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q-DIFFERENCE OPERATOR ON $L_q^2(0,+\infty)$

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Abstract. In this research, the minimal and maximal operators defined by q- difference expression are given in the Hilbert space $L_q^2(0,\infty)$. The existence problem of a q^{-1}-normal extension for the minimal operator is mentioned. In addition, the sets of the minimal operator spectrum and the maximal operator spectrum are examined.

1. Introduction

The q-analysis first appeared in the 1740s, when Euler launched the division theory, also called the total analytic number theory, Euler wrote and compiled works in the early 1800s [4]. The advancement of q-calculus continued in 1813 under the study of Gauss, who gave the hypergeometric series and their interrelationships [5].

The study of quantum calculus, or q-calculus, which has been going on for 300 years since Euler, has often been regarded as one of the most difficult topics to deal with in mathematics. Today, due to its use in a variety of areas, such as mathematics, physics, rapid progress is being made in studies in the field of q-calculus. The working history of q- analysis, quantum mechanics, theta functions, hypergeometric functions, analytic number theory, finite difference theory, Mock theta functions, Bernoulli and Euler polynomials, gamma function theory has a wide variety of applications in combinatorics. Moreover, there is the application of the q-difference operator to thermodynamics. It has been demonstrated that the formalization of the q-calculus may be used to realize the thermodynamics of

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the q-deformed algebra. It is found that if it is used a suitable Jackson derivative instead of the ordinary thermodynamic derivative, then the entire structure of thermodynamics is maintained [9]. For some numerous contributions the history of q-calculus, fundamental principles, and fundamentals of q-differential equations, the key books [3], [8] and [1] can be cited.

Moreover, a closed linear operator T with dense domain on any Hilbert space is said formally q-normal operator iff $D(T) \subset D(T^*)$ and

$$TT^* = qT^*T.$$

When $D(T) = D(T^*)$ is satisfied for a formally q-normal operator, then T said a q-normal operator. Moreover, q-normal operators appear in quantum group theory in the study of the hermitean quantum plane and of quantum groups. For instance, the q-deformed quantum plane C_q^1 is a $*$-algebra with one generator T such that $TT^* = qT^*T$ [10]. Definitions of these and other classes which are called q-deformed operators was given and investigated by Ota [10], for detail analysis see [2, 11–14].

2. The Minimal and Maximal Operators $L_q^2(0,+\infty)$

Suppose that $L_q^2(0,+\infty)$ is defined as

$$L_q^2(0,+\infty) = \left\{ u : [0,+\infty) \to \mathbb{C} \ : \ \int\limits_0^{+\infty} |u(t)|^2 d_q t = (1-q) \sum_{k=-\infty}^{+\infty} q^k \left|u\left(q^k\right)\right|^2 < +\infty \right\}.$$

$L_q^2(0,+\infty)$ is a linear vector space with equivalent classes, which are defined for two functions u and v in the same equivalent class iff $u\left(q^k\right) = v\left(q^k\right),\ k \in \mathbb{Z}$. Also $L_q^2(0,+\infty)$ is separable and its the inner product is follows [1]

$$(u,v)_{L_q^2(0,+\infty)} := \int\limits_0^{+\infty} u(t)\,\overline{v(t)} d_q t\ ,\ u, v \in L_q^2(0,+\infty).$$

In addition, Jackson reintroduced the q-difference operator [7] and he defined as

$$D_q u(t) = \frac{u(t) - u(qt)}{(1-q)t}, \quad t \neq 0$$

and also the q-derivative for $t = 0$ is defined for $|q| < 1$ as

$$D_q u(0) = \lim_{n\to+\infty} \frac{u(tq^n) - u(0)}{tq^n},\ t = 0,$$

if there is the limit and it is independent of t.

Note that we have assume $0 < q < 1$ for this paper.

Corollary 1. *If* $u \in L_q^2(0,+\infty)$, *then* $\lim\limits_{n\to+\infty} u\left(\frac{1}{q^n}\right) = 0$.

