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Existence of mild solutions for semilinear $\psi-$Caputo-type fractional evolution equations with nonlocal conditions in Banach spaces

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Abstract

The main crux of this manuscript is to investigate the existence of mild solutions for a class of semilinear $\psi-$Caputo-type fractional evolution equations in Banach spaces with nonlocal conditions. The proofs are based on Krasnoselskii fixed point theorem, compact semigroup, generalized $\psi-$Laplace transform and certain fundamental $\psi-$fractional calculus tools. As application, a nontrivial example is given to illustrate our theoretical results.

Keywords: $\psi-$fractional integral $\psi-$Caputo fractional derivative fixed point compact semigroup.
2010 MSC: 34A08, 34A45, 34K37, 90C32, 26A33.

1. Introduction

Fractional differential equations have grown in importance and popularity over the last three decades, owing to their use in a wide variety of scientific and engineering domains, as well as their potential to simulate a diverse range of processes and phenomena with memory effects. Indeed, fractional-order models have been proven to be more appropriate for several real-world situations than integer-order models, as fractional derivatives are a useful tool for describing memory and hereditary features of many materials and processes. Material theory, transport processes, earthquakes, electrochemical processes, wave propagation, signal theory, biology, electromagnetic theory, fluid flow phenomena, thermodynamics, mechanics, geology, astrophysics, economics, and control theory are all applications of fractional differential equations theory (see[1, 8, 9, 12, 14, 16, 25, 29]);In comparison to traditional integer-order models, this is the main advantage

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of fractional differential equations.Recent research has focused on fractional differential equations, particularly boundary value problems for nonlinear fractional differential equations, which can be used to model and describe non-homogeneous physical events. Many researchers have obtained some interesting results on the existence and uniqueness of solutions of boundary value problems for fractional differential equations involving different fractional derivatives such as Riemann-Liouville [22], Caputo [3], Hilfer [20], Erdelyi-Kober [24] and Hadamard [2]. There is a certain type of kernel dependency included in all those definitions. Therefore, a fractional derivative with respect to another function known as the ψ-Caputo derivative was introduced in order to study fractional differential equations in a general manner. For specific selections of ψ, we can obtain some well-known fractional derivatives, such as the Caputo, Caputo-Hadamard, or Caputo-Erdélyi-Kober fractional derivatives, which are dependent on a kernal. From the viewpoint of applications, this approach also seems appropriate. With the help of a good selection of a "trial" function ψ, the $\psi-$Caputo fractional derivative allows some measure of control over the modeling of the phenomenon under consideration. Using fixed point theorems and the Picard iteration approach, Almeida examined the existence and uniqueness results of nonlinear fractional differential equations using a Caputo-type fractional derivative with respect to another function. Using various fixed point theorems, Zhang in [35] showed the existence and uniqueness results for nonlinear fractional boundary value problems involving Caputo type fractional derivatives. Yong Zhou in [33], on the other hand, investigated the nonlocal Cauchy problem for fractional evolution equations in an arbitrary Banach space and proposed a number of conditions for the existence and uniqueness of mild solutions. In [32], Suechoei et all studied the local and global existence of mild solutions to initial value problems for fractional semilinear evolution equations with local conditions in Banach spaces. To solve these classes of fractional evolution equations, in this paper, we introduce the generalized Laplace transform and compact semigroup in Banach spaces, but with a nonlocal term in their initial conditions. The reader can consult articles as well [11, 13, 15, 17, 18, 23, 27, 31, 34, 36] and the references therein for more details on fractional evolution and differential equations.
Motivated by all previous works, the purpose of this paper is to investigate the existence results obtained in [33] by aplaying ψ-Caputo fractional derivatives of order $\beta \in (0,1)$. To be more precise, we establish the existence of mild solutions for the following fractional evolution equation with nonlocal conditions in a Banach space X:

$$\begin{cases} {}^{C}D_{0^+}^{\beta,\psi}x(t) = \mathcal{A}x(t) + f(t,x(t)), \quad t \in \Delta = [0,T], \\ \\ x(0) + \Phi(x) = x_0. \end{cases} \tag{1}$$

Where ${}^{C}D_{0^+}^{\beta,\psi}$ is the $\psi-$Caputo fractional derivative, $T > 0$, $x_0 \in X$, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of operators on X, $f : [0,\infty)\times X \to X$ and the nonlocal term $\Phi : C([0,\infty), X) \to X$ are given functions satisfying some assumptions.

The outline of the paper is the following. In Section 2, we give some basic definitions and properties of $\psi-$fractional integral and $\psi-$Caputo fractional derivatives. In Section 3, we establish the existence of mild solutions for $\psi-$Caputo type fractional evolution problem (1) by using Krasnoselskii fixed point theorem and compact semigroups. As application, an illustrative example is presented in Section 4 followed by conclusion in Section 5.

2. Preliminaries

In this section, we give some notations, definitions and results on ψ-fractional derivatives and ψ-fractional integrals, for more details we refer the reader to [4, 6, 19].

Notations

• We denote by X a Banach space with the norm $\| \, . \, \|$.

• We denote by $C(\Delta, X)$ the Banach space of continuous functions from Δ to X provided with the topology of the norm

$$\|x\|_\infty = \sup_{t\in\Delta} \|x(t)\|.$$

• We denote by B_r the closed ball centered at 0 with radius $r > 0$.
• We denote by $L^\infty(\Delta, X)$ the space of all essentially bounded functions equipped with the essential supremum norm $\| \, . \, \|_{L^\infty}$.

Remark 2.1. *Sometimes in this article, we can note $\| \, . \, \|_\infty$ by $\| \, . \, \|$ without ambiguity to avoid weighing down the calculations.*

Throughout this paper, let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup of uniformly bounded linear operators $\{T(t)\}_{t\geq 0}$ on X. Then there exists $M \geq 1$ such that $M = \sup\limits_{t\in[0,\infty)} \|T(t)\|$.
Some results about the semigroups of linear operators can be found in [28].

