Proceedings of International Mathematical Sciences
ISSN: 2717-6355, URL: https://dergipark.org.tr/tr/pub/pims
Volume 4 Issue 1 (2022), Pages 15-30.
Doi: https://doi.org/ 10.47086/pims.1120339

STABILITY ANALYSIS OF TWO PREDATORS-ONE PREY MODEL WITH FEEDBACK CONTROL AND TIME FRACTIONAL DERIVATIVE

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Abstract. The interaction between prey and predator is one of the most fundamental processes in ecology. In this paper, we first consider the system incorporating a feedback control and we discuss the dynamic behavior of prey-predator interaction model that includes two competitive predators and one prey with a generalized interaction functional. The primary presumption in the model construction is the effects of feedback control and the competition between two predators on the only prey which gives a strong implication of the real-world situation. By analyzing characteristic equations, we carry out detailed discussion with respect to stability of equilibrium points of the considered model. Further, we investigate the impact of the memory measured by fractional time derivative on the temporal behavior.

1. Introduction

Mathematical modeling of the real-world phenomenon is a potent tool for predicting some ecological and biological components. The validity of this mathematical approximation depends on the model itself. The crucial component that describes the interaction between different species in a certain environment is the interaction functional. There are many types of these functionals in the literature [9,10,14,15,19]. Each one describes a specific manner of intermingling between two species. The reason for this great diversity in functionals is due to the variety of environmental conditions in the problem. Some of the factors that influence the selection of these parameters are the behavior of the prey and predator and the studied area. For the last factor, many components play a crucial role such as

2020 *Mathematics Subject Classification.* Primary: 34A08,37N25,39A28 ; Secondary: 39A30, 92D25 .

Key words and phrases. Predator-prey model; Stability; Fractional-order; Feedback control; Population dynamics; Predator competition.

Submitted on 23.05.2022, Accepted on 30.06.2022.
Communicated by Huseyin CAKALLI and Nazlım Deniz ARAL.

rivers (water availability), food (for the prey) and the density of prey and predator. Overall, the functional selection depends on many factors.

In the environment, the intermingling is not limited to just two populations, but interactions can be defined between more than two species in one single place. The scientists interested in this point of view have put efforts to model such complex interactions in the last few decades. We can take as an example two types of prey and one predator [5], where the predator has the capability of hunting both prey populations. Moreover, in prey–predator–super predator models the predator feeds the prey only, and the super predator feeds both prey and predator. In some models, it is studied the interaction between two predators and one prey model where two types of predators are fed the same prey. Due to the intrinsic nature of the predators, there will always be a constant struggle to capture this one prey. The predator-prey models with three species have been attracted many researchers. In [8], it is highlighted and studied the intermingling and competition between two competitive predators on one prey with a generalized class of interaction functionals in the presence of the time-fractional derivative. Fractional ordinal systems are not just an extension of traditional integer ordinal systems in mathematics but also have some merits that integer-order systems do not have, such as memory and hereditary properties [11,21]. As known, many biological systems have memory [18]. Fractional order systems compared to integer order systems can more accurately describe population patterns and reveal the relationships between prey species and predatory species [1,4].

In real situations, it is seen that one predator determines its own hunting territory. The presence of other predators in such territories is entirely unacceptable. This situation is called competition. The models in which competition is found, have also received much attention in many research papers such as.

When examining the local asymptotic stability of the equilibrium points of dynamic systems, note that the equilibrium value of the considered system is sometimes not as we would like, and maybe in some cases what we need is a smaller value. In this case, we may change the system structurally by introducing a feedback control variable [2,12], which can be implemented by employing biological control strategy. In [13], the dynamic behavior of fractional-order predator–prey model incorporating a constant prey refuge and feedback control has investigated.

In this paper, we are interested in studying the intermingling and competition between two competitive predators on one prey with a generalized class of interaction functionals in the presence of the time-fractional derivative. By summarizing all the previously mentioned components let us focus on the following incorporating feedback control time-fractional formulation with a generalized consumption functional:

$$\begin{cases} {}_0^cD_t^q x(t) = x(r - ax - \frac{rx}{k}) - f(x)y - g(x)z - cu, & x(0) = x_0 \\ {}_0^cD_t^q y(t) = e_1 f(x)y - \mu_1 y - \beta yz, & y(0) = y_0 \\ {}_0^cD_t^q z(t) = e_2 g(x)z - \mu_2 z - \gamma yz, & z(0) = z_0 \\ {}_0^cD_t^q u(t) = -hu + mx, & u(0) = u_0 \end{cases} \tag{1.1}$$

where $0 < q < 1$, ${}_0^cD_t^q$ is the Caputo $q-$order fractional derivative. The conditions on the functionals f and g are defined as

(A_1) $f(0) = 0,\ g(0) = 0,$
(A_2) $f'(x) > 0,\ g'(x) > 0\ for\ x > 0.$

In the system (1.1), $x(t), y(t)$ and $z(t)$ are the densities of prey, first predator and second predator populations at time t, respectively; $u(t)$ denotes the feedback control variable for prey population at time t. We assume that the prey population reproduces logistically with the increasing rate r, a is the intraspecific competition coefficient of prey population and the carrying capacity k of the space, e_1 and e_2 are respectively the conversion rate of the prey biomass into the first predator population and the diversion of the prey biomass into the second predator biomass, μ_1 and μ_2 are the mortality rates of the first and second predators, respectively, $\beta(resp., \gamma)$ is the competition rate of the first predator with the second one (resp., of the second predator with the first one). The functionals f and g are respectively the interaction functionals for the first and second predator populations with the prey population. Here all the parameters are assumed to be positive.

The rest of this paper is organized as follows. In section 2, we introduce some notations, definitions and lemmas. In section 3, we give the equilibrium points of fractional-order predator–prey model (1.1), and we discuss their stability. The concluding section of the paper is intended to highlight the biological meanings of the acquired numerical results.

2. Preliminaries

We introduce some useful definitions and lemmas in this section which are necessary for our latter study.

Definition 2.1. *[11] The $q-$order fractional integral for a function ζ is defined as*

$$ {}_0I_t^q \zeta(s) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \zeta(s) ds, \qquad q > 0$$

where $\Gamma(.)$ is the well-known Gamma function which is defined by $\Gamma(q) = \int_0^\infty e^{-t} t^{z-1} dt$.