Proposition 1. *If* $D_q u(t) \in L_q^2(0,+\infty)$, *then the limit* $\lim\limits_{n\to+\infty} u(q^n)$ *exists.*

Proof. Let $D_q u(t)$ be in $L_q^2(0,+\infty)$. Because the characteristic function $\chi_{[0,1]} \in L_q^2(0,+\infty)$ and

$$\begin{aligned}
\left(D_q u, \chi_{[0,1]}\right)_{L_q^2(0,+\infty)} &= \int_0^{+\infty} \chi_{[0,1]}(t)\, D_q u(t)\, d_q t \\
&= (1-q)\sum_{k=0}^{+\infty} q^k \frac{u\left(q^k\right)-u\left(q^{k+1}\right)}{(1-q)\,q^k} \\
&= \sum_{k=0}^{+\infty} u\left(q^k\right)-u\left(q^{k+1}\right) \\
&= \lim_{n\to+\infty} \sum_{k=0}^{n} u\left(q^k\right)-u\left(q^{k+1}\right) \\
&= u(1) - \lim_{n\to+\infty} u\left(q^n\right),
\end{aligned}$$

are true, the limit $\lim\limits_{n\to+\infty} u\left(q^n\right)$ exists. □

First of all, we give the abstract definition of maximal and minimal operators for differential operators [6]. Suppose that Ω is an n-dimensional infinitely differentiable manifold and a differential expression

$$p(.) = \sum_{|\alpha|\leqslant m} a_\alpha D^\alpha,$$

where the coefficients a_α are infinitely differentiable functions of $x=(x_1,\ldots,x_n)$. Also, $\alpha\in\mathbb{C}^n$, $|\alpha| = \alpha_1+\ldots+\alpha_n$, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}\ldots D_n^{\alpha_n}$ and $D_k = \frac{1}{i}\frac{\partial}{\partial x_k}$ are denoted. The formal adjoint of the expression $p(.)$ is the form $p^+(.) = \sum\limits_{|\alpha|\leqslant m} (-1)^{|\alpha|}\overline{a_\alpha} D^\alpha$ in $L^2(\Omega)$. In this case, two operators

$$\begin{aligned}
&{P_0}' u = p(u), \;\; {P_0}' : C_0^\infty(\Omega)\subset L^2(\Omega)\to L^2(\Omega), \\
&{P_0^+}' u = p^+(u), \;\; {P_0^+}' : C_0^\infty(\Omega)\subset L^2(\Omega)\to L^2(\Omega)
\end{aligned}$$

have closures in $L^2(\Omega)$ and these closures are denoted by P_0 and P_0^+ respectively. The operator P_0 is said as the minimal operator defined by the expression p. Similarly, P_0^+ is called the minimal operator defined by the differential expression p^+. The adjoint P of P_0^+ is said the maximal operator generated by p. It is easy seen that $D(P_0) = D(P^+)$ and $D(P) = D(P_0^+)$.

The q-derivative for multiplication of two functions $u(t)$ and $v(t)$ defined on $[0,+\infty)$ is follows for all $t\in(0,+\infty)$

$$D_q(uv)(t) = v(t)\, D_q u(t) + u(qt)\, D_q v(t).$$

This relation said q-product rule. It is obtain that

$$\begin{aligned}
\int\limits_0^{+\infty} D_q(uv)(t)\,d_qt &= (1-q)\sum_{k=-\infty}^{+\infty} q^k\left(\frac{u\left(q^k\right)v\left(q^k\right)-u\left(q^{k+1}\right)v\left(q^{k+1}\right)}{(1-q)\,q^k}\right)\\
&= \sum_{k=-\infty}^{+\infty} u\left(q^k\right)v\left(q^k\right)-u\left(q^{k+1}\right)v\left(q^{k+1}\right)\\
&= \lim_{n,m\to+\infty}\sum_{k=-m}^{n} u\left(q^k\right)v\left(q^k\right)-u\left(q^{k+1}\right)v\left(q^{k+1}\right)\\
&= \lim_{n,m\to+\infty} u\left(q^{-m}\right)v\left(q^{-m}\right)-u\left(q^{-m+1}\right)v\left(q^{-m+1}\right)\\
&\quad +u\left(q^{-m+1}\right)v\left(q^{-m+1}\right)-u\left(q^{-m+2}\right)v\left(q^{-m+2}\right)\\
&\quad +u\left(q^{-m+2}\right)v\left(q^{-m+2}\right)-\ldots+u\left(q^{-1}\right)v\left(q^{-1}\right)\\
&\quad -u(1)\,v(1)+u(q)\,v(q)+\ldots+u\left(q^n\right)v\left(q^n\right)-u\left(q^{n+1}\right)v\left(q^{n+1}\right)\\
&= \lim_{n,m\to+\infty} u\left(q^{-m}\right)v\left(q^{-m}\right)-u\left(q^n\right)v\left(q^n\right)\\
&= -\lim_{n\to+\infty} u\left(q^n\right)v\left(q^n\right)
\end{aligned}$$