Definition 2.2. *[5] Let $q > 0$, $g \in L^1([\Delta, \mathbb{R})$ and $\psi \in C^n(\Delta, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in \Delta$. The ψ-Riemann-Liouville fractional integral at order q of the function g is given by*

$$I_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{q-1} g(s)ds. \tag{2}$$

Remark 2.3. *Note that if $\psi(t) = t$ and $\psi(t) = log(t)$, then the equation* (2) *is reduced to the Riemann-Liouville and Hadamard fractional integrals respectively.*

Definition 2.4. *[5] Let $q > 0$, $g \in C^{n-1}(\Delta, \mathbb{R})$ and $\psi \in C^n(\Delta, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in \Delta$. The $\psi-$Caputo fractional derivative at order q of the function g is given by*

$${}^C D_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(n-q)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-q-1} g_\psi^{[n]}(s)ds, \tag{3}$$

where

$$g_\psi^{[n]}(s) = \left(\frac{1}{\psi'(s)} \frac{d}{ds}\right)^n g(s) \quad \text{and} \quad n = [q] + 1,$$

and $[q]$ denotes the integer part of the real number q.

Remark 2.5. *In particular, note that if $\psi(t) = t$ and $\psi(t) = log(t)$, then the equation* (3) *is reduced to the the Caputo fractional derivative and Caputo-Hadamard fractional derivative respectively.*

Remark 2.6. *In particular, if $q \in]0, 1[$, then we have*

$${}^C D_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t (\psi(t) - \psi(s))^{q-1} g'(s)ds,$$

and

$${}^C D_{0^+}^{q,\psi} g(t) = I_{0^+}^{1-q,\psi} \left(\frac{g'(t)}{\psi'(t)}\right).$$

Proposition 2.7. *[5] Let $q > 0$, if $g \in C^{n-1}(\Delta, \mathbb{R})$, then we have*

1) ${}^C D_{0^+}^{q,\psi} I_{0^+}^{q,\psi} g(t) = g(t)$.
2) $I_{0^+}^{q,\psi} \, {}^C D_{0^+}^{q,\psi} g(t) = g(t) - \sum\limits_{k=0}^{n-1} \dfrac{g_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k$.
3) $I_{a^+}^{q,\psi}$ *is linear and bounded from $C(\Delta, \mathbb{R})$ to $C(\Delta, \mathbb{R})$.*

Proposition 2.8. *[5] Let $\mu > \nu > 0$ and $t \in \Delta$, then*

1) $I_{0^+}^{\mu,\psi}(\psi(t)-\psi(0))^{\nu-1} = \dfrac{\Gamma(\nu)}{\Gamma(\mu+\nu)}(\psi(t)-\psi(0))^{\mu+\nu-1}$.
2) $D_{0^+}^{\mu,\psi}(\psi(t)-\psi(0))^{\nu-1} = \dfrac{\Gamma(\nu)}{\Gamma(\nu-\mu)}(\psi(t)-\psi(0))^{\nu-\mu-1}$.
3) $D_{0^+}^{\mu,\psi}(\psi(t)-\psi(0))^{k} = 0, \quad \forall k < n \in \mathbb{N}$.

Definition 2.9. *(See[21]). Let $y : \Delta \to \mathbb{R}$ be real valued function. The generalized Laplace transform of y is given by*

$$\mathcal{L}_\psi\{y(t)\}(s) := \widehat{Y}(s) = \int_0^\infty \psi'(t)e^{-s(\psi(t)-\psi(0))}y(t)dt, \tag{4}$$

for all s.

Definition 2.10. *[21] Let f and g be two functions which are piecewise continuous on Δ and of exponential order . The generalized $\psi-$convolution of f and g is defined by*

$$(f \underset{\psi}{*} g)(t) = \int_0^t f(s)g(\psi^{-1}(\psi(t)+\psi(0)-\psi(s)))\psi'(s)ds.$$

Lemma 2.11. *(See [21]). Let $q > 0$ and y be a piecewise continuous function on each interval $[0,t]$ and $\psi(t)$-exponential order. Then we have*

1. $\mathcal{L}_\psi\{I_{0^+}^{q,\psi}y(t)\}(s) = \dfrac{\widehat{Y}(s)}{s^q}$.
2. $\mathcal{L}_\psi\{{}^C D_{0^+}^{q,\psi}y(t)\}(s) = s^q\left[\mathcal{L}_\psi\{y(t)\} - \displaystyle\sum_{k=0}^{n-1} s^{-k-1}f^{(k)}(0)\right]$, *where $n = [q]+1$.*

Definition 2.12. *(See[22, 30]). The Wright function is defined by*

$$W_\beta(z) = \sum_{k=0}^\infty \frac{z^k}{k!\Gamma(\beta(k+1))} = \sum_{k=0}^\infty \frac{(-z)^k\Gamma(\beta(k+1))\sin(\pi(k+1)\beta)}{k!},$$

for $\beta > 0$ and $z \in \mathbb{C}$.

Remark 2.13. *We see that the Wright function is a generalization of the expoilential function. Indeed for $\beta = 0$, then we have*

$$W_0(z) = e^z.$$

Proposition 2.14. *(See [22, 30]). The Wright function W_β satisfies the following interesting properties:*

1. $W_\beta(t) \geq 0$, *for $t \geq 0$ and we have* $\displaystyle\int_0^\infty W_\beta(t)dt = 1$,
2. $\displaystyle\int_0^\infty W_\beta(z)z^r dz = \frac{\Gamma(1+r)}{\Gamma(1+\beta r)}$ *for $r > -1$,*
3. $\displaystyle\int_0^\infty W_\beta(t)e^{-zt}dt = E_\beta(-z)$, $z \in \mathbb{C}$,
4. $\beta\displaystyle\int_0^\infty tW_\beta(t)e^{-zt}dt = E_{\beta,\beta}(-z)$, $z \in \mathbb{C}$,
   where $E_\beta(.)$ and $E_{\beta,\beta}(.)$ are Mittag-Leffler functions given in [22].

