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Araştırma Makalesi / Research Article

Exact Solutions of the Oskolkov Equation in Fluid Dynamics

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Keywords
Sub equation method; Oskolkov equation; Nonlinear partial differential equation; Exact solution

Abstract

Traveling wave solutions of the Oskolkov equation, which is a model describing the dynamics of an incompressible visco-elastic Kelvin-Voigt fluid, are investigated in this study. Complex trigonometric and complex hyperbolic solutions of Oskolkov equation are obtained using the sub equation method. In these obtained solutions, graphs are presented by assigning special values to the parameters. The presented graphics are drawn with a computer package program. Implemented method is powerful and an effective method to achieve the exact solutions of nonlinear partial differential equations (NPDEs).

Akışkanlar Dinamiğinde Oskolkov Denkleminin Tam Çözümleri

Anahtar Kelimeler
Alt denklem metodu; Oskolkov denklemi; Lineer olmayan kısmi diferansiyel denklem; Tam çözüm

Öz

Bu çalışmada, sıkıştırılamaz bir visko-elastik Kelvin-Voigt akışkanının dinamiklerini tanımlayan bir model olan Oskolkov denkleminin gezici dalga çözümleri araştırıldı. Alt denklem yöntemini kullanarak Oskolkov denkleminin karmaşık trigonometrik ve karmaşık hiperbolik çözümleri elde edildi. Bu elde edilen çözümlerde parametrelere özel değerler atanarak grafikler sunuldu. Sunulan grafikler bir bilgisayar paket programı ile çizildi. Uygulanan yöntem, lineer olmayan kısmi diferansiyel denklemlerin tam çözümlerini üretmek için güçlü ve etkili bir yöntemdir.

1. Introduction

Mathematical models, called NPDEs include plasma physics, chemistry, quantum mechanics, hydro-dynamic molecular biology, nonlinear optics, sheet water wave, biological science, optical fibers, etc. as seen in various fields of nonlinear science. Investigating NPDEs provides a clearer understanding of complex events. Recently, today by experts from around the world drew attention to several new mathematical models used to describe real-world problems.

In that sense, some methods are new direct algebraic method Kurt *et al*. (2020), Extended trial equation method Gurefe *et al*. (2013), simplest equation method Chen and Jiang (2018), modified Kudryashov method Yokus *et al*. (2021), functional variable method Liu and Chen (2013), Hirota bilinear method Zhang *et al*. (2021), modified (1/G')-expansion method (Yokus *et al*. 2022, Duran *et al*. 2021), first integral method Raslan (2008), modified expansion method Duran and Kaya (2021), the Laplace method Akgül and Modanlı (2022), (G'/G)-expansion method (Zayed and Gepreel 2009).

The Oskolkov equation appears in various study, such as exact solutions of the Oskolkov equation have been presented by using the modified (G'⁄G)-method Alam *et al*. (2022), Ghanbari has been obtained exact solutions for two Oskolkov-type equations Ghanbari (2021), anayltical solutions have been obtained by aid of the modified simple equation method for Oskolkov equation Roshid and Roshid (2018), Thabet *et al*. have been presented exact solutions for Oskolkov equations with exponential rational function method Thabet *et al*. (2022), a set of shock wave solutions for generalised Oskolkov equation is obtained using the unified method Ak *et al*. (2018), Gözükızıl and Akçağıl have been attained the analytical solutions for the Oskolkov equation via tanh-coth method (Gözükızıl and Akçağıl 2013).

The goal of this study is to use the sub equation method to determine the traveling wave solution of the (1+1)-D Oskolkov equation. The Oskolkov equation can be shown in the form of

$$u_t - \beta u_{xxt} - \alpha u_{xx} + u u_x = 0. \quad (1)$$

Here α, β are constants and u is a function of x and t. The solution of the Oskolkov equation with the sub equation method is not available in the literature. In this study, different solutions of the Oskolkov equation from the literature are presented using the sub equation method.