is finite for any $u(t)\,,v(t)\,,D_qu(t)\,,D_qv(t)\in L_q^2\left((0,+\infty)\right)$. Because

$$\begin{aligned}
(D_qu,v)_{L_q^2(0,+\infty)} &= \int\limits_0^{+\infty} D_qu(t)\,\overline{v(t)}d_qt \qquad (1)\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}-\int\limits_0^{+\infty} u(t)\,\overline{D_qu(t)}d_qt\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}-(1-q)\sum_{k=-\infty}^{+\infty} q^ku\left(q^{k+1}\right)\frac{u\left(q^k\right)-u\left(q^{k+1}\right)}{(1-q)\,q^k}\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}+(1-q)\sum_{k=-\infty}^{+\infty} q^{k+1}u\left(q^{k+1}\right)\frac{u\left(q^{k+1}\right)-u\left(q^k\right)}{(1-q)\,q^{k+1}}\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}-(1-q)\sum_{k=-\infty}^{+\infty} q^ku\left(q^k\right)\frac{1}{q}D_{q^{-1}}u(t)\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}+\int\limits_0^{+\infty} u(t)\,\overline{-\frac{1}{q}D_{q^{-1}}v(t)}d_qt\\
&= -\lim_{k\to+\infty} u\left(q^k\right)\overline{v\left(q^k\right)}+\left(u,-\frac{1}{q}D_{q^{-1}}v\right)_{L_q^2(0,+\infty)}, \qquad (2)
\end{aligned}$$

the formal adjoint expression of the expression D_q is $-\frac{1}{q}D_{q^{-1}}$ on $L_q^2(0,+\infty)$.

Now, let's define the linear operators $L_0 : D_0 \subset L_q^2(0,+\infty) \to L_q^2(0,+\infty)$ of the form $L_0 u(t) = D_q u(t)$ where its domain is

$$D_0 = \left\{ u \in L_q^2(0,+\infty) \ : \ D_q u(t) \in L_q^2(0,+\infty) \ \ \textit{and} \ \lim_{k\to+\infty} u\left(q^k\right) = 0 \right\}$$

and $L : D \subset L_q^2(0,+\infty) \to L_q^2(0,+\infty)$ of the form $L_0 u(t) = D_q u(t)$ where

$$D = \left\{ u \in L_q^2(0,+\infty) \ : \ D_q u(t) \in L_q^2(0,+\infty) \right\}.$$

We say that these operators are the minimal operator and the maximal operator generated by the q-difference expression, respectively. Moreover, $L_0 \subset L$ is obvious, i.e. the maximal operator L is an extension of the minimal operator L_0.

Theorem 1. *The operator L_0 is a formally q^{-1}-normal operator on $L_q^2(0,+\infty)$.*

Proof. The set of functions

$$\varphi_m(t) := \begin{cases} \dfrac{1}{q^{\frac{m}{2}}\sqrt{1-q}}, & t = q^m \\ 0, & ,\textit{otherwise} \end{cases}, \quad m \in \mathbb{Z}$$

is an orthogonal basis of $L_q^2(0,+\infty)$ and this basis is clearly contained in D_0. Therefore, the minimal linear operator L_0 has dense domain.

Now let's show that the minimal operator is closed. Suppose that any sequence $\{u_n\} \subset D_0$ such that $u_n \xrightarrow[n\to\infty]{} u$ and $L_0 u_n \xrightarrow[n\to\infty]{} f$. In this case,

$$\|u_n - u\|^2_{L_q^2(0,+\infty)} = (1-q) \sum_{k=-\infty}^{+\infty} q^k \left|u_n\left(q^k\right) - u\left(q^k\right)\right|^2 \xrightarrow[n\to\infty]{} 0.$$