Definition 2.15. *(See [26]) Let $\rho \in [0, \infty)$. The one-sided stable probability density is defined by*

$$\omega_\beta(\rho) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}(\psi(\rho)-\psi(0))^{-\beta n-1}\frac{\Gamma(\beta n+1)}{n!}\sin(n\pi\beta).$$

Lemma 2.16. *[26] The Laplace transform of $\omega_\beta(t)$ is given by*

$$\int_0^\infty e^{-\lambda(\psi(t)-\psi(0))}\omega_\beta(t)\psi'(t)dt = e^{-\lambda^\beta}. \tag{5}$$

Lemma 2.17. *(See [7]). Let Ω be a non-empty, closed convex and bounded subset of a Banach algebra X and Let $\mathcal{T}_1 : \Omega \to X$ and $\mathcal{T}_2 : \Omega \to X$ be two operators such that*

(1) $\mathcal{T}_1$ is a contraction,
(2) $\mathcal{T}_2$ is completely continuous,
(3) $\mathcal{T}_1 x + \mathcal{T}_2 y \in \Omega$ for all $x, y \in \Omega$.
Then the equation $\mathcal{T}_1 x + \mathcal{T}_2 x = x$ has a solution in Ω.

3. Main results

In this section, before we give the main result of our paper, first of all we should define what we mean by mild solution for the problem (1) and prove the fundamental Lemmas 3.2 and 3.3. For this purpose, we assume the following assumptions throughout the rest of our paper.

(A_1) $T(t)$ is a compact operator for every $t > 0$,

(A_2) f is a Caratheódory function on $\Delta \times X$. i.e. f satisfies the following properties:
- the map $t \mapsto f(t, x(t))$ is strongly measurable for each $x \in C(\Delta, X)$, and
- the map $x \mapsto f(t, x(t))$ is continuous for each $t \in \Delta$.

(A_3) There exists a constant $L_\Phi \in (0, \frac{1}{M})$ such that

$$\| \Phi(x) - \Phi(y) \| \ \leq L_\Phi \|x - y\|, \quad for \ \ each \ \ x, y \in C(\Delta, X).$$

(A_4) For any $r > 0$, there exists a function $h_r(t) \in L^\infty(\Delta, X)$ such that for any $x \in C(\Delta, X)$ satisfying $\|x\| < r$,

$$\| f(t, x(t)) \| \ \leq h_r(t), \quad \forall t \in \Delta.$$

Definition 3.1. *A function $x \in C(\Delta, X)$ such that its $\beta-$derivative existing on Δ is said te be a solution of the problem* (1) *if x satisfies the equation ${}^C D_{0^+}^{\beta,\psi} x(t) = \mathcal{A}x(t) + f(t, x(t))$ on Δ and the condition $x(0) + \Phi(x) = x_0$.*

Lemma 3.2. *A function $x(t) \in C(\Delta, X)$ is a solution of the fractional differential equation (1) if and only if x satisfies the following fractional integral equation*

$$x(t) = x_0 - \Phi(x) + \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}[\mathcal{A}x(s) + f(s, x(s))]ds. \tag{6}$$

Proof. Let x be a solution of the problem (1), then we apply the $\psi-$fractional integral $I_{0^+}^{\beta,\psi}$ on both sides of (1) we get

$$I_{0^+}^{\beta,\psi}\, {}^C D_{0^+}^{\beta,\psi} x(t) = I_{0^+}^{\beta,\psi}[\mathcal{A}x(t) + f(s, x(t))],$$

and by using Proposition 2.7 we obtain

$$x(t) - x(0) = I_{0^+}^{\beta,\psi}[\mathcal{A}x(t) + f(t, x(t)],$$

since $x(0) + \Phi(x) = x_0$, it follows that

$$x(t) = x_0 - \Phi(x) + \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1}[\mathcal{A}x(s) + f(s, x(s))]ds.$$

Hence the integral equation (6) holds.
Conversely, by direct computation, it is clear that if x satisfies the integral equation (6), then the equation (1) holds which completes the proof. □

Lemma 3.3. *If the fractional integral equation (6) holds, then we have*

$$\begin{aligned} x(t) = &\int_0^\infty \varphi_\beta(\rho) T((\psi(t) - \psi(0))^\beta \rho)(x_0 - \Phi(x))d\rho \\ &+\beta \int_0^t \int_0^\infty \rho\varphi_\beta(\rho)(\psi(t) - \psi(s))^{\beta-1} T((\psi(t) - \psi(0))^\beta \rho) f(s, x(s))\psi'(s)d\rho ds, \end{aligned} \tag{7}$$

where $\varphi_\beta(\rho) = \frac{1}{\beta}\rho^{-1-\frac{1}{\beta}}\omega_\beta(\rho^{-\frac{1}{\beta}})$ *is the probability density function defined on* $(0, \infty)$.