Definition 2.2. *[11] The Caputo $q-$order fractional derivative for a function ζ is defined as*

$$ {}_0^cD_t^q \zeta(s) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} \zeta^n(s) ds,$$

where n is a positive integer, $n-1 < q < n$. Particularly, when $0 < q < 1$,

$$ {}_0^cD_t^q \zeta(s) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \zeta^{'}(s) ds.$$

Lemma 2.1. *[11] If the Caputo $q-$order fractional derivative ${}_0^cD_t^q$ is integrable then*

$$ {}_0I_t^q\ {}_0^cD_t^q \zeta(s) = \zeta(t) - \sum_{k=0}^{n-1} \frac{\zeta^k(0)}{k!} t^k.$$

Especially, for $0 < q \leq 1$, one can obtain

$$ {}_0I_t^q \, {}_0^cD_t^q \zeta(s) = \zeta(t) - \zeta(0). $$

Lemma 2.2. *[3] Let $V(t)$ be a continuous function on $[0,+\infty)$ and satisfying*

$$ {}_0^cD_t^q V(t) \leq \theta \; V(t), $$

where $0 < q < 1$ and θ is a constant. Then

$$ V(t) \leq V(0) \; E_q(\theta t^q) \qquad \forall t \geq 0. $$

Lemma 2.3. *[20] Consider the following $q-$order fractional system:*

$$ \begin{cases} {}_0^cD_t^q z(t) = f(z), \\ z(0) = z_0, \end{cases} \tag{2.1} $$

where $0 < q < 1$ and $z \in R^n$. The equilibrium points of the system (2.1) can be calculated by solving the following equation: $f(z) = 0$. These points are locally asymptotically stable if all eigenvalues λ_i of the Jacobian matrix $J = \frac{\partial f}{\partial z}$ evaluated at the equilibrium points satisfy the Matignon conditions:

$$ |\arg(\lambda_i)| > \frac{q\pi}{2}. $$

Theorem 2.4. *[13] The trivial equilibrium point of the system attained by the $\lambda^2 + (k-r)\lambda + cm - rk = 0$ characteristic equation is locally asymptotically stable if either of the following criteria is satisfied.*

(H_1) $\quad k \geq r$ *and* $rk < cm$,

(H_2) $\quad k < r$, $rk < cm$, $(k+r)^2 < 4cm$ *and* $0 < q < \frac{2}{\pi}\arctan\left(\frac{\sqrt{4(cm-rk)-(k-r)^2}}{r-k}\right)$.

Theorem 2.5. *[13] The predator-extinction equilibrium point of the system attained by the $\lambda^2 + \left(r - \frac{2cm}{k} + k\right)\lambda + rk - cm = 0$ characteristic equation is locally asymptotically stable if either of the following criteria is satisfied.*

(H_3) $\quad k^2 + rk - 2cm \geq 0$ *and* $rk > cm$,

(H_4) $\quad k^2 + rk - 2cm < 0$, $rk > cm$, $\left(k^2 + rk - 2cm\right)^2 < 4k^2(rk - cm)$ *and* $0 < q < \frac{2}{\pi}\arctan\left(\frac{\sqrt{4k^2(rk-cm)-(k^2+rk-2cm)^2}}{2cm-k^2-rk}\right)$.

3. Mathematical analysis and asymptotic behavior of the solution

3.1. **Equilibria of the model.** In this subsection, we determine the local behavior of the system (1.1). First, we determine the equilibria of the system (1.1), which are the solutions of the following system:

$$ \begin{aligned} 0 &= x(r - ax - \frac{rx}{k}) - f(x)y - g(x)z - cu, \\ 0 &= e_1 f(x)y - \mu_1 y - \beta yz, \\ 0 &= e_2 g(x)z - \mu_2 z - \gamma yz, \\ 0 &= -hu + mx. \end{aligned} \tag{3.1} $$

As a first remark, we deduce that the system 3.1 has the following particular cases:

(i) For the system (1.1) there always exists the trivial equilibrium point $E_0(0,0,0,0)$, which represents the extinction of the three populations.

(ii) $E_1(x_1, 0, 0, u_1)$ which implies the extinction of two types of predators, where $x_1 = \frac{k(rh-cm)}{h(ak+r)}$, $u_1 = \frac{km(rh-cm)}{h^2(ak+r)}$. This point is called the predator-free equilibrium (PFE).

(iii) Searching for the first predator-free equilibrium (FPFE) as $E_2(x_2, 0, z_2, u_2)$, we insert $y = 0$. By replacing this result in the third equation of system (3.1) we get, $x_2 = g^{-1}\left(\frac{\mu_2}{e_2}\right)$, $u_2 = \frac{m}{h}g^{-1}(\frac{\mu_2}{e_2})$ where g^{-1} is the inverse function of g, which exists since g is a bijective function from the conditions (A_1) and (A_2). Substituting this last result into the first equation of (3.1) yields

$$z_2 = \frac{e_2 x_2\left(r - ax_2 - \frac{rx_2}{k} - \frac{cm}{h}\right)}{\mu_2},$$

which is positive if $x_2 < \frac{k(rh-cm)}{h(ak+r)}$. Summarizing all the results, we can conclude that FPFE $E_2(x_2, 0, z_2, u_2)$ exists if $x_2 < \frac{k(rh-cm)}{h(ak+r)}$.

(iv) Seeking for the second predator-free equilibrium (SPFE) as $E_3(x_3, y_3, 0, u_3)$ by replacing $z = 0$ in (3.1). By substituting this result into the second equation of system (3.1) we get $x_3 = f^{-1}(\frac{\mu_1}{e_1})$, $u_3 = \frac{m}{h}f^{-1}(\frac{\mu_1}{e_1})$ where f^{-1} is the inverse function of f , which exists since f is a bijective function function from the conditions (A_1) and (A_2). Taking this last result along with the first equation of (3.1), we get

$$y_3 = \frac{e_1 x_3\left(r - ax_3 - \frac{rx_3}{k} - \frac{cm}{h}\right)}{\mu_1},$$

which is biologically relevant if $x_3 < \frac{k(rh-cm)}{h(ak+r)}$. Summarizing all the results, we can deduce that SPFE as $E_3(x_3, y_3, 0, u_3)$ exists if $x_3 < \frac{k(rh-cm)}{h(ak+r)}$.