Because of the last relation, we have

$$\lim_{n\to\infty} u_n\left(q^k\right) = u\left(q^k\right) \tag{3}$$

From this relation,

$$\lim_{n\to+\infty} \frac{u_n\left(q^k\right) - u_n\left(q^{k+1}\right)}{(1-q)\,q^k} = \frac{u\left(q^k\right) - u\left(q^{k+1}\right)}{(1-q)\,q^k} = f\left(q^k\right),\ k \in \mathbb{Z}.$$

is attained. Also, from (3) and the boundary condition at $t = 0$

$$\left|u\left(q^k\right)\right| \leqslant \left|u_n\left(q^k\right) - u\left(q^k\right)\right| + \left|u_n\left(q^k\right)\right| \xrightarrow[n,k\to+\infty]{} 0$$

is true. This means that $u \in D(L_0)$ and $Lu(t) = f$. Therefore, the minimal linear operator L_0 is closed. On the other hand, $D(L_0^*) = D$ and the following equations can be easily obtained

$$\|L_0 u(t)\|^2_{L_q^2(0,+\infty)} \quad = \int_0^{+\infty} |D_q u(t)|^2 d_q t$$

$$= (1-q) \sum_{k=-\infty}^{+\infty} q^k \left| D_q u\left(q^k\right) \right|^2$$

$$= (1-q) \sum_{k=-\infty}^{+\infty} q^k \left| \frac{u\left(q^k\right) - u\left(q^{k+1}\right)}{(1-q)\, q^k} \right|^2$$

for any $u \in D(L_0)$. Also,

$$\begin{aligned}
\|L_0^* u(t)\|^2_{L_q^2(0,+\infty)} &= \int_0^{+\infty} \left| -\frac{1}{q} D_{q^{-1}} u(t) \right|^2 d_q t \\
&= (1-q) \sum_{k=-\infty}^{+\infty} q^k \left| -\frac{1}{q} D_{q^{-1}} u\left(q^k\right) \right|^2 \\
&= \frac{1}{q^2} (1-q) \sum_{k=-\infty}^{+\infty} q^k \left| \frac{u\left(q^k\right) - u\left(q^{k-1}\right)}{\left(1-\frac{1}{q}\right) q^k} \right|^2 \\
&= \frac{1}{q} (1-q) \sum_{k=-\infty}^{+\infty} q^{k-1} \left| \frac{u\left(q^{k-1}\right) - u\left(q^k\right)}{(1-q)\, q^{k-1}} \right|^2 \\
&= \frac{1}{q} (1-q) \sum_{k=-\infty}^{+\infty} q^k \left| \frac{u\left(q^k\right) - u\left(q^{k+1}\right)}{(1-q)\, q^k} \right|^2 .
\end{aligned}$$

is hold. As a result of the last equations for all $u \in D(L_0) \subset D(L_0^*)$

$$\|L_0^* u\| = \sqrt{q^{-1}}\, \|L_0 u\|$$

is seen. This is completed the proof. □

Corollary 2. *The minimal operator L_0 is a maximal formally q-normal in $L_q^2(0,+\infty)$.*

Proof. Assume that $\tilde{L_0}$ is a q-normal extension of L_0, i.e. $L_0 \subset \tilde{L_0}$. Therefore, for all $u \in D(\tilde{L_0}) = D(\tilde{L_0}^*)$

$$\begin{aligned}
\left(\tilde{L_0} u, u\right)_{L_q^2(0,+\infty)} - \left(u, \tilde{L_0}^* u\right)_{L_q^2(0,+\infty)} &= (D_q u, u)_{L_q^2(0,+\infty)} - \left(u, -\frac{1}{q} D_{q^{-1}} u\right)_{L_q^2(0,+\infty)} \\
&= -\lim_{k \to +\infty} \left|u\left(q^k\right)\right|^2 = 0
\end{aligned}$$

is obtained from the equation (1). This means that $D(\tilde{L_0}) = D(L_0)$ and $\tilde{L_0} = L_0$. However this is a contradiction. According to this result and Theorem 2.1, the minimal operator L_0 is a maximal formally q-normal operator in $L_q^2(0,+\infty)$. □

3. Spectrum Sets of the operators L_0 and L

Theorem 2. *The point spectrum set of L_0 is*

$$\sigma_p(L_0)=\left\{\frac{q^m}{1-q}:\ m\in\mathbb{Z}\right\}.$$

Proof. Suppose that a complex number λ is an element of the point spectrum of L_0. Therefore, there is a non-zero element $u(t)$ corresponding to a complex number λ in $D(L_0)$, which that satisfies the following equation

$$\frac{u\left(q^k\right)-u\left(q^{k+1}\right)}{(1-q)\,q^k}=\lambda u\left(q^k\right),\quad k\in\mathbb{Z}.$$