Proof. Let $\lambda > 0$. By applying the generalized Laplace transforms on the fractional integral equation (7) and using Lemma 2.11, we obtain

$$\widehat{X}(\lambda) = \frac{1}{\lambda}(x_0 - \Phi(x)) + \frac{1}{\lambda^\beta}(\mathcal{A}\widehat{X}(\lambda) + \widehat{F}(\lambda)) \ ,$$

where

$$\widehat{X}(\lambda) = \int_0^\infty e^{-\lambda(\psi(\tau)-\psi(0))} x(\tau)\psi'(\tau)d\tau,$$

and

$$\widehat{F}(\lambda) = \int_0^\infty e^{-\lambda(\psi(\tau)-\psi(0))} f(\tau, x(\tau))\psi'(\tau)d\tau,$$

it follows that

$$\widehat{X}(\lambda) = \lambda^{\beta-1}(\lambda^\beta I - \mathcal{A})^{-1}(x_0 - \Phi(x)) + (\lambda^\beta I - \mathcal{A})^{-1}\widehat{F}(\lambda),$$

thus

$$\widehat{X}(\lambda) = \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta s} T(s)(x_0 - \Phi(x))ds + \int_0^\infty e^{-\lambda^\beta s} T(s)\widehat{F}(\lambda)ds,$$

choosing $\xi = \psi(t) - \psi(0)$, we get

$$\widehat{X}(\lambda) = \beta \int_0^\infty (\lambda\xi)^{\beta-1} e^{(-\lambda\xi)^\beta} T(\xi^\beta)(x_0 - \Phi(x))d\xi + \beta \int_0^\infty \xi^{\beta-1} e^{-(\lambda\xi)^\beta} T(\hat{t}^\beta)\widehat{F}(\lambda)d\xi := J_1 + J_2.$$

Where

$$\begin{aligned} J_1 &= \beta \int_0^\infty \lambda^{\beta-1}\psi'(t)(\psi(t) - \psi(0))^{\beta-1} e^{-(\lambda(\psi(t)-\psi(0)))^\beta} T((\psi(t) - \psi(0))^\beta)(x_0 - \Phi(x))dt, \\ &= \int_0^\infty \frac{-1}{\lambda} \left(e^{-(\lambda(\psi(t)-\psi(0)))^\beta}\right)' T((\psi(t) - \psi(0))^\beta)(x_0 - \Phi(x))dt, \end{aligned}$$

and

$$\begin{aligned} J_2 &= \beta \int_0^\infty (\psi(t) - \psi(0))^{\beta-1} e^{-(\lambda(\psi(t)-\psi(0)))^\beta} T((\psi(t) - \psi(0))^\beta)\widehat{F}(\lambda)\psi'(t)dt, \\ &= \int_0^\infty \beta(\psi(t) - \psi(0))^{\beta-1} e^{-(\lambda(\psi(t)-\psi(0)))^\beta} T((\psi(t) - \psi(0))^\beta)\psi'(t) \\ &\times \int_0^\infty e^{-(\lambda(\psi(s)-\psi(0)))} f(s, \ u(s))\psi'(s)dsdt. \end{aligned}$$

From (5) it follows that

$$J_1 = \int_0^\infty -\frac{1}{\lambda}\frac{d}{dt}(e^{-(\lambda(\psi(t)-\psi(0)))^\beta})T((\psi(t)-\psi(0))^\beta)(x_0-\Phi(x))dt$$
$$= \int_0^\infty \int_0^\infty \rho\omega_\beta(\rho)e^{-\lambda(\psi(t)-\psi(0))\rho}T((\psi(t)-\psi(0))^\beta)(x_0-\Phi(x))\psi'(t)d\rho dt,$$

thus

$$J_1 = \int_0^\infty e^{-\lambda(\psi(t)-\psi(0))}\left(\int_0^\infty \omega_\beta(\rho)T(\frac{(\psi(t)-\psi(0))^\beta}{\rho^\beta})(x_0-\Phi(x))d\rho\right)\psi'(t)dt.$$

And

$$\begin{aligned}
J_2 = &\int_0^\infty\int_0^\infty \beta(\psi(t)-\psi(0))^{\beta-1}e^{-(\lambda(\psi(t)-\psi(0)))^\beta}T((\psi(t)-\psi(0))^\beta)e^{-(\lambda(\psi(s)-\psi(0)))}\\
&\times f(s,x(s))\psi'(s)\psi'(t)dsdt,\\
= &\int_0^\infty\int_0^\infty\int_0^\infty \beta(\psi(t)-\psi(0))^{\beta-1}\omega_\beta(\rho)e^{-\lambda(\psi(t)-\psi(0))\rho}T((\psi(t)-\psi(0))^\beta)\\
&\times e^{-\lambda(\psi(s)-\psi(0))}f(s,x(s))\psi'(s)\psi'(t)d\rho dsdt,
\end{aligned}$$

$$\begin{aligned}
= &\int_0^\infty\int_0^\infty\int_0^\infty \beta e^{-\lambda(\psi(t)+\psi(s)-2\psi(0))}\frac{(\psi(t)-\psi(0))^{\beta-1}}{\rho^\beta}\omega_\beta(\rho)T\left(\frac{(\psi(t)-\psi(0))^\beta}{\rho^\beta}\right)\\
&\times f(s,x(s))\psi'(s)\psi'(t)d\rho dsdt,\\
= &\int_0^\infty\int_t^\infty\int_0^\infty \beta e^{-\lambda(\psi(\tau)-\psi(0))}\omega_\beta(\rho)\frac{(\psi(t)-\psi(0))^{\beta-1}}{\rho^\beta}T\left(\frac{(\psi(t)-\psi(0))^\beta}{\rho^\beta}\right)\\
&\times f(\psi^{-1}(\psi(\tau)-\psi(t)+\psi(0)),x(\psi^{-1}(\psi(\tau)-\psi(t)+\psi(0))))\psi'(\tau)\psi'(t)d\rho d\tau dt,\\
= &\int_0^\infty\int_0^\tau\int_0^\infty \beta e^{-\lambda(\psi(\tau)-\psi(0))}\omega_\beta(\rho)\frac{(\psi(t)-\psi(0))^{\beta-1}}{\rho^\beta}T\left(\frac{(\psi(t)-\psi(0))^\beta}{\rho^\beta}\right)\\
&\times f(\psi^{-1}(\psi(\tau)-\psi(t)+\psi(0)),x(\psi^{-1}(\psi(\tau)-\psi(t)+\psi(0))))\psi'(\tau)\psi'(t)d\rho dtd\tau,
\end{aligned}$$

hence

$$J_2 = \int_0^\infty e^{-\lambda(\psi(\tau)-\psi(0))}\left(\int_0^\tau\int_0^\infty \beta\omega_\beta(\rho)\frac{(\psi(\tau)-\psi(s))^{\beta-1}}{\rho^\beta}T(\frac{(\psi(\tau)-\psi(s))^\beta}{\rho^\beta}\lambda(s,x(s))\psi'(s)d\rho ds\right)\psi'(\tau)d\tau.$$