Remark. *It is assumed that both functional f and g are increasing in x. From x_3 and x_2, if $\lim_{x\to\infty} f(x) = a$ (resp.,$\lim_{x\to\infty} g(x) = b$) then another condition on the parameters arises, $\frac{\mu_1}{e_1} < a$ (resp., $\frac{\mu_2}{e_2} < b$), which is a necessary condition for having a solution for the equation $f(x) = \frac{\mu_1}{e_1}$ (resp., $g(x) = \frac{\mu_2}{e_2}$).*

(v) Now we are in a position to seek the coexistence equilibrium point $E_4(x^*, y^*, z^*, u^*)$, which is the positive solution of the following system:

$$\begin{aligned} 0 &= x(r - ax - \frac{rx}{k}) - f(x)y - g(x)z - cu, \\ 0 &= e_1 f(x) - \mu_1 - \beta z, \\ 0 &= e_2 g(x) - \mu_2 - \gamma y, \\ 0 &= -hu + mx. \end{aligned} \tag{3.2}$$

From $0 = e_2 g(x) - \mu_2 - \gamma y$ we obtain,

$$y^* = \frac{e_2}{\gamma}g(x) - \frac{\mu_2}{\gamma}. \tag{3.3}$$

Moreover, from $0 = e_1 f(x) - \mu_1 - \beta z$ we find that

$$z^* = \frac{e_1}{\beta}f(x) - \frac{\mu_1}{\beta}. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), from the first equation, we get $F_1(x) = F_2(x)$, where

$$\begin{aligned} F_1(x) &= x(r - ax - \frac{rx}{k}) - \frac{cmx}{h}, \\ F_2(x) &= f(x)g(x)\left(\frac{e_2}{\gamma} - \frac{e_1}{\beta}\right) - \left(\frac{\mu_2}{\gamma}f(x) - \frac{\mu_1}{\beta}g(x)\right). \end{aligned} \tag{3.5}$$

Some straightforward calculations suggest that

$$F_1(0) = F_1\left(\frac{k(rh-cm)}{h(ak+r)}\right) = 0, \qquad F_1(x) = \begin{cases} >0 & for \quad x < \frac{k(rh-cm)}{h(ak+r)} \\ <0 & for \quad x > \frac{k(rh-cm)}{h(ak+r)}. \end{cases}$$

To guarantee at least one nontrivial intersection between two curves of the functionals F_1 and F_2, we introduce the following assumption:

$$F_1(\widetilde{x}) > F_2(\widetilde{x}), \qquad F_2\left(\frac{k(rh-cm)}{h(ak+r)}\right) > 0 \qquad with \qquad \widetilde{x} = \max\{x_2, x_3\},$$

which it can be rewritten as

$$\widetilde{x} < \frac{k(rh-cm)}{h(ak+r)}, \qquad r > r_\varepsilon := \frac{k\left(\frac{f(x)g(x)\left(\frac{e_2}{\gamma}-\frac{e_1}{\beta}\right)-\left(\frac{\mu_2}{\gamma}f(x)-\frac{\mu_1}{\beta}g(x)\right)}{x} + \frac{cm}{h} + ax\right)}{(k-x)}.$$

Under this condition , we get the existence of at least one nonnegative solution of system.

3.2. **Asymptotic behavior of the system (1.1).** In this part, we are interested in determining the asymptotic stability of the equilibria obtained in the previous section. For the time-fractional-order derivative, the concept of the local stability is very different from the first-order derivative, where in this case, we have an expansion of the stability region in comparison with the first-order derivative.

Let $E(x, y, z, u)$ be an equilibrium for the system (1.1). The Jacobian matrix of system (1.1) at $E(x, y, z, u)$ is expressed as

$$J(E) = \begin{pmatrix} r - 2ax - \frac{2rx}{k} - f^{'}(x)y - g^{'}(x)z & -f(x) & -g(x) & -c \\ e_1 f^{'}(x)y & e_1 f(x) - \mu_1 - \beta z & -\beta y & 0 \\ e_2 g^{'}(x)z & -\gamma z & e_2 g(x) - \mu_2 - \gamma y & 0 \\ m & 0 & 0 & -h \end{pmatrix} \tag{3.6}$$

At $E_0(0, 0, 0, 0)$, the Jacobian matrix of the system (1.1) is

$$J(E_0) = \begin{pmatrix} r & -f(0) & -g(0) & -c \\ 0 & e_1 f(0) - \mu_1 & 0 & 0 \\ 0 & 0 & e_2 g(0) - \mu_2 & 0 \\ m & 0 & 0 & -h \end{pmatrix},$$

and the characteristic equation for J(E_0) is

$$(\lambda - (e_1 f(0) - \mu_1))\,(\lambda - (e_2 g(0) - \mu_2))\,(\lambda^2 + (h-r)\lambda + cm - rh) = 0. \tag{3.7}$$

The eigenvalues of (3.7) are

$$\lambda_2 = e_1 f(0) - \mu_1, \quad \lambda_3 = e_2 g(0) - \mu_2, \quad \lambda_{1,4} = \frac{-(h-r) \pm \sqrt{\Delta_1}}{2}, \tag{3.8}$$

where $\Delta_1 = (h-r)^2 - 4(cm - rh)$.

Obviously, $\lambda_2 = e_1 f(0) - \mu_1 < 0$ *and* $\lambda_3 = e_2 g(0) - \mu_2 < 0$ are always negative. Now we discuss the eigenvalues λ_1 and λ_4 , it is clear that the cases $h > r$, $h = r$ and $h < r$ are possible, so we consider three separate cases.

Case **1.** $h > r$

(1a) $rh < cm$. If $\Delta_1 \geq 0$, we can derive from (3.8) that four eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are negative, which imply that the equilibrium point E_0 is locally asymptotically stable for all $0 < q < 1$. In fact, $|\arg(\lambda_{1,2,3,4})| = \pi > \frac{q\pi}{2}$ for all $0 < q < 1$, which satisfy the condition of Lemma 2.3. If $\Delta_1 < 0$, then λ_1 and λ_4 are complex conjugates with negative real parts, which imply $|\arg(\lambda_{1,4})| = \arctan\left(\frac{\sqrt{-\Delta_1}}{r-h}\right) + \pi > \frac{q\pi}{2}$ for all $0 < q < 1$. According to Lemma 2.3, we know that the equilibrium point E_0 is locally asymptotically stable.