The outline of this article is as follows: in Section 2 introduces the sub-equation method for differential equations. In Section 3, traveling wave solutions of (1+1) dimensional Oskolkov equation are generated using the sub-equation method. Some of the important results of the article are presented in Section 4.

2. Sub Equation Method

Consider the following general form of a NPDE with u as the dependent variable and x,t as the independent variables Duran *et al*. (2021),

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots\right) = 0. \quad (2)$$

By applying the traditional wave conversion

$$u = u(x,t) = u(\xi), \ \ \xi = x - ct, c \neq 0, \quad (3)$$

c is the velocity of the wave and a constant.
Equation (2) converts into ODE

$$T(u, u', u'', \dots) = 0. \quad (4)$$

It is assumed that the solution of equation (4) has the form

$$u(\xi) = \sum_{i=0}^{n} a_i \, G^i(\xi), \ \ a_n \neq 0, \quad (5)$$

here $a_i, (i = (0,1,\dots,n))$ are constants to be determined. n denotes a constant to be found by using the balancing process Equation (4) and

$G = G(\xi)$ gratifies the ODE below

$$G'(\xi) = \mu + (G(\xi))^2 = 0, \ \ \mu \in R. \quad (6)$$

Some specific solutions for equation (6) are presented in the formulas below,

$$\mathrm{G}(\xi) = \begin{cases} -\sqrt{-\mu}\tanh(\sqrt{-\mu}\xi), \ \mu < 0, \\ -\sqrt{-\mu}\coth(\sqrt{-\mu}\xi), \mu < 0, \\ \sqrt{\mu}\tan(\sqrt{\mu}\xi), \qquad \mu > 0, \\ -\sqrt{\mu}\cot(\sqrt{\mu}\xi), \qquad \mu > 0, \\ \frac{-1}{\xi + r}, \ r \ is \ constant, \ \mu = 0. \end{cases} \quad (7)$$

Equations (5) and (6) are substituted into equation (4) and the $G^i(\xi)$ coefficients are equal to zero. A nonlinear algebraic system is produced by this procedure $a_i, i = (0,1,\dots,n)$. Finally, constants are determined by solving nonlinear algebraic equations. Replacing attained constants from nonlinear algebraic system equation (6) into equation (5) via a solution of equation (7). This provides the exact solutions for equation (2).

One of the significant advantages of the method is that it produces three different types of traveling wave solutions: trigonometric, hyperbolic and rational forms. These solutions are in Eq. (7) formats. At the same time, classical wave solution is applied in this method and the balancing term is used. Also, its difference from other methods: in the sub-equation method, base equation is an ordinary differential equation, and since the base equations in the expansion methods are different, they have different properties from the solutions produced by other methods.

3. The solutions of Oskolkov Equation

Considering Oskolkov equation (1) and using transformation $u = u(x,t) = u(\xi)$, $\xi = x - ct$, $c \neq 0$, we obtain

$$-cu' + \beta c u''' - \alpha u'' + uu' = 0. \qquad (8)$$

Considering the Eq. (8), we get the balancing term $n = 2$ and taking into account the series given in equation (5),

$$u(\xi) = a_0 + a_1(\mathrm{G}(\xi)) + a_2(\mathrm{G}(\xi))^2. \qquad (9)$$

Here $a_1 \neq 0$ or $a_2 \neq 0$. If we substitute the equation (9) in the equation (8) and necessary adjustments are made. Thus, we may write equation system as:

$$G(\xi)^0: -c\mu a_1 + 2c\beta\mu^2 a_1 = 0,$$

$$G(\xi)^1: -2\alpha\mu a_1 + \mu a_1^2 - 2c\mu a_2 + 16c\beta\mu^2 a_2 + 2\mu a_0 a_2 = 0,$$

$$G(\xi)^2: -ca_1 + 8c\beta\mu a_1 + a_0 a_1 - 8\alpha\mu a_2 + 3\mu a_1 a_2 = 0,$$

$$G(\xi)^3: -2\alpha a_1 + a_1^2 - 2ca_2 + 40c\beta\mu a_2 + 2a_0 a_2 + 2\mu a_2^2 = 0,$$

$$G(\xi)^4: 6c\beta a_1 - 6\alpha a_2 + 3a_1 a_2 = 0,$$

$$G(\xi)^5: 24c\beta a_2 + 2a_2^2 = 0. \qquad (10)$$

α, β, μ, c and a_0, a_1, a_2 constants are attained from equation (10) the system via a package program.