It follows that

$$\widehat{X}(\lambda) = \int_0^\infty e^{-\lambda(\psi(t)-\psi(0))}\left(\int_0^\infty \omega_\beta(\rho)T(\frac{(\psi(t)-\psi(0))^\beta}{\rho^\beta})(x_0-\Phi(x))d\rho\right)\psi'(t)dt$$
$$+\int_0^\infty e^{-\lambda(\psi(\tau)-\psi(0))}\left(\int_0^\tau\int_0^\infty \beta\omega_\beta(\rho)\frac{(\psi(\tau)-\psi(s))^{\beta-1}}{\rho^\beta}T(\frac{(\psi(\tau)-\psi(s))^\beta}{\rho^\beta}\lambda(s,x(s))\psi'(s)d\rho ds\right)\psi'(\tau)d\tau.$$

By using the inverse Laplace transform we obtain

$$\begin{aligned}
x(t) = &\int_0^\infty \varphi_\beta(\rho)T((\psi(t)-\psi(0))^\beta\rho)(x_0-\Phi(x))d\rho\\
&+\beta\int_0^t\int_0^\infty \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds.
\end{aligned}$$

Which complets the proof . □

Theorem 3.4. *Assume that the hypotheses $(A_1)-(A_4)$ are satisfied, then the nonlocal fractional Cauchy problem (1) has at least one mild solution defined on Δ.*

Proof. In order to prove the Theorem 3.4, let $E = C(\Delta, X)$ and let B_r be a subset of the space E defined by

$$B_r = \{x \in E : \ \| x \| \leq r\}.$$

where

$$r := \frac{M}{1 - ML_\Phi}\left(\|x_0\| + \|\Phi(0)\| + \frac{(\psi(T)-\psi(0))^\beta}{\Gamma(\beta+1)}\|h_r\|_{L^\infty} \right).$$

It is easy te see that B_r is a closed, convex and bounded subset of the Banach space E.
To show that the fractional integral equation (7) has at least one mild solution $x \in C(\Delta, X)$, we consider two operators $\mathcal{T}^\beta_{1,\psi}, \mathcal{T}^\beta_{2,\psi} : C(\Delta, X) \to C(\Delta, X)$ defined as follow:

$$\mathcal{T}^\beta_{1,\psi}x(t) = \int_0^\infty \varphi_\beta(\rho)T((\psi(t)-\psi(0))^\beta\rho)(x_0 - \Phi(x))d\rho, \tag{8}$$

and

$$\mathcal{T}^\beta_{2,\psi}x(t) = \beta\int_0^t\int_0^\infty \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds. \tag{9}$$

We can transforme the fractional integral equation (7) into the operator equation as follows:

$$x(t) = \mathcal{T}^\beta_{1,\psi}x(t) + \mathcal{T}^\beta_{2,\psi}x(t), \quad t \in \Delta.$$

Now, we will show that the operators $\mathcal{T}^\beta_{1,\psi}$ and $\mathcal{T}^\beta_{2,\psi}$ satisfy all the conditions of Lemma 2.17.
First, we prove that $\mathcal{T}^\beta_{1,\psi}$ is a contraction on E.
Let $x, y \in E$, then from hypothesis (A_3) we get

$$\|\mathcal{T}^\beta_{1,\psi}x(t) - \mathcal{T}^\beta_{1,\psi}y(t)\| \leq \int_0^\infty \Big\|\varphi_\beta(\rho)T((\psi(t)-\psi(0))^\beta\rho)\Big\| \ \Big\|\Phi(x)) - \Phi(y)\Big\|d\rho,$$

Taking supremum over t, we obtain

$$\|\mathcal{T}^\beta_{1,\psi}x - \mathcal{T}^\beta_{1,\psi}y\| \leq ML_\Phi\|x-y\|, \quad for \quad all \quad x, y \in E.$$

Therefore, $\mathcal{T}^\beta_{1,\psi}$ is a contractive mapping with constant $ML_\Phi < 1$.
Secondly, we show the operator $\mathcal{T}^\beta_{2,\psi}$ is completely continuous.
For this purpose, it is enough to prove that the operator $\mathcal{T}^\beta_{2,\psi}$ is continuous and $\mathcal{T}^\beta_{2,\psi}(B_r)$ is uniformly bounded and equicontinuous.
Let us show that the operator $\mathcal{T}^\beta_{2,\psi}$ is continuous.
Let x_n be a sequence in B_r converging to $x \in B_r$, then we have

$$\begin{aligned}\Big\|\mathcal{T}^\beta_{2,\psi}x_n(t) - \mathcal{T}^\beta_{2,\psi}x(t)\Big\| &= \Big\|\beta\int_0^t\int_0^\infty \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho) \\ &\quad \times [f(s,x_n(s)) - f(s,x(s))]\psi'(s)d\rho ds\Big\|, \\ &\leq \frac{M}{\Gamma(\beta+1)}\int_0^t (\psi(t)-\psi(s))^{\beta-1}\Big\|f(s,x_n(s)) - f(s,x(s))\Big\|\psi'(s)ds, \\ &\leq \frac{M(\psi(T)-\psi(0))^\beta}{\Gamma(\beta+1)}\sup_{s\in\Delta}\|f(s,x_n(s)) - f(s,x(s))\|,\end{aligned}$$

by using (A_2) we obtain

$$\lim_{n\mapsto+\infty}\mathcal{T}^{\beta}_{2,\psi}x_n(t)=\mathcal{T}^{\beta}_{2,\psi}x(t),\quad for\quad all\quad t\in\Delta.$$

Which shows that $\mathcal{T}^{\beta}_{2,\psi}$ is a continuous operator on B_r.