(1b) $rh = cm$. From (3.7) we know that one eigenvalue must be zero and remaining three eigenvalues are negative. Then E_0 is marginally stable.

(1c) $rh > cm$. Then $\Delta_1 = (h+r)^2 - 4(cm) > 0$. From (3.8), we see that one of the eigenvalues λ_1 and λ_4 is positive and the other eigenvalue is negative. Let $\lambda_4 < 0$ and $\lambda_1 > 0$, which imply $|\arg(\lambda_4)| = \pi > \frac{q\pi}{2}$ and $|\arg(\lambda_1)| = 0 < \frac{q\pi}{2}$ for all $0 < q < 1$. Hence E_0 is unstable.

Case **2**. $h = r$.

(2a) $rh < cm$. Then $\Delta_1 < 0$ and (3.7) has pure imaginary roots $\lambda_1 = 2\sqrt{cm - rh}i$ and $\lambda_4 = -2\sqrt{cm - rh}i$, which mean $|\arg(\lambda_{1,4})| = \frac{\pi}{2} > \frac{q\pi}{2}$ for all $0 < q < 1$. Since $\lambda_2 < 0$, $\lambda_3 < 0$ according to Lemma 2.3, we know that the equilibrium point E_0 is locally asymptotically stable.

(2b) $rh = cm$. From (3.7), we see that $\lambda_1 = \lambda_4 = 0$, λ_2 and λ_3 is negative. Then E_0 is marginally stable.

(2c) $rh > cm$. From (3.7), we know that $\lambda_1 = 2\sqrt{cm - rh}$ and $\lambda_4 = -2\sqrt{cm - rh}$, which imply $|\arg(\lambda_4)| = \pi > \frac{q\pi}{2}$ and $|\arg(\lambda_1)| = 0 < \frac{q\pi}{2}$ for all $0 < q < 1$. Hence E_0 is unstable.

Case **3**. $h < r$.

(3a) $rh < cm$. If $\Delta_1 \geq 0$, then the two eigenvalues λ_1 and λ_4 are positive which imply $|\arg(\lambda_{1,4})| = 0 < \frac{q\pi}{2}$ for all $0 < q < 1$. Thus the equilibrium point E_0 is unstable. If $\Delta_1 < 0$, then λ_1 and λ_4 are complex conjugates with positive real parts. According to Lemma 2.3, we know that the equilibrium point E_0 is locally asymptotically stable if $|\arg(\lambda_{1,4})| = \arctan\left(\frac{\sqrt{-\Delta_1}}{r-h}\right) > \frac{q\pi}{2}$ is satisfied.

(3b) $rh = cm$. It is clear that (3.7) has a positive eigenvalue $\lambda_1 = r - h$, which means $|\arg(\lambda_1)| = 0 < \frac{q\pi}{2}$ for all $0 < q < 1$. Hence E_0 is unstable.

(3c) $rh > cm$. Then $\Delta_1 = (h+r)^2 - 4(cm) > 0$. From (3.7) , we see that one of the eigenvalues λ_1 and λ_4 is positive and the other eigenvalue is negative. Thus the equilibrium point E_0 is unstable.

If $h<r$, $rh<cm$, $(h+r)^2<4(cm)$, one has $|\arg(\lambda_{1,4})|=\arctan\left(\frac{\sqrt{-\Delta_1}}{r-h}\right)<\frac{\pi}{2}$, where $\Delta_1=(h-r)^2-4(cm-rh)$, thus $q<\frac{2}{\pi}\arctan\left(\frac{\sqrt{4(cm-rh)-(h-r)^2}}{r-h}\right)<\frac{2}{\pi}\times\frac{\pi}{2}=1$.

Hence we resume the stability conditions for the equilibrium $E_0(0,0,0,0)$ by the following theorem.

Theorem 3.1. *The trivial equilibrium point $E_0(0,0,0,0)$ representing the extinction of the three populations of the system (1.1) is locally asymptotically stable if either of the following criteria is satisfied:*
(i) $h\geq r$ *and* $rh<cm$,
(ii) $h<r,\ rh<cm,\ (h+r)^2<4cm$ *and* $0<q<\frac{2}{\pi}\arctan\left(\frac{\sqrt{4(cm-rh)-(h-r)^2}}{r-h}\right)$.

At the predator-free equilibrium $E_1(x_1,0,0,u_1)$, the Jacobian matrix of the system (1.1) is

$$J(E_1)=\begin{pmatrix} r-\frac{2cm}{h} & -f\left(\frac{k(rh-cm)}{h(ak+r)}\right) & -g\left(\frac{k(rh-cm)}{h(ak+r)}\right) & -c \\ 0 & e_1f\left(\frac{k(rh-cm)}{h(ak+r)}\right)-\mu_1 & 0 & 0 \\ 0 & 0 & e_2g\left(\frac{k(rh-cm)}{h(ak+r)}\right)-\mu_2 & 0 \\ m & 0 & 0 & -h \end{pmatrix} \tag{3.9}$$

and the characteristic equation for $E_1(x_1,0,0,u_1)$ is

$$(\lambda-(e_1f(x_1)-\mu_1))(\lambda-(e_2g(x_1)-\mu_2))\left(\lambda^2+\left(r-\frac{2cm}{h}+h\right)\lambda+rh-cm\right)=0. \tag{3.10}$$

The Jacobian matrix (3.9) has the eigenvalues

$$\begin{aligned}\lambda_2 &= e_1f\left(\frac{k(rh-cm)}{h(ak+r)}\right)-\mu_1, \qquad \lambda_3=e_2g\left(\frac{k(rh-cm)}{h(ak+r)}\right)-\mu_2,\\ \lambda_{1,4} &= \frac{-\frac{h^2+rh-2cm}{h}\pm\sqrt{\Delta_2}}{2},\end{aligned} \tag{3.11}$$

where

$$\Delta_2=\frac{\left(h^2+rh-2cm\right)^2-4h^2(rh-cm)}{h^2}.$$

Then, we have

$$\lambda_2=\begin{cases}<0 & for \quad \frac{k(rh-cm)}{h(ak+r)}<x_3\\ >0 & for \quad \frac{k(rh-cm)}{h(ak+r)}>x_3\end{cases}$$

and

$$\lambda_3=\begin{cases}<0 & for \quad \frac{k(rh-cm)}{h(ak+r)}<x_2\\ >0 & for \quad \frac{k(rh-cm)}{h(ak+r)}>x_2.\end{cases}$$

Obviously, λ_2 and λ_3 are negative if $x < \widetilde{x} = \min\{x_2, x_3\}$. Now we discuss the eigenvalues λ_1 and λ_4, it is clear that the cases $h^2+rh-2cm > 0$, $h^2+rh-2cm = 0$ *and* $h^2+rh-2cm < 0$ are possible, respectively, so we consider three separate cases.

