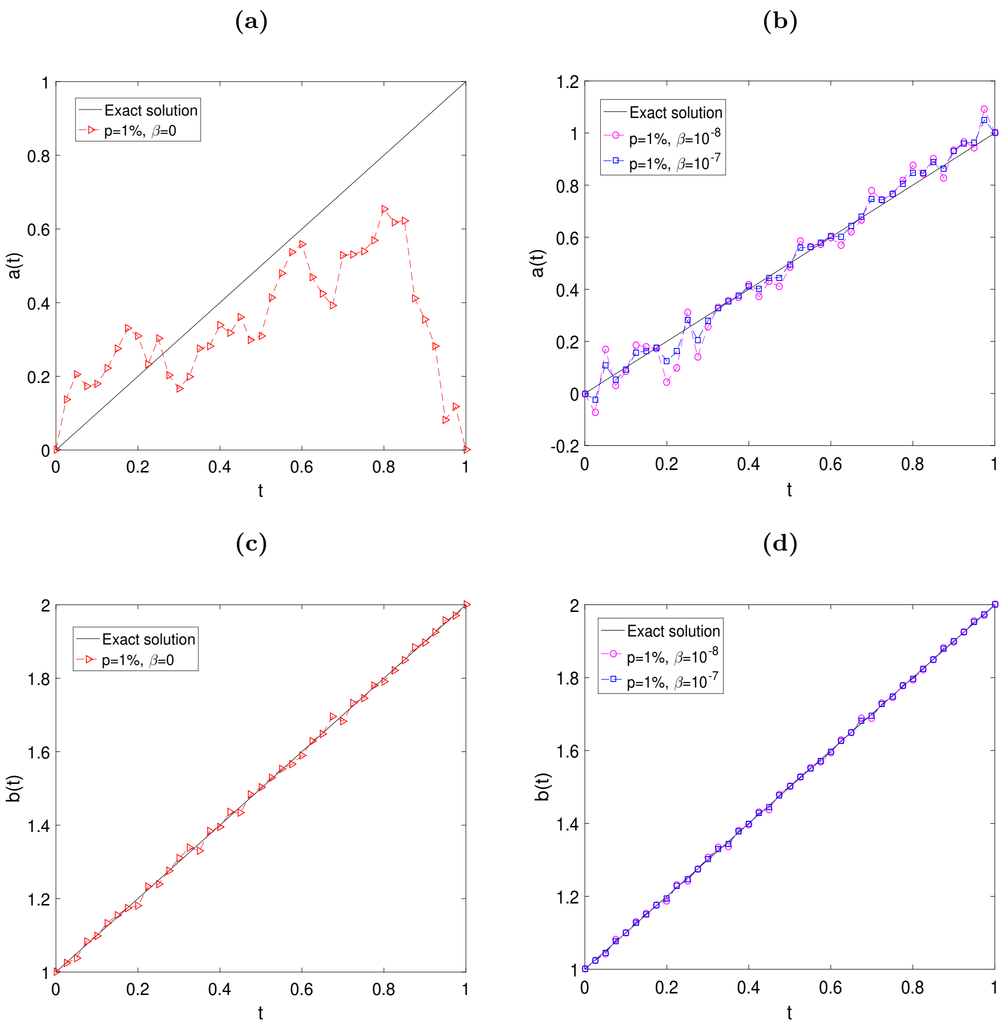

Figure 3. The approximate and analytical (6.10) solutions of the potential $a(t)$, and the heat source $b(t)$, for $p = 1\%$, with $\beta \in \{0, 10^{-8}, 10^{-7}\}$.

The regularization parameter β is chosen to be $10^{-8}, 10^{-7}$ for $p = 1\%$ noise (see Figures 3(b) and 3(d) obtaining RMSE$(a) \in \{0.0605, 0.0329\}$ and RMSE$(b) \in \{0.0063, 0.0031\}$, and $\beta \in \{10^{-6}, 10^{-5}\}$ for $p = 3\%$ noise (see Figures 4(b) and 4(d) obtaining RMSE$(a) \in \{0.1491, 0.1189\}$ and RMSE$(b) \in \{0.0087, 0.0081\}$, which provide stable and comparatively accurate approximations for the functions $a(t)$ and $b(t)$. We have also investigated higher amounts of noise p in (6.4), but the results obtained were less accurate and therefore, they are not presented. Although not presented, it is illustrated that the regularized cost function F_β versus no. of iterations monotonically decreasing convergence is observed. Other details about the RMSE values ((6.1) and 6.2)), and the no. of iterations, with and without regularization are listed in Table 2. Eventually, from Figures 2-4 and Table 2, it is observed that the MATLAB simulation results are fairly stable and accurate.