Next we show that $\mathcal{T}^{\beta}_{2,\psi}(B_r)$ is a uniformly bounded.
Let $x\in B_r$, then we have

$$\left\|\mathcal{T}^{\beta}_{2,\psi}x(t)\right\| = \left\|\beta\int_0^t\int_0^\infty\rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\right\|,$$

$$\left\|\mathcal{T}^{\beta}_{2,\psi}x(t)\right\| \leq\beta\int_0^t\int_0^\infty\rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho)\left\|f(s,x(s))\right\|\psi'(s)d\rho ds,$$

by using (A_4), we obtain

$$\|\mathcal{T}^{\beta}_{2,\psi}x\|\leq\frac{M(\psi(T)-\psi(0))^\beta}{\Gamma(\beta+1)}\|h_r\|_{L^\infty}\quad for\quad all\quad x\in B_r.$$

This shows that $\mathcal{T}^{\beta}_{2,\psi}$ is uniformly bounded on B_r.

Now, let us also show that $\mathcal{T}^{\beta}_{2,\psi}(B_r)$ is equicontinuous on Δ.

Let $x\in B_r$ and $t_1,t_2\in\Delta$ such that $t_1<t_2$, then we have

$$\begin{aligned}
\|\mathcal{T}^{\beta}_{2,\psi}x(t_2)-\mathcal{T}^{\beta}_{2,\psi}x(t_1)\| &= \Big\|\beta\int_0^{t_2}\int_0^\infty\rho\varphi_\beta(\rho)(\psi(t_2)-\psi(s))^{\beta-1}T((\psi(t_2)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\\
&-\beta\int_0^{t_1}\int_0^\infty\rho\varphi_\beta(\rho)(\psi(t_1)-\psi(s))^{\beta-1}T((\psi(t_1)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\Big\|,\\
&\leq\beta\Big\|\int_0^{t_2}\int_0^\infty\rho\varphi_\beta(\rho)(\psi(t_2)-\psi(s))^{\beta-1}T((\psi(t_2)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\\
&+\int_{t_1}^{t_2}\int_0^\infty\rho\varphi_\beta(\rho)(\psi(t_2)-\psi(s))^{\beta-1}T((\psi(t_2)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\\
&+\int_0^{t_1}\int_0^\infty\rho\varphi_\beta(\rho)(\psi(t_1)-\psi(s))^{\beta-1}T((\psi(t_2)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\\
&-\int_0^{t_1}\int_0^\infty\rho\varphi_\beta(\rho)(\psi(t_1)-\psi(s))^{\beta-1}T((\psi(t_2)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\\
&-\int_0^{t_1}\int_0^\infty\rho\varphi_\beta(\rho)(\psi(t_1)-\psi(s))^{\beta-1}T((\psi(t_1)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\Big\|,
\end{aligned}$$

it follows that

$$\begin{aligned}
\left\|\mathcal{T}^{\beta}_{2,\psi}x(t_2)-\mathcal{T}^{\beta}_{2,\psi}x(t_1)\right\| &\leq\beta\Big\|\int_{t_1}^{t_2}\int_0^\infty\rho\varphi_\beta(\rho)(\psi(t_2)-\psi(s))^{\beta-1}T((\psi(t_2)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\Big\|\\
&+\beta\Big\|\int_{t_1}^{t_2}\int_0^\infty\rho\varphi_\beta(\rho)\Big[(\psi(t_2)-\psi(s))^{\beta-1}-(\psi(t_2)-\psi(s))^{\beta-1}\Big]T((\psi(t_2)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\Big\|\\
&+\beta\Big\|\int_0^{t_1}\int_0^\infty\rho\varphi_\beta(\rho)(\psi(t_1)-\psi(s))^{\beta-1}\Big[T((\psi(t_2)-\psi(0))^\beta\rho)-T((\psi(t_1)-\psi(0))^\beta\rho)\Big]f(s,x(s))\psi'(s)d\rho ds\Big\|,\\
&:=\beta\Big(\Upsilon_1(t_2)+\Upsilon_2(t_2)+\Upsilon_3(t_2)\Big).
\end{aligned}$$

We have

$$\begin{aligned}\Upsilon_1(t_2) &\leq \Big\| \int_{t_1}^{t_2} \int_0^{\infty} \rho\varphi_\beta(\rho)(\psi(t_2)-\psi(s))^{\beta-1}\Big\|T((\psi(t_2)-\psi(0))^\beta\rho)f(s,x(s))\Big\|\psi'(s)d\rho ds,\\ &\leq \frac{M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}\Big(\psi(t_2)-\psi(t_1)\Big)^\beta.\end{aligned}$$

Since ψ is a continuous function, then we obatin

$$\lim_{t_2\to t_1}\Upsilon_1(t_2)=0.$$

On the other hand we have

$$\begin{aligned}\Upsilon_2(t_2) &\leq \int_{t_1}^{t_2} \int_0^{\infty} \rho\varphi_\beta(\rho)\Big[(\psi(t_2)-\psi(s))^{\beta-1}-(\psi(t_2)-\psi(s))^{\beta-1}\Big]\Big\|T((\psi(t_2)-\psi(0))^\beta\rho)f(s,x(s))\Big\|\psi'(s)d\rho ds,\\ &\leq \frac{M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}\Big(\psi^\beta(t_2)-\psi^\beta(t_1)-(\psi(t_2)-\psi(t_1))^\beta\Big).\end{aligned}$$