Case **4**. $h^2 + rh - 2cm > 0$.

$rh > cm$. If $\Delta_2 \geq 0$, we can derive from (3.11) that four eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are negative if $x < \widetilde{x} = \min\{x_2, x_3\}$, which imply that the equilibrium point E_1 is locally asymptotically stable for all $0 < q < 1$. In fact, $|\arg(\lambda_{1,2,3,4})| = \pi > \frac{q\pi}{2}$ for all $0 < q < 1$, which satisfy the condition of Lemma 2.3. If $\Delta_2 < 0$, then λ_1 and λ_4 are complex conjugates with negative real parts, which imply $|\arg(\lambda_{1,4})| = \arctan\left(\frac{h\sqrt{-\Delta_2}}{2cm-h^2-rh}\right) + \pi > \frac{q\pi}{2}$ for all $0 < q < 1$. According to Lemma 2.3, we know that the equilibrium point E_1 is locally asymptotically stable

Case **5**. $h^2 + rh - 2cm = 0$.

$rh > cm$. Then $\Delta_2 < 0$ and (3.10) has pure imaginary roots $\lambda_1 = 2\sqrt{rh-cm}i$ and $\lambda_4 = -2\sqrt{rh-cm}i$ which means that $|\arg(\lambda_{1,4})| = \frac{\pi}{2} > \frac{q\pi}{2}$ for all $0 < q < 1$. If $x < \widetilde{x} = \min\{x_2, x_3\}$ holds, then we have $\lambda_2 < 0$ and $\lambda_3 < 0$. According to Lemma 2.3, we know that the equilibrium point E_1 is locally asymptotically stable.

Case **6**. $h^2 + rh - 2cm < 0$.

$rh > cm$, If $\Delta_2 \geq 0$, then the two eigenvalues λ_1 and λ_4 are positive which imply $|\arg(\lambda_{1,4})| = 0 < \frac{q\pi}{2}$ for all $0 < q < 1$. Thus the equilibrium point E_1 is unstable. If $\Delta_2 < 0$ then λ_1 and λ_4 are complex conjugates with positive real parts. In addition, $x < \widetilde{x} = \min\{x_2, x_3\}$ holds, then we have $\lambda_2 < 0$ and $\lambda_3 < 0$. According to Lemma 2.3, we know that the equilibrium point E_1 is locally asymptotically stable if $|\arg(\lambda_{1,4})| = \arctan\left(\frac{h\sqrt{-\Delta_2}}{2cm-h^2-rh}\right) > \frac{q\pi}{2}$ is satisfied.

When $h^2 + rh - 2cm < 0$, $rh > cm$, $\left(h^2 + rh - 2cm\right)^2 < 4h^2(rh - cm)$, one has $|\arg(\lambda_{1,4})| = \arctan\left(\frac{\sqrt{4h^2(rh-cm)-(h^2+rh-2cm)^2}}{2cm-h^2-rh}\right) < \frac{\pi}{2}$, thus

$q < \frac{2}{\pi}\arctan\left(\frac{\sqrt{4h^2(rh-cm)-(h^2+rh-2cm)^2}}{2cm-h^2-rh}\right) < \frac{2}{\pi} \times \frac{\pi}{2} = 1.$

Hence we resume the stability conditions for the equilibrium $E_1(x_1, 0, 0, u_1)$ by the following theorem.

Theorem 3.2. *The predator-extinction equilibrium point of the system is locally asymptotically stable if either of the following criteria is satisfied:*

(i) $h^2 + rh - 2cm \geq 0$, $rh > cm$ *and* $x < \widetilde{x} = \min\{x_2, x_3\}$,

(ii) $h^2 + rh - 2cm < 0$, $rh > cm$, $\left(h^2 + rh - 2cm\right)^2 < 4h^2(rh - cm)$,

$0 < q < \frac{2}{\pi}\arctan\left(\frac{\sqrt{4h^2(rh-cm)-(h^2+rh-2cm)^2}}{2cm-h^2-rh}\right)$ *and* $x < \widetilde{x} = \min\{x_2, x_3\}$.

Now we analyze the linear stability of FPFE point of $E_2(x_2, 0, z_2, u_2)$. The Jacobian matrix corresponding to the equilibrium FPFE is evaluated as

$$J(E_2) = \begin{pmatrix} r - 2ax_2 - \frac{2rx_2}{k} - g'(x_2)z_2 & -f(x_2) & -g(x_2) & -c \\ 0 & e_1 f(x_2) - \mu_1 - \beta z_2 & 0 & 0 \\ e_2 g'(x_2)z_2 & -\gamma z_2 & 0 & 0 \\ m & 0 & 0 & -h \end{pmatrix}. \tag{3.12}$$

As a first look, we can deduce that $\lambda_2 = e_1 f(x_2) - \mu_1 - \beta z_2$ is an eigenvalue of the Jacobian matrix (3.12). By replacing the explicit formula of $z_2 = \frac{e_2 x_2\left(r - a x_2 - \frac{r x_2}{k} - \frac{cm}{h}\right)}{\mu_2}$ we obtain $\lambda_2 = e_1 f(x_2) - \mu_1 - \frac{\beta e_2 x_2\left(r - a x_2 - \frac{r x_2}{k} - \frac{cm}{h}\right)}{\mu_2}$. Obviously, if $e_1 f(x_2) - \mu_1 < 0$ (equivalent to $x_2 < x_3$), then $|\arg(\lambda_2)| > \frac{q\pi}{2}$. Now we presume that if $e_1 f(x_2) - \mu_1 > 0$ (equivalent to $x_2 > x_3$). Then

$$\lambda_2 = \begin{cases} > 0 & for \quad r < r_1 := \frac{k\left(\frac{(e_1 f(x_2) - \mu_1)\mu_2}{\beta e_2 x_2} + a x_2 + \frac{cm}{h}\right)}{k - x_2}, \\ < 0 & for \quad r > r_1. \end{cases}$$