We gain that

$$u\left(q^{k+1}\right)=\left(1-\lambda\,(1-q)\,q^k\right)u\left(q^k\right)\tag{4}$$

for all $k\in\mathbb{Z}$. If $\lambda=\frac{1}{(1-q)q^m}$ for any $m\in\mathbb{Z}$ is true, then the eigenvector $u(t)$ should be defined as

$$\begin{aligned}
&u\left(q^k\right)=0,\quad k\geqslant m+1\\
&u\left(q^k\right)=\left(\prod_{i=k-m}^{-1}\frac{1}{1-q^i}\right)u\left(q^m\right),\quad k\leqslant m-1.
\end{aligned}$$

Since $0<q<1$ and the limit

$$\lim_{k\to-\infty}\left|1-q^k\right|=+\infty$$

is true, a negative integer k_0 is exist such that

$$\prod_{n=k_0+1-m}^{-1}\frac{1}{|1-q^n|}\leqslant 1.$$

From this result and $0<q<1$ it is get that

$$\begin{aligned}
\|u\|^2_{L_q^2(0,+\infty)}&=\sum_{k=-\infty}^{+\infty}q^k\left|u\left(q^k\right)\right|^2\\
&=\sum_{k=k_0}^{m}q^k\left|u\left(q^k\right)\right|^2+\sum_{k=-\infty}^{k_0-1}q^k\left|u\left(q^k\right)\right|^2\\
&=\sum_{k=k_0}^{m}q^k\left|u\left(q^k\right)\right|^2+\sum_{k=-\infty}^{k_0-1}q^k\left(\prod_{i=k-m}^{-1}\left|\frac{1}{1-q^i}\right|^2\right)\left|u\left(q^m\right)\right|^2\\
&\leqslant\sum_{k=k_0}^{m}q^k\left|u\left(q^k\right)\right|^2+\sum_{k=-\infty}^{k_0-1}q^k\left|\frac{1}{1-q^{k-m}}\right|^2\left|u\left(q^m\right)\right|^2
\end{aligned}$$

$$= \sum_{k=k_0}^{m} q^k \left|u\left(q^k\right)\right|^2 + \sum_{k=-\infty}^{k_0-1} q^k \left|\frac{q^{-k}}{q^{-k}-q^{-m}}\right|^2 \left|u\left(q^m\right)\right|^2$$
$$= \sum_{k=k_0}^{m} q^k \left|u\left(q^k\right)\right|^2 + \sum_{k=-\infty}^{k_0-1} q^{-k} \left|\frac{1}{q^{-k}-q^{-m}}\right|^2 \left|u\left(q^m\right)\right|^2 < +\infty.$$

These prove that $u(t)$ is an eigenvector corresponding to $\frac{q^m}{1-q}$ for $m \in \mathbb{Z}$.

On the other hand, λ is different from $\frac{1}{(1-q)q^m}$ for any $m \in \mathbb{Z}$, then

$$u\left(q^k\right) = \left(\prod_{i=0}^{k-1} \left(1-\lambda\left(1-q\right)q^i\right)\right) u\left(1\right), \quad k \in \mathbb{N}.$$

Hence, $u \in D_0$, $u\left(q^k\right) \xrightarrow[k \to +\infty]{} 0$ iff there exists $m \in \mathbb{N}$ satisfied the following equality

$$1-\lambda\left(1-q\right)q^m = 0$$

must be supplied [15] or $u(1) = 0$. In this case, $u(1) = 0$ and so $u = 0$ is obtained from the equation (4). These results imply that $\sigma_r(L_0) = \left\{\frac{q^m}{1-q} : \ m \in \mathbb{Z}\right\}$. □