Since ψ is a continuous function, then we obatin

$$\lim_{t_2\to t_1}\Upsilon_2(t_2)=0.$$

For $t_1>0$ and $\epsilon>0$ we have

$$\begin{aligned}\Upsilon_3(t_2) \leq &\Big\| \int_0^{t_1-\epsilon} \int_0^{\infty} \rho\varphi_\beta(\rho)(\psi(t_1)-\psi(s))^{\beta-1}\Big[T((\psi(t_2)-\psi(0))^\beta\rho)-T((\psi(t_1)-\psi(0))^\beta\rho)\Big]f(s,x(s))\psi'(s)d\rho ds\\ &+\int_{t_1-\epsilon}^{t_1} \int_0^{\infty} \rho\varphi_\beta(\rho)(\psi(t_1)-\psi(s))^{\beta-1}\Big[T((\psi(t_2)-\psi(0))^\beta\rho)-T((\psi(t_1)-\psi(0))^\beta\rho)\Big]f(s,x(s))\psi'(s)d\rho ds\Big\|,\\ &\leq \frac{M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}\Big(\psi^\beta(t_2)-\psi^\beta(t_1)-(\psi(t_2)-\psi(t_1))^\beta\Big)\sup_{s\in[0,t_1-\epsilon]}\Big[T((\psi(t_2)-\psi(s))^\beta\rho)-T((\psi(t_1)-\psi(s))^\beta\rho)\Big]\\ &+\frac{M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)}\Big(\psi(t_1)-\psi(t_1-\epsilon)\Big)^\beta,\end{aligned}$$

by using (A_1) and since ψ is a continuous function, then for $t_2\to t_1$ and $\epsilon\to 0$ it follows that $\Upsilon_3(t_2)\to 0$.

Which shows that $\mathcal{T}^\beta_{2,\psi}(B_r)$ is equicontinuous.

It remains to prove that for any $t\in\Delta$ the set $\mathcal{S}(t)=\Big\{(\mathcal{T}^\beta_{1,\psi}x)(t): x\in B_r\Big\}$ is relatively compact in the space E.

Obviously, $\mathcal{S}(0)$ is relatively compact in E. Let $0<t\leq T$ be fixed. Then, for every $\epsilon>0$ and $\eta>0$, let $x\in B_r$ and define an operator $\mathcal{F}_{\epsilon,\eta}$ on B_r by

$$\begin{aligned}(\mathcal{F}_{\epsilon,\eta}x)(t) &= \beta\int_0^{t-\epsilon}\int_\eta^{\infty} \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds.\\ &= \beta\int_0^{t}\int_0^{\infty} \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho+\epsilon^\beta\eta-\epsilon^\beta\eta)f(s,x(s))\psi'(s)d\rho ds.\\ &= \beta\int_0^{t}\int_0^{\infty} \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}\Big[T(\epsilon^\beta\eta)T((\psi(t)-\psi(0))^\beta\rho-\epsilon^\beta\eta)\Big]f(s,x(s))\psi'(s)d\rho ds.\\ &= \beta T(\epsilon^\beta\eta)\int_0^{t}\int_0^{\infty} \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}\Big[T((\psi(t)-\psi(0))^\beta\rho-\epsilon^\beta\eta)\Big]f(s,x(s))\psi'(s)d\rho ds.\end{aligned}$$

By using (A_1) then $T(\epsilon^\beta \eta)$ is compact and hence the set $F_{\epsilon,\eta}(t) = \Big\{(\mathcal{F}_{\epsilon,\eta}x)(t) : x \in B_r\Big\}$ is relatively compact in E for all $\epsilon \in (0,t)$ and $\eta > 0$. Moreover, for every $x \in B_r$, we have

$$\begin{aligned}\Big\|(\mathcal{T}^\beta_{2,\psi}x)(t) - (\mathcal{F}_{\epsilon,\eta}x)(t)\Big\| &= \beta\Big\| \int_0^t \int_0^\eta \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds \\ &+ \int_0^t \int_\eta^\infty \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds. \\ &- \int_0^{t-\epsilon} \int_\eta^\infty \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\Big\|, \\ &\leq \beta\Big\| \int_0^t \int_0^\eta \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds \\ &+ \int_{t-\epsilon}^t \int_\eta^\infty \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho)f(s,x(s))\psi'(s)d\rho ds\Big\|,\end{aligned}$$

it follows that

$$\Big\|(\mathcal{T}^\beta_{2,\psi}x)(t) - (\mathcal{F}_{\epsilon,\eta}x)(t)\Big\| \leq M\|h_r\|_{L^\infty}(\psi(T)-\psi(0))^\beta\Big(\int_0^\eta \rho\varphi_\beta(\rho)d\rho\Big) + \frac{M(\psi(t)-\psi(t-\epsilon))^\beta}{\Gamma(\beta+1)}\|h_r\|_{L^\infty},$$

thus

$$\lim_{(\epsilon,\eta)\to(0,0)} \mathcal{F}_{\epsilon,\eta}x = \mathcal{T}^\beta_{2,\psi}x.$$

Therefore, there are relatively compact sets arbitrarily close to the set $\mathcal{S}(t)$ for $t > 0$. Hence, $\mathcal{S}(t)$ is relatively compact in E. Therefore, by Arzelà–Ascoli Theorem [10] we deduce that $\mathcal{T}^\beta_{2,\psi}(B_r)$ is relatively compact in E. Thus, the continuity of $\mathcal{T}^\beta_{2,\psi}$ and relatively compactness of $\mathcal{T}^\beta_{2,\psi}(B_r)$ imply that $\mathcal{T}^\beta_{2,\psi}$ is a completely continuous.