Case 1. If $\mu < 0$,

$$\beta = \frac{5a_2}{12a_0},\ a_1 = 2i\sqrt{\frac{6}{5}}\sqrt{a_0}\sqrt{a_2},\ \mu = \frac{6a_0}{5a_2},$$

$$c = -\frac{a_0}{5},\ \alpha = i\sqrt{\frac{5}{6}}\sqrt{a_0}\sqrt{a_2}, \qquad (11)$$

replacing values equation (11) into equation (9) and we get complex hyperbolic solution for equation (1):

$$u_1(x,t) = a_0 - \frac{12}{5} i\sqrt{a_0}\sqrt{-\frac{a_0}{a_2}}\sqrt{a_2}\tanh\Big[\sqrt{\frac{6}{5}}\Big(x + \frac{ta_0}{5}\Big)\sqrt{-\frac{a_0}{a_2}}\Big] - \frac{6}{5}a_0\tanh\Big[\sqrt{\frac{6}{5}}\Big(x + \frac{ta_0}{5}\Big)\sqrt{-\frac{a_0}{a_2}}\Big]^2. \qquad (12)$$

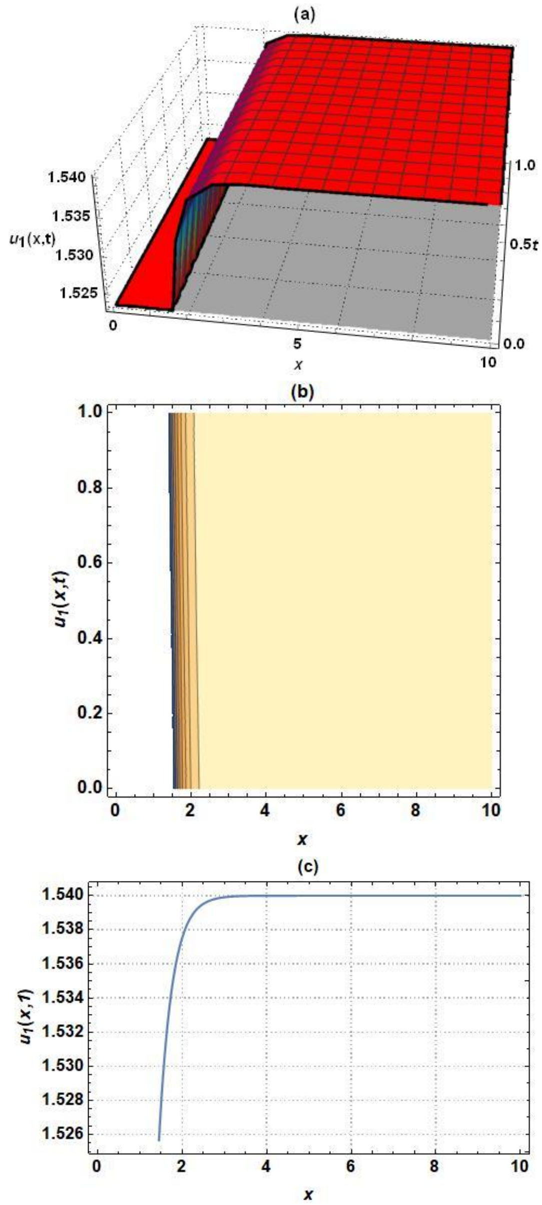


Figure 1. 3D, contour, and 2D graphs of in the $u_1(x,t)$ obtained of the equation (1) for $\mu = -0.7, a_2 = -1.2, a_0 = 0.7$.