Under the condition $\lambda_2 > 0$, we get $|\arg(\lambda_2| < \frac{q\pi}{2}$. This means that FPFE is an unstable equilibrium point. Besides, from $\lambda_2 > 0$, we conclude that $|\arg(\lambda_2)| < \frac{q\pi}{2}$. This means that three remaining eigenvalues of the Jacobian matrix (3.12) determine the stability (resp., instability) of this equilibrium. Note that these significant eigenvalues are the eigenvalues of the matrix

$$\widetilde{J} = \begin{pmatrix} r - 2a x_2 - \frac{2 r x_2}{k} - g^{'}(x_2) z_2 & -g(x_2) & -c \\ e_2 g^{'}(x_2) z_2 & 0 & 0 \\ m & 0 & -h \end{pmatrix}. \tag{3.13}$$

To determine the nature of the eigenvalues of the reduced matrix (3.13), we define the characteristic equation of (3.13) as

$$P(\lambda) = \lambda^3 + \vartheta_1 \lambda^2 + \vartheta_2 \lambda + \vartheta_3,$$

where

$$\begin{aligned} \vartheta_1 &= h - r + 2a x_2 + \frac{2 r x_2}{k} + g^{'}(x_2) z_2, \\ \vartheta_2 &= -cm - hr + 2ha x_2 + \frac{2hr x_2}{k} + h g^{'}(x_2) z_2 + e_2 g^{'}(x_2) g(x_2) z_2, \\ \vartheta_3 &= h e_2 g^{'}(x_2) g(x_2) z_2. \end{aligned}$$

$D(P)$ denotes the discriminant of the cubic polynomial $P(\lambda)$, as follows:

$$\begin{aligned} D(P) &= \begin{vmatrix} 1 & \vartheta_1 & \vartheta_2 & \vartheta_3 & 0 \\ 0 & 1 & \vartheta_1 & \vartheta_2 & \vartheta_3 \\ 3 & 2\vartheta_1 & \vartheta_2 & 0 & 0 \\ 0 & 3 & 2\vartheta_1 & \vartheta_2 & 0 \\ 0 & 0 & 3 & 2\vartheta_1 & \vartheta_2 \end{vmatrix} \\ &= 18\vartheta_1\vartheta_2\vartheta_3 + (\vartheta_1\vartheta_2)^2 - 4\vartheta_3(\vartheta_1)^2 - 4(\vartheta_2)^2 - 27(\vartheta_3)^2 \end{aligned}$$

Using the Routh-Hurwitz stability criterion for fractional calculus defined in [7], [16] and [17] we get the stability conditions for the nontrivial equilibrium.

Theorem 3.3. *The positive equilibrium point is asymptotically stable if either of the following criteria is satisfied:*

(i) $D(P) > 0$, $\vartheta_1 > 0$, $\vartheta_3 > 0$, $\vartheta_1\vartheta_2 - \vartheta_3 > 0$ for all $q \in (0,1)$,
(ii) $D(P) < 0$, $\vartheta_1 \geq 0$, $\vartheta_2 \geq 0$, $\vartheta_3 > 0$, $0 < q < \frac{2}{3}$,
(iii) $D(P) < 0$, $\vartheta_1 > 0$, $\vartheta_3 > 0$, $\vartheta_1\vartheta_2 = \vartheta_3$ for all $q \in (0,1)$.

Hence we resume the stability conditions for the equilibrium $E_2(x_2, 0, z_2, u_2)$ by the following theorem. Therefore,

Theorem 3.4. *For FPFE, if* $x_2 < \frac{k(rh-cm)}{h(ak+r)}$*, then we have;*
(i) *If* $x_2 > x_3$ *and* $r < r_1$*, then the FPFE is unstable,*
(ii) *For* $x_2 < x_3$ *or* $(x_2 > x_3$ *and* $r > r_1)$ *if one of the condition* $(i), (ii)$ *or* (iii) *in Theorem 3.3 holds we get the local stability of FPFE.*

To study the stability of the SPFE of $E_3(x_3, y_3, 0, u_3)$, the Jacobian matrix corresponding to the equilibrium SPFE is evaluated as

$$J(E_3) = \begin{pmatrix} r - 2ax_3 - \frac{2rx_3}{k} - f^{'}(x_3)y_3 & -f(x_3) & -g(x_3) & -c \\ e_1 f^{'}(x_3)y_3 & 0 & -\beta y_3 & 0 \\ 0 & 0 & e_2 g(x_3) - \mu_2 - \gamma y_3 & 0 \\ m & 0 & 0 & -h \end{pmatrix}. \tag{3.14}$$

As a first look, we can deduce that $\lambda_3 = e_2 g(x_3) - \mu_2 - \gamma y_3$ is an eigenvalue of the Jacobian matrix (3.14). By replacing the explicit formula of $y_3 = \frac{e_1 x_3\left(r - ax_3 - \frac{rx_3}{k} - \frac{cm}{h}\right)}{\mu_1}$ we obtain $\lambda_3 = e_2 g(x_3) - \mu_2 - \frac{\gamma e_1 x_3\left(r - ax_3 - \frac{rx_3}{k} - \frac{cm}{h}\right)}{\mu_1}$. Obviously, if $e_2 g(x_3) - \mu_2 < 0$ (equivalent to $x_2 > x_3$), then $|\arg(\lambda_3)| > \frac{q\pi}{2}$. Now we presume that if $e_1 f(x_2) - \mu_1 > 0$ (equivalent to $x_2 < x_3$). Then

$$\lambda_3 = \begin{cases} > 0 & for \quad r < r_2 := \frac{k\left(\frac{(e_2 g(x_3) - \mu_2)\mu_1}{\gamma e_1 x_3} + ax_3 + \frac{cm}{h}\right)}{k - x_3}, \\ < 0 & for \quad r > r_2. \end{cases}$$

Under the condition $\lambda_3 > 0$, we get $|\arg(\lambda_3)| < \frac{q\pi}{2}$. This means that FPFE is an unstable equilibrium point. Besides, from $\lambda_3 > 0$, we conclude that $|\arg(\lambda_3)| < \frac{q\pi}{2}$. This means that three remaining eigenvalues of the Jacobian matrix (3.14) determine the stability (resp., instability) of this equilibrium. Note that these significant eigenvalues are the eigenvalues of the matrix

$$\overline{\widetilde{J}} = \begin{pmatrix} r - 2ax_3 - \frac{2rx_3}{k} - f^{'}(x_3)y_3 & -f(x_3) & -c \\ e_1 f^{'}(x_3)y_3 & 0 & 0 \\ m & 0 & -h \end{pmatrix}. \tag{3.15}$$