Now it remains to show that the assumption (3) in Lemma 2.17 is verified.
Let $x, y \in B_r$ be arbitrary , then by hypothesis (A_2), we have

$$\begin{aligned}\Big\| \mathcal{T}^\beta_{1,\psi}x(t) + \mathcal{T}^\beta_{2,\psi}y(t)\Big\| &\leq \Big\| \mathcal{T}^\beta_{1,\psi}x(t) \Big\| + \Big\| \mathcal{T}^\beta_{2,\psi}y(t) \Big\|, \\ &\leq \Big\| \int_0^\infty \varphi_\beta(\rho)T((\psi(t)-\psi(0))^\beta\rho)(x_0 - \Phi(x))d\rho \Big\| \\ &+ \Big\| \beta\int_0^t \int_0^\infty \rho\varphi_\beta(\rho)(\psi(t)-\psi(s))^{\beta-1}T((\psi(t)-\psi(0))^\beta\rho)f(s,y(s))\psi'(s)d\rho ds \Big\|,\end{aligned}$$

by using (A_3), we obtain

$$\Big\|\mathcal{T}^\beta_{1,\psi}x + \mathcal{T}^\beta_{2,\psi}y\Big\| \leq \frac{M}{1-ML_\Phi}\left(\|x_0\| + \|\Phi(0)\| + \frac{(\psi(T)-\psi(0))^\beta}{\Gamma(\beta+1)}\|h_r\|_{L^\infty}\right) = r.$$

Finally, all conditions of Lemma 2.17 are satisfied for the operators $\mathcal{T}^\beta_{1,\psi}$ and $\mathcal{T}^\beta_{2,\psi}$. Hence the opertor $\mathcal{T}^\beta_{1,\psi}+\mathcal{T}^\beta_{2,\psi}$ has a fixed point on B_r. Therefore, the nonlocal Cauchy problem (1) has a mild solution defined on Δ. The proof is completed.

□

4. An illustrative example

In this section, we give a nontrivial example to illustrate our main result.
Consider the following fractional evolution equation in the space $X = L^2([0,\pi])$:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2},e^t} v(t,x) = \dfrac{\partial^2}{\partial x^2} v(t,x) + \dfrac{e^{-t}}{9+e^t} cos(v(t,x)), \quad (t,x) \in [0,1] \times [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \quad t \in [0,1], \\ v(0,x) = \dfrac{1}{20} \displaystyle\sum_{i=1}^{10} log\Big(1 + |v(t_i,x)|\Big), \quad 0 < t_i < 1, \quad i = 1,2,..,10. \end{cases} \tag{10}$$

In this example we choose $\beta = \frac{1}{2}$, $T = 1$, $\psi(t) = e^t$, $f(t,v) = \dfrac{e^{-t}}{9+e^t} cos(u(x,t))$ and $\Phi(v) = \dfrac{1}{20} \displaystyle\sum_{i=1}^{10} log\Big(1 + |v(t_i,x)|\Big)$.

We define the operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ by

$$D(\mathcal{A}) := \Big\{ v \in X : v, v' \ are \ \ absolutely \ continuous \ and \ \ v'' \in X, v(0) = v(\pi) = 0 \Big\},$$

and

$$Av = \frac{\partial^2}{\partial x^2} v.$$

It is well knowledge that $\mathcal{A}$ has a discrete spectrum.
The eigenvalue are $-n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors
$e_n(z) = \sqrt{\frac{2}{\pi}} \sin(nz)$. Then

$$\mathcal{A}v = \sum_{n=1}^{\infty} -n^2 \langle v, e_n \rangle e_n, \quad v \in D(\mathcal{A}) \ .$$

Furthermore, $\mathcal{A}$ generates a uniformly bounded analytic semigroup $\{T(t)\}_{t \geq 0}$ in X and is given by

$$T(t)v = \sum_{n=1}^{\infty} e^{-n^2 t} \langle v, e_n \rangle e_n, v \in X$$

where

$$\langle v, e_n \rangle = \int_0^{\pi} v(t) e_n(t) dt.$$

Since $\|T(t)\| \leq e^{-t}$ for all $t \geq 0$, then $M = 1$. Which implies that $\sup\limits_{t \in [0,\infty)} \|T(t)\| = 1$ and (A_1) holds.

It is clear that the assumption (A_2) and (A_4) are satisfied . Indeed we have

$$| \ f(t,v) \ | \ = \ \Big| \ \frac{e^{-t}}{1+e^t} cos(u(t,x) \ \Big| \ \leq \frac{1}{2} e^{-t}.$$

To prove the assumption (A_3), let $u, v \in X$, then we have

$$|\Phi(u) - \Phi(v)| = \Big| \frac{1}{20} \sum_{i=1}^{10} log\Big(1 + |u(t_i,x)|\Big) - \frac{1}{20} \sum_{i=1}^{10} log\Big(1 + |v(t_i,x)|\Big) \Big|,$$

from which, we have

$$|\Phi(u) - \Phi(v)| \leq \frac{1}{2}|u - v|,$$

thus (A_3) hold with $L_\Phi = \frac{1}{2}$.

Finally, all the conditions of Theorem 3.4 are satisfied, thus the fractional evolution equation (10) has a mild solution.

5. Conclusion

In the current manuscript, we studied the existence of mild solutions for fractional Cauchy problem of nonlinear fractional evolution equations with nonlocal conditions involving $\psi-$Caputo type fractional derivatives. As a preliminary step, we construct a generic structure of mild solutions associated with our proposed model utilizing generalized $\psi-$Laplace transform, semigroups and some basic properties of $\psi-$fractional calculus. The existence result is established by using Krasnoselskii fixed point theorem. Finally, by using an appropriate example, the investigation of our theoritical result has been illustrated.

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