If $a_2 < 0$ and $\frac{a_0}{a_2} < 0$ are selected, our complex-valued traveling wave solution turns into a real-valued form. Similar situations can be encountered in other cases.

Case 2. If $\mu < 0$,

$$\beta = \frac{5a_2}{12a_0},\ a_1 = 2i\sqrt{\frac{6}{5}}\sqrt{a_0}\sqrt{a_2},\ \mu = \frac{6a_0}{5a_2},$$

$$c = -\frac{a_0}{5},\ \alpha = i\sqrt{\frac{5}{6}}\sqrt{a_0}\sqrt{a_2}, \qquad (13)$$

replacing values equation (13) into equation (9), we attain complex hyperbolic solution for equation (1):

$$u_2(x,t) = a_0 - \frac{6}{5}\coth[\sqrt{\frac{6}{5}}(x+\frac{ta_0}{5})\sqrt{-\frac{a_0}{a_2}}]^2 a_0 - \frac{12}{5}i\coth[\sqrt{\frac{6}{5}}(x+\frac{ta_0}{5})\sqrt{-\frac{a_0}{a_2}}]\sqrt{a_0}\sqrt{-\frac{a_0}{a_2}}\sqrt{a_2}. \quad (14)$$

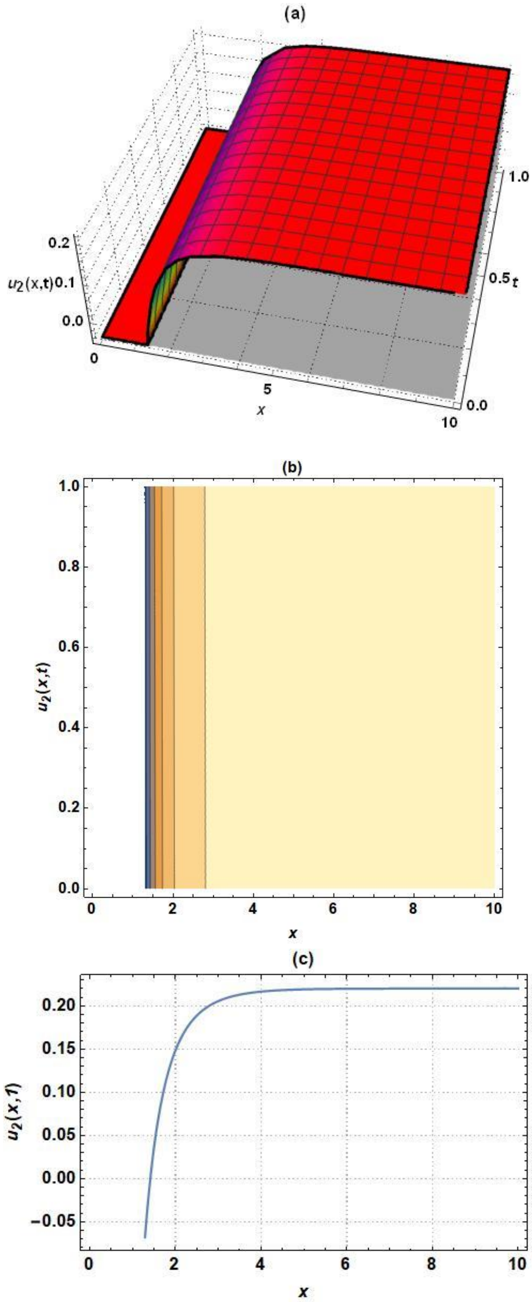


Figure 2. 3D, contour, and 2D graphs of the $u_2(x,t)$ obtained of the equation (1) for $\mu = -0.1, a_2 = = -1.2, a_0 = 0.1$.