Theorem 3. *The set of L_0 residual spectrum is empty.*

Proof. Assume that $\lambda \in \mathbb{C}$ is in $\sigma_r\left(L_0\right)$. Since $L_q^2\left(0,+\infty\right) = \overline{R(L_0-\lambda E)} \oplus Ker\left(L_0^* - \bar{\lambda}E\right)$ is provided, where E is the identity operator in $L_q^2\left(0,+\infty\right)$, it is clear that $\bar{\lambda} \in \sigma_p\left(L_0^*\right)$. Therefore, there exists an element $u \in L_q^2\left(0,+\infty\right), u \neq 0$ and

$$L_0^* u\left(t\right) = \bar{\lambda} u\left(t\right).$$

Therefore, we have

$$-\frac{1}{q}\frac{u\left(q^k\right)-u\left(q^{k-1}\right)}{\left(1-\frac{1}{q}\right)q^k} = \frac{u\left(q^k\right)-u\left(q^{k-1}\right)}{\left(1-q\right)q^k} = \bar{\lambda} u\left(q^k\right)$$

for all $k \in \mathbb{Z}$. The following equation is obtained from this equation

$$u\left(q^{k-1}\right) = \left(1-\bar{\lambda}\left(1-q\right)q^k\right) u\left(q^k\right)$$

for all $k \in \mathbb{Z}$. If $\bar{\lambda}$ is equal to $\frac{1}{(1-q)q^m}$ for any $m \in \mathbb{Z}$, then

$$u\left(q^k\right) = 0, \quad k \leqslant m-1$$
$$u\left(q^k\right) = \left(\prod_{i=m+1}^{k} \frac{1}{1-q^{i-m}}\right) u\left(q^m\right), \quad k \geqslant m+1$$

is holds. Because $\sum\limits_{k=m+1}^{+\infty} q^k \left(\prod\limits_{i=m+1}^{k} \frac{1}{1-q^{i-m}} \right)^2$ converges to a complex number, the function $u(t)$ defined as above is an element of $L_q^2(0,+\infty)$.

Otherwise, if $\bar{\lambda}$ is not equal to $\frac{1}{(1-q)q^m}$ for any $m \in \mathbb{Z}$, then it must be $u(1) \neq 0$ and

$$u\left(q^k\right) = \left(\prod_{i=0}^{-k} \left(1 - \bar{\lambda}\left(1-q\right) q^{-i}\right) \right) u(1), \quad k \leqslant 0.$$

But the limit $\lim\limits_{k \to -\infty} u\left(q^k\right)$ does not exist when $\bar{\lambda}$ is not equal to $\frac{1}{(1-q)q^m}$ for any $m \in \mathbb{Z}$. As a result of these,

$$\sigma_r\left(L_0\right) = \emptyset$$

is obtained.

□

Corollary 3. *It is held that* $0 \in \sigma_c(L_0)$ *for the minimal operator* L_0*.*

Corollary 4. *The point spectrum and residual spectrum of* L_0^* *are as follows*

$$\sigma_p(L_0^*) = \left\{ \frac{q^m}{1-q} : \; m \in \mathbb{Z} \right\} \quad \text{and} \quad \sigma_r(L_0^*) = \emptyset.$$

Theorem 4. *The point and continuous spectrum sets of the maximal operator are in the form*

$$\sigma_p\left(L\right) = \mathbb{C} \backslash \left\{0\right\} \quad \text{and} \quad \sigma_c\left(L\right) = \left\{0\right\}.$$

Proof. Suppose that λ is a nonzero complex number. We deal with the solution of following problem

$$\left(L - \lambda E\right) u\left(q^k\right) = 0, \;\; k \in \mathbb{Z}.$$

It is written for any $k \in \mathbb{Z}$

$$u\left(q^{k+1}\right) = \left(1 - \lambda\left(1-q\right) q^k\right) u\left(q^k\right). \tag{5}$$

If $u\left(q^k\right)$ are different from zero for all $k \in \mathbb{Z}$, then we have

$$u\left(q^{k+1}\right) = \left(\prod_{n=0}^{k} \left(1 - \lambda\left(1-q\right) q^n\right) \right) u\left(1\right)$$

for all positive integer k. Since the infinite product $\prod\limits_{k=0}^{+\infty} \left(1 - \lambda\left(1-q\right) q^k\right)$ converges, the sequence $\left\{u\left(q^k\right)\right\}_{k \in \mathbb{N}}$ is bounded. From this result the series $\sum\limits_{k=0}^{+\infty} q^k \left|u\left(q^k\right)\right|^2$ is finite.