To determine the nature of the eigenvalues of the reduced matrix (3.15), we define the characteristic equation of (3.15) as

$$P^*(\lambda) = \lambda^3 + \theta_1\lambda^2 + \theta_2\lambda + \theta_3,$$

where

$$\theta_1 = h - r + 2ax_3 + \frac{2rx_3}{k} + f^{'}(x_3)y_3,$$
$$\theta_2 = -cm - hr - 2hax_3 + \frac{2hrx_3}{k} + hf^{'}(x_3)y_3 + e_1 f^{'}(x_3)f(x_3)y_3,$$
$$\theta_3 = he_1 f^{'}(x_3)f(x_3)y_3.$$

$D(P^*)$ denotes the discriminant of the cubic polynomial
$D(P^*) \ = 18\theta_1\theta_2\theta_3 + (\theta_1\theta_2)^2 - 4\theta_3(\theta_1)^2 - 4(\theta_2)^2 - 27(\theta_3)^2$

With the same technics in Theorem 3.3, we get the stability conditions for the nontrivial equilibrium. Therefore

Theorem 3.5. *For SPFE if $x_2 < \frac{k(rh-cm)}{h(ak+r)}$, then we have;*
(i) If $x_2 < x_3$ and $r < r_2$, then the FPFE is unstable.
(ii) For $x_2 > x_3$ or $(x_2 < x_3$ and $r > r_2)$ if one of the conditions in (i), (ii) or (iii) in Theorem 3.3 holds, we get the local stability of FPFE.

Now we are in a position to focus on studying the local behavior of the coexistence equilibrium. For this positive equilibrium point, we have that assumption for the existence of at least one non-negative solution of the system (1.1). The Jacobian matrix of the system (1.1) evaluated at the equilibrium $E_4(x^*, y^*, z^*, u^*)$ is given by

$$J(E_4) = \begin{pmatrix} r - 2ax^* - \frac{2rx^*}{k} - f^{'}(x^*)y^* - g^{'}(x^*)z^* & -f(x^*) & -g(x^*) & -c \\ e_1 f^{'}(x^*)y^* & e_1 f(x^*) - \mu_1 - \beta z^* & -\beta y^* & 0 \\ e_2 g^{'}(x^*)z^* & -\gamma z^* & e_2 g(x^*) - \mu_2 - \gamma y^* & 0 \\ m & 0 & 0 & -h \end{pmatrix}. \tag{3.16}$$

Therefore, the characteristic equation associated with Jacobian (3.16) is

$$\Delta(\lambda) = \lambda^4 + \Phi_1\lambda^3 + \Phi_2\lambda^2 + \Phi_3\lambda + \Phi_4,$$

where

$$\begin{aligned}
\Phi_1 &= h - r + 2ax^* + \frac{2rx^*}{k} + g^{'}(x^*)z^* + f^{'}(x^*)y^*, \\
\Phi_2 &= cm - rh + 2hax^* + \frac{2rhx^*}{k} + hg^{'}(x^*)z^* + hf^{'}(x^*)y^* + \beta y^*\gamma z^* + e_1 f^{'}(x^*)f(x^*)y^* \\
&\quad + e_2 g^{'}(x^*)g(x^*)z^*, \\
\Phi_3 &= h\beta y^*\gamma z^* + he_1 f^{'}(x^*)f(x^*)y + he_2 g^{'}(x^*)g(x^*)z^* - r\beta y^*\gamma z^* + 2ax^*\beta y^*\gamma z^* \\
&\quad + \frac{2rx^*}{k}\beta y^*\gamma z^* + g^{'}(x^*)\left(z^*\right)^2\beta y^*\gamma + f^{'}(x^*)\left(y^*\right)^2\beta\gamma z^* - f(x^*)\beta y^* e_2 g^{'}(x^*)z^* \\
&\quad - g(x^*)e_1 f^{'}(x^*)y^*\gamma z^*, \\
\Phi_4 &= cm\beta y^*\gamma z^* - rh\beta y^*\gamma z^* + h\beta y^*\gamma z^* 2ax^* + \frac{2hrx^*}{k}\beta y^*\gamma z^* + hg^{'}(x^*)\left(z^*\right)^2\beta y^*\gamma \\
&\quad + hf^{'}(x^*)\left(y^*\right)^2\beta\gamma z^* - hf(x^*)\beta y^* e_2 g^{'}(x^*)z^* - hg(x^*)e_1 f^{'}(x^*)y^*\gamma z^*.
\end{aligned}$$

and $D(\triangle)$ denotes the discriminant of the polinom $\triangle(\lambda)$ as follows,

$$
\begin{aligned}
D(\Delta) &= \begin{vmatrix}
1 & 0 & 0 & 4 & 0 & 0 & 0 \\
\Phi_1 & 1 & 0 & 3\Phi_1 & 4 & 0 & 0 \\
\Phi_2 & \Phi_1 & 1 & 2\Phi_2 & 3\Phi_1 & 4 & 0 \\
\Phi_3 & \Phi_2 & \Phi_1 & \Phi_3 & 2\Phi_2 & 3\Phi_1 & 4 \\
\Phi_4 & \Phi_3 & \Phi_2 & 0 & \Phi_3 & 2\Phi_2 & 3\Phi_1 \\
0 & \Phi_4 & \Phi_3 & 0 & 0 & \Phi_3 & 2\Phi_2 \\
0 & 0 & \Phi_4 & 0 & 0 & 0 & \Phi_3
\end{vmatrix} \\
&= 256\left(\Phi_4\right)^3 - 192\Phi_1\Phi_3\left(\Phi_4\right)^2 - 128\left(\Phi_4\right)^2\left(\Phi_2\right)^2 + 144\Phi_2(\Phi_3)^2\Phi_4 \\
&\quad -27(\Phi_3)^4 + 144(\Phi_1)^2\Phi_2(\Phi_4)^2 - 6(\Phi_1)^2(\Phi_3)^2\Phi_4 - 80\Phi_1(\Phi_2)^2\Phi_3\Phi_4 \\
&\quad +18\Phi_1\Phi_2(\Phi_3)^3 + 16(\Phi_2)^4\Phi_4 - 4(\Phi_2)^3(\Phi_3)^2 - 27(\Phi_1)^4(\Phi_4)^2 \\
&\quad +18(\Phi_1)^3\Phi_2\Phi_3\Phi_4 - 4(\Phi_1)^3(\Phi_3)^3 - 4(\Phi_1)^2(\Phi_2)^3\Phi_4 + (\Phi_1)^2(\Phi_2)^2(\Phi_3)^2.
\end{aligned}
$$

Using the Routh-Hurwitz stability criterion for fractional calculus, we get the stability conditions for the nontrivial positive equilibrium.