Case 3. If $\mu > 0$,

$$\beta = \frac{5a_2}{12a_0},\ a_1 = 2i\sqrt{\frac{6}{5}}\sqrt{a_0}\sqrt{a_2},\ \mu = \frac{6a_0}{5a_2},$$

$$c = -\frac{a_0}{5},\ \alpha = i\sqrt{\frac{5}{6}}\sqrt{a_0}\sqrt{a_2}, \quad (15)$$

replacing values equation (15) into equation (9), we get complex trigonometric solutions for equation (1):

$$u_3(x,t) = a_0 + \frac{12}{5}i\sqrt{a_0}\sqrt{\frac{a_0}{a_2}}\sqrt{a_2}\tan[\sqrt{\frac{6}{5}}(x+\frac{ta_0}{5})\sqrt{\frac{a_0}{a_2}}] + \frac{6}{5}a_0\tan[\sqrt{\frac{6}{5}}(x+\frac{ta_0}{5})\sqrt{\frac{a_0}{a_2}}]^2. \quad (16)$$

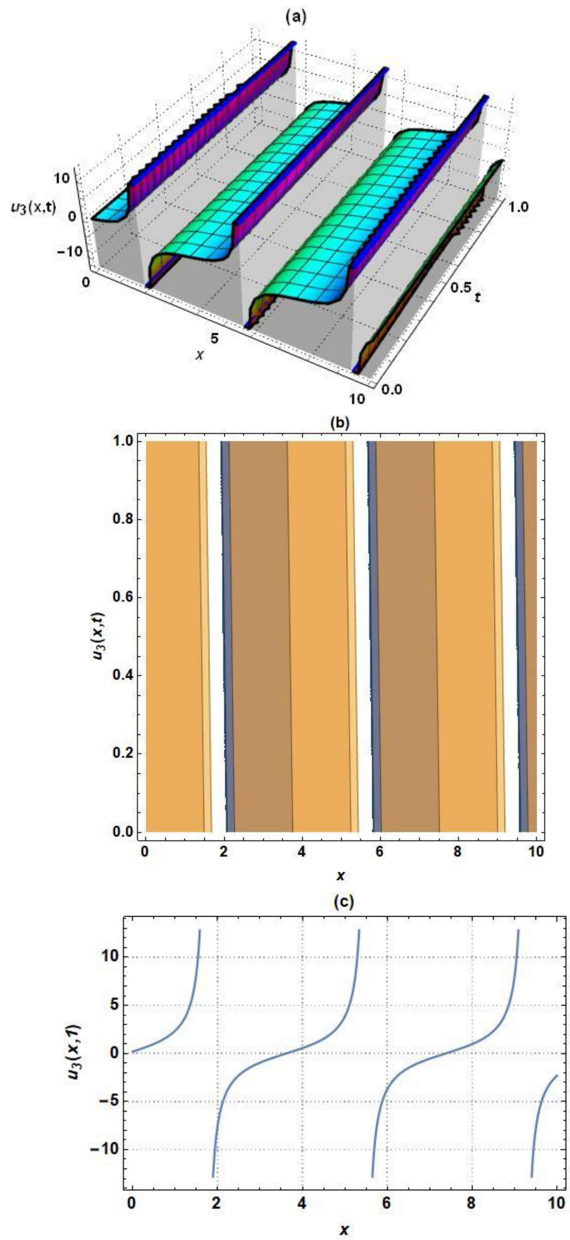


Figure 3. 3D, contour, and 2D graphs of the $u_3(x,t)$ obtained of the equation (1) for $\mu = 0.7, a_2 = 1.2, a_0 = 0.7$.