In the case of negative integers, we gain

$$u\left(q^{k}\right)=\left(\prod_{n=k}^{-1}\left(1-\lambda\left(1-q\right)q^{k}\right)^{-1}\right)u\left(1\right)$$

for all $k\leqslant -1$. Because the limit

$$\lim_{k\to-\infty}\left|1-\lambda\left(1-q\right)q^{k}\right|=+\infty \tag{6}$$

is true, it is clear that

$$\prod_{n=k-1}^{-1}\frac{1}{\left|1-\lambda\left(1-q\right)q^{n}\right|}\leqslant 1$$

for small enough negative integers k. This result give us the following inequality

$$\begin{aligned}
q^{k}\left|u\left(q^{k}\right)\right|^{2} &= q^{k}\left(\prod_{n=k}^{-1}\frac{1}{\left|1-\lambda\left(1-q\right)q^{n}\right|^{2}}\right)\left|u\left(1\right)\right|^{2}\\
&\leqslant q^{k}\frac{1}{\left|1-\lambda\left(1-q\right)q^{k}\right|^{2}}\left|u\left(1\right)\right|^{2}\\
&= q^{k}\frac{q^{-2k}}{\left|q^{-k}-\lambda\left(1-q\right)\right|^{2}}\left|u\left(1\right)\right|^{2}\\
&= \frac{q^{-k}}{\left|q^{-k}-\lambda\left(1-q\right)\right|^{2}}\left|u\left(1\right)\right|^{2}
\end{aligned}$$

for small enough negative integers k. Because of the limit (6) and the fact that the geometric series $\sum\limits_{k=-\infty}^{0}\alpha q^{-k}$ converges for $0<q<1$, these results allow us that the series $\sum\limits_{k=-\infty}^{0}q^{k}\left|u\left(q^{k}\right)\right|^{2}$ converges absolutely. These show us to conclude that $\sum\limits_{k=-\infty}^{+\infty}q^{k}\left|u\left(q^{k}\right)\right|^{2}$ is convergent.

When $u(q^{m+1})$ is equal to zero for an integer $m\in\mathbb{Z}$, it is obtained that $u(q^{k})=0$ for all $k\geqslant m+1$. We note that this condition includes the case of $\lambda=\frac{q^{-m}}{1-q},\ m\in\mathbb{Z}$. Moreover, the equation

$$u\left(q^{k}\right)=\left(\prod_{n=k}^{m-1}\left(1-\lambda\left(1-q\right)q^{n}\right)^{-1}\right)u\left(q^{m}\right)$$

is easily checked for all $k<m$. We already know that

$$\sum_{k=-\infty}^{m-1}q^{k}\left(\prod_{n=k}^{m-1}\left|\left(1-\lambda\left(1-q\right)q^{n}\right)^{-1}\right|^{2}\right)\left|u\left(q^{m}\right)\right|^{2}$$

is convergent. Because of all these reasons, we get that $u(t)$ is an eigenvector of the maximal operator L for $\lambda \in \mathbb{C} \setminus \{0\}$.

If $\lambda = 0$, then returning to the equation (5) it must be $u(t) = 0$. This means that zero is not an eigenvalue. Also, if $0 \in \sigma_r(L)$, then it must be $0 \in \sigma_p(L^*)$ because of $L_q^2(0,+\infty) = \overline{R(L)} \oplus Ker(L^*)$. But, it can be easily proved that $0 \notin \sigma_p(L^*)$. Therefore, it must be $\sigma_c(L) = \{0\}$ from the fact of the closeness of the spectrum. $\square$

Remark 1. *It can be defined the two operators P_0 and P defined by $p(.) = \frac{d}{dt}$ in $L^2(0,+\infty)$ and these operators are called the minimal and maximal operators, respectively. Also, their domains are as follows*

$$\begin{aligned} D(P_0) &= \left\{u \in L^2(0,+\infty) : \; u' \in L^2(0,+\infty) \;\; and \;\; u(0) = 0\right\}, \\ D(P) &= \left\{u \in L^2(0,+\infty) : \; u' \in L^2(0,+\infty)\right\}. \end{aligned}$$

The operator P_0 is maximal formal normal. It means that there is not any normal extension of L_0. Moreover, the point and residual spectrum sets of P_0 are $\sigma_p(P_0) = \emptyset$ and $\sigma_r(P_0) = \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}$ and the spectrum parts of the maximal operator P are $\sigma_p(P) = \{\lambda \in \mathbb{C} : Re(\lambda) < 0\}$, $\sigma_r(P) = \emptyset$ and $\sigma_c(P) = \{\lambda \in \mathbb{C} : Re(\lambda) = 0\}$.

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