Theorem 3.6. *The positive equilibrium point $E_4(x^*, y^*, z^*, u^*)$ is asymptotically stable if either of the following criteria is satisfied:*

(i) $D(\Delta) > 0, \Phi_1 > 0,\ \ \Phi_3 > 0,\ \ \Phi_4 > 0,\ \ \Phi_1\Phi_2 - \Phi_3 > 0,$ $\Phi_3(\Phi_1\Phi_2 - \Phi_3) - (\Phi 1)^2\,\Phi_4 > 0$

(ii) $D(\Delta) < 0,\ \Phi_1 \geq 0,\ \Phi_2 \geq 0,\ \Phi_3 \geq 0,\ \Phi_4 \geq 0,\ \ 0 < q < \frac{2}{3}$.

(iii) $D(\Delta) < 0,\ \Phi_1 > 0,\ \Phi_3 > 0,\ \ \Phi_4 > 0,\ \Phi_1\Phi_2 = \Phi_3,\ \ \Phi_3(\Phi_1\Phi_2 - \Phi_3) = (\Phi_1)^2\,\Phi_4)$ *for all* $q \in (0, 1)$.

4. Numerical analysis of the system (1.1)

The main purpose of this section is to solve the following fractal problem numerically:

$$ {}_0^cD_t^qV(t) = P\left(t, V(t)\right). \tag{4.1} $$

By applying the fundamental theorem of fractional calculus on (1.1), we get

$$ V(t) - V(0) = \frac{1}{\Gamma(q)}\int_0^t P\left(s, V(s)\right)(t-s)^{q-1}ds. \tag{4.2} $$

Letting $t = t_n = nh$ in (4.2), we arrive at

$$ V(t_n) = V(0) + \frac{1}{\Gamma(q)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} P\left(s, V(s)\right)(t_n - s)^{q-1}ds. $$

Now we can approximate the function $P(t, V(t))$ by the following linear approximation:

$$P\left(t,K(t)\right) \approx P\left(t_{i+1},V_{i+1}\right) + \frac{t-t_{i+1}}{h}\left(P\left(t_{i+1},V_{i+1}\right) - P\left(t_i,V_i\right)\right), \qquad t\epsilon[t_i,t_{i+1}] \tag{4.3}$$

with the notation $V_i = V(t_i)$.

By substituting equation (4.2) into (4.3) and applying some algebra (for more detail, see [6]) we get

$$V_n = V_0 + h^q\left(\Phi_n P(t_0,V_0) + \sum_{i=1}^{n}\Psi_{n-i}P\left(t_i,V_i\right)\right) \tag{4.4}$$

with

$$\Phi_n = \frac{(n-1)^{q+1} - n^q(n-q-1)}{\Gamma(q+2)},$$

$$\Psi_n = \begin{cases} \frac{1}{\Gamma(q+2)}, & n=0, \\ \frac{(n-1)^q - 2n^q + (n+1)^q}{\Gamma(q+2)} & n = 1,2,3,... \end{cases}$$

Using the numerical method presented in the formula (4.4) to solve the problem (4.1), we obtain the following iterative schemes:

$$\begin{aligned}
x_n &= x_0 + h^q\left(\Phi_n P_1\left(x_0,y_0,z_0,u_0\right) + \sum_{i=1}^{n}\Psi_{n-i}P_1\left(x_i,y_i,z_i,u_i\right)\right),\\
y_n &= y_0 + h^q\left(\Phi_n P_2\left(x_0,y_0,z_0,u_0\right) + \sum_{i=1}^{n}\Psi_{n-i}P_2\left(x_i,y_i,z_i,u_i\right)\right),\\
z_n &= z_0 + h^q\left(\Phi_n P_3\left(x_0,y_0,z_0,u_0\right) + \sum_{i=1}^{n}\Psi_{n-i}P_3\left(x_i,y_i,z_i,u_i\right)\right),\\
u_n &= u_0 + h^q\left(\Phi_n P_4\left(x_0,y_0,z_0,u_0\right) + \sum_{i=1}^{n}\Psi_{n-i}P_4\left(x_i,y_i,z_i,u_i\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
P_1(x,y,z,u) &= x(r - ax - \frac{rx}{k}) - f(x)y - g(x)z - cu,\\
P_2(x,y,z,u) &= e_1 f(x)y - \mu_1 y - \beta yz,\\
P_3(x,y,z,u) &= e_2 g(x)z - \mu_2 z - \gamma yz,\\
P_4(x,y,z,u) &= -hu + mx.
\end{aligned}$$

5. Conclusion

In this research, we studied an ecological model with two predators fighting on one prey with a generalized functional response. We consider a fractional-order predator–prey model incorporating feedback control. The reason behind considering a comprehensive generalized class of functional interaction is to model the diversity in predator–prey interaction with the environment. These interactions can be affected by many factors, such as the environment and the adaptation of the three species. We analyzed the existence of different equilibrium points and some criteria were derived to ensure the asymptotical stability of these equilibrium points. In the first section, we studied the existence of the equilibria of the system (1.1), where we can have many equilibrium points next to the predator-free equilibrium. By analyzing the existence of the equilibria we obtained that these populations may have many scenarios. They include the extinction of three populations, two types of predators, the extinction of each population of predators, and finally the coexistence of the three populations. For the coexistence stage, we provided some conditions on the model parameters for the existence of this equilibrium. The theoretical results show that feedback control play important roles in adjusting coexistence of prey species and predator species. To determine which scenario will prevail, we have utilized the local asymptotic stability using the Jacobian matrix.

Acknowledgments. The authors are thankful to the referees for their critical remarks to improve this paper.

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