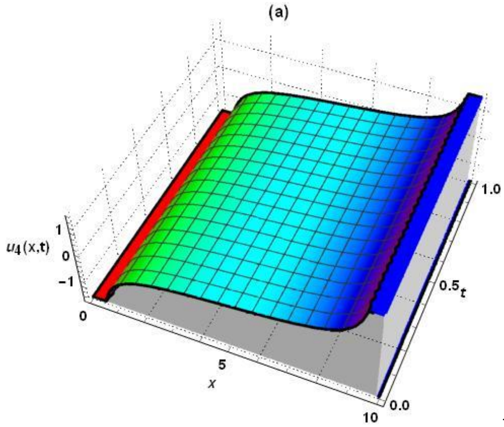
Case 4. If $\mu > 0$,

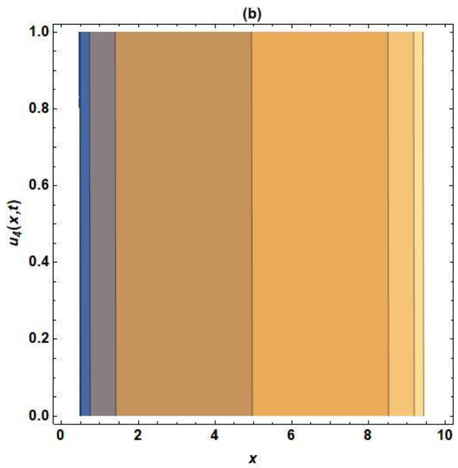
$$\beta = \frac{5a_2}{12a_0},\ a_1 = 2i\sqrt{\frac{6}{5}}\sqrt{a_0}\sqrt{a_2},\ \mu = \frac{6a_0}{5a_2},$$

$$c = -\frac{a_0}{5},\ \alpha = i\sqrt{\frac{5}{6}}\sqrt{a_0}\sqrt{a_2}, \tag{17}$$

replacing values equation (17) into equation (9), we get complex trigonometric solutions for equation (1):

$$u_4(x,t) = a_0 + \frac{6}{5}\cot[\sqrt{\frac{6}{5}}(x + \frac{ta_0}{5})\sqrt{\frac{a_0}{a_2}}]^2 a_0 - \frac{12}{5}i\cot[\sqrt{\frac{6}{5}}(x + \frac{ta_0}{5})\sqrt{\frac{a_0}{a_2}}]\sqrt{a_0}\sqrt{\frac{a_0}{a_2}}\sqrt{a_2}. \tag{18}$$





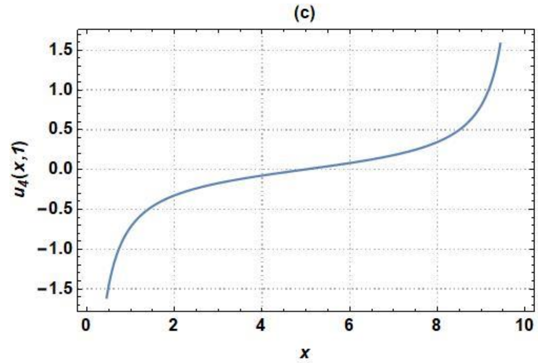


Figure 4. 3D, contour, and 2D graphs of the $u_4(x,t)$ obtained of the equation (1) for $\mu = 0.1, a_2 = 1.2, a_0 = 0.1$.

Case 5. If $\mu = 0$,

$$\beta = \frac{5a_2}{12a_0},\ a_1 = 2i\sqrt{\frac{6}{5}}\sqrt{a_0}\sqrt{a_2},\ \mu = \frac{6a_0}{5a_2},$$

$$c = -\frac{a_0}{5},\ \alpha = i\sqrt{\frac{5}{6}}\sqrt{a_0}\sqrt{a_2}, \tag{19}$$

For an algebraic solution to exist, $\mu = 0$ had to be. μ is not zero so algebraic solution cannot be written.

4. Conclusion

In the literature, different types of solutions of the Oskolkov equation have been presented with the help of different methods. For example; Ghanbari has been presented the exponential and hyperbolic type solutions of the Oskolkov equation (Ghanbari 2021). Alam *et al*. have been presented kinkwave, periodic respiratory waves, cuspwave and periodic wave solutions in their studies (Alam *et al*. 2022). Roshid and Bashar have been presented on kinky periodic wave and breather wave (Roshid and Bashar 2019).

In this study, solutions of Oskolkov equation in complex hyperbolic and complex trigonometric form are produced. In the solutions obtained, the 3D, contour and 2D graphs are presented by giving special values to the parameters. The results attained here show that sub equation method is reliable, powerful and can be used to process other NPDEs. In addition, in this study, a computer package program was used for graphs and calculations.

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