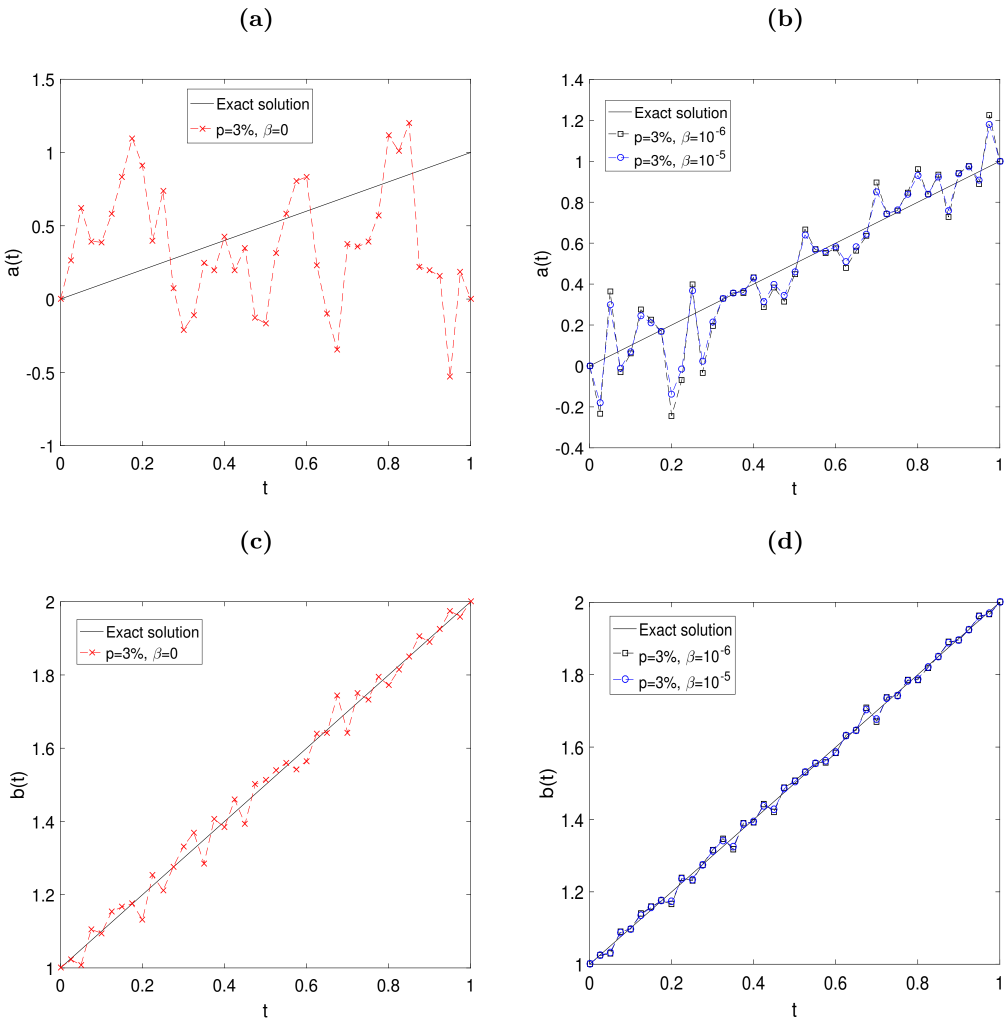

Figure 4. The approximate and analytical (6.10) solutions of the potential $a(t)$, and the heat source $b(t)$, for $p = 3\%$, with $\beta \in \{0, 10^{-6}, 10^{-5}\}$.

Table 2. RMSE values, and no. of iterations, with $p \in \{1\%, 3\%\}$ noise, with $\delta \in \{0, 10^{-9}, 10^{-8}, 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}\}$.

p	β	RMSE(a)	RMSE(b)	Iter
1%	0	0.3209	0.0086	30
	10^{-9}	0.0893	0.0080	10
	10^{-8}	0.0605	0.0063	10
	10^{-7}	0.0329	0.0031	10
	10^{-6}	0.0472	0.0043	10
3%	0	0.5520	0.0100	40
	10^{-7}	0.2251	0.0099	20
	10^{-6}	0.1491	0.0087	20
	10^{-5}	0.1198	0.0081	20
	10^{-4}	0.1237	0.0089	20

7. Conclusions

The article considers the problem of identifying the time-dependent potential and force terms in the two-dimensional parabolic equation with homogeneous boundary conditions and the time-average temperature observations. The unique solvability of the solution of the inverse problem on a sufficiently small time interval has been proved by using of the contraction mapping. The proposed work is novel and has never been solved theoretically nor numerically before. The direct solver based on the ADE technique was employed. The resulting non-linear optimization problem was solved computationally by means of the MATLAB subroutine *lsqnonlin*. Since the problem under consideration was ill-posed, therefore, the Tikhonov regularization was utilized in order to tackle the stability. The numerical results show that ADE is an accurate, stable and robust regularization method for reconstructing the time-dependent potential and force terms from time-average temperature observations. The main difficulty in regularization when we solve the inverse problem is how to choose an appropriate regularization parameter β which compromises between accuracy and stability. However, one can use techniques such as the L-curve method [19] or Morozov's discrepancy principle [44] to find such a parameter, but in our work we have used trial and error. As mentioned in [16], the regularization parameter β is selected based on experience by first choosing a small value and gradually increasing it until any numerical oscillations in the unknown coefficients disappear.

Future work will concern inversion of real physical measurements data (5) and (6) to reconstruct the time-dependent lowest and source terms in a 2D parabolic equation in the context or the heat transfer coefficient in the context of fins used in condensers and evaporators,[42].

Acknowledgment. The authors have no competing interests to declare.

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