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$\Psi_\Gamma - C$ Sets in Ideal Topological Spaces

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Abstract. In this paper, we present a new type of set called $\Psi_\Gamma - C$ set by using the operator Ψ_Γ. We investigate the relationships of these sets with some special sets which were studied in the literature. For instance θ-open set, semi θ-open set, θ-semiopen set, regular θ-closed set. In particular, we show that $\Psi_\Gamma - C$ set is weaker than θ-open set. Furthermore, we prove that the collection of $\Psi_\Gamma - C$ set is closed under arbitrary union. Finally, we obtain the conclusion that the collection of $\Psi_\Gamma - C$ set forms a supratopology.

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1. Introduction

After the concepts of ideal and local function were presented by Kuratowski in [8], many authors have studied about these concepts in the literature. Among these studies, Natkaniec presented the set operator Ψ [14] in 1986. Then, Ψ-set [3], $\Psi - C$ set [13], $*^{\Psi}$-set [6] and Ψ^*-set [12] were studied by using Ψ operator. Furthermore, in [1] Al-Omari and Noiri studied the local closure function and the operator Ψ_Γ in ideal topological spaces. They also obtained new topologies by using the operator Ψ_Γ in [1]. Moreover, Islam and Modak defined the concept of semi-closure local function [7] and they obtained a new topology via this function.

On the other hand, Pavlović showed that under what conditions local function and local closure function are coincide in [16]. Then, Tunç and Özen Yıldırım presented the I_Γ-dense, Γ-dense-in-itself and I_Γ-perfect sets by using local closure function in [18].

In this study, we present the concept of $\Psi_\Gamma - C$ set by using the operator Ψ_Γ. We research the relationships of these sets with L_Γ-perfect [18], R_Γ-perfect [18], I_Γ-perfect and some other sets which were studied before in the literature [1, 2, 4, 5, 13, 15, 18, 19]. We also research some properties of such sets and we obtain new results.

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2. Preliminaries

In this paper, (X, τ) (shortly X) represents a topological space. In a topological space (X, τ), the closure and the interior of a subset A of X are denoted by $cl(A)$ and $int(A)$, respectively. $P(X)$ represents the family of all subsets of X.

An ideal I [8] on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following conditions:

(i) if $A \in I$ and $B \subseteq A$, then $B \in I$ (heredity),

(ii) if $A \in I$ and $B \in I$, then $A \cup B \in I$ (finite additivity).

An ideal topological space (X, τ, I) is a topological space (X, τ) with an ideal I on X. For a subset A of X, $A^*(I, \tau) = \{x \in X \mid U \cap A \notin I \text{ for each } U \in \tau(x)\}$ is called the local function [8] of A with respect to τ and I, where $\tau(x) = \{U \in \tau \mid x \in U\}$. We use A^* instead of $A^*(I, \tau)$. For a subset A of X, $\Gamma(A)(I, \tau) = \{x \in X \mid A \cap cl(U) \notin I \text{ for every } U \in \tau(x)\}$ is called the local closure function [1] of A with respect to I and τ. It is shortly denoted by $\Gamma(A)$ instead of $\Gamma(A)(I, \tau)$. An operator Ψ is defined as $\Psi(A) = X \setminus (X \setminus A)^*$ by using the $()^*$-operator in [14]. Another operator $\Psi_\Gamma : P(X) \mapsto \tau$ is defined as $\Psi_\Gamma(A) = X \setminus \Gamma(X \setminus A)$ for each $A \in P(X)$ in [1].

A subset A of X is called I_Γ-perfect [18] (resp. Γ-dense-in-itself [18], L_Γ-perfect [18], R_Γ-perfect [18], I_Γ-dense [18]) if $A = \Gamma(A)$ (resp. $A \subseteq \Gamma(A)$, $A \setminus \Gamma(A) \in I$, $\Gamma(A) \setminus A \in I$, $\Gamma(A) = X$). A subset A of X is called $\Psi - C$ set [13] if $A \subseteq cl(\Psi(A))$.

Theorem 2.1 ([17]). *In an ideal topological space (X, τ, I), $\Psi_\Gamma(A) \subseteq \Gamma(A)$ for each subset A of X and $\Psi_\Gamma(A) = \varnothing$ for $A \in I$ such that $cl(\tau) \cap I = \{\varnothing\}$ where $cl(\tau) = \{cl(G) : G \in \tau\}$.*

For a topological space (X, τ) and a subset A of X, $cl_\theta(A) = \{x \in X : cl(U) \cap A \neq \varnothing \text{ for each } U \in \tau(x)\}$ is called the θ-closure of A [19]. The θ-interior of A [19], denoted $int_\theta(A)$, consists of those points x of A such that $U \subseteq cl(U) \subseteq A$ for some open set U containing x. A subset A is called θ-closed [19] if $A = cl_\theta(A)$. The complement of a θ-closed set is called θ-open. The family of all θ-open sets in (X, τ) is denoted by τ_θ. Moreover, τ_θ is a topology on X. Al-Omari and Noiri defined the topologies on X in [1] as follows: $\sigma = \{A \subseteq X : A \subseteq \Psi_\Gamma(A)\}$ and $\sigma_0 = \{A \subseteq X : A \subseteq int(cl(\Psi_\Gamma(A)))\}$ and $\tau_\theta \subseteq \sigma \subseteq \sigma_0$. A subset A of X is called σ-open [1] (resp. σ_0-open [1]) set, if $A \in \sigma$ (resp. $A \in \sigma_0$). A subset A of X is called θ^I-closed [15] if $\Gamma(A) \subseteq A$. A subset A of X is called regular θ-closed [2] if $A = cl_\theta(int_\theta(A))$. A subset A of X is called semi θ-open [2] if $A \subseteq cl_\theta(int_\theta(A))$. A subset A of X is called θ-semiopen [4] if there exists a θ-open set U of X such that $U \subseteq A \subseteq cl(U)$. $SO_{\theta s}(X, \tau)$ represents the collection of all θ-semiopen sets in a topological space (X, τ) [4]. A subset A of X is called an M^*-open set [5] if $A \subseteq int(cl(int_\theta(A)))$. A subset A of X is called preopen [10] if $A \subseteq int(cl(A))$. The complement of a preopen set is called a preclosed [10] set. A subset A of X is called generalized closed (briefly, g-closed) [9] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is open.

Definition 2.2 ([11]). Let Y be a nonempty set and τ' be a collection of subsets of Y. If $Y \in \tau'$ and τ' is closed under arbitrary union, then τ' is called a supratopology on Y. (Y, τ') is called a supratopological space (or supraspace).

3. $\Psi_\Gamma - C$ Sets and Their Relationships

Definition 3.1. Let (X, τ, I) be an ideal topological space and $A \subseteq X$. A set A is said to be a $\Psi_\Gamma - C$ set if $A \subseteq cl(\Psi_\Gamma(A))$. The collection of all $\Psi_\Gamma - C$ sets in (X, τ, I) is denoted by $\Psi_\Gamma(X, \tau, I)$.

Theorem 3.2. *In an ideal topological space (X, τ, I), $int_\theta(A) \subseteq \Psi_\Gamma(A)$ for each subset A of X.*

Proof. Let A be a subset of X in an ideal topological space (X, τ, I). Assume that an element x of X is not in $\Psi_\Gamma(A)$. Then, $x \notin X \setminus \Gamma(X \setminus A)$ and so $x \in \Gamma(X \setminus A)$. It implies that $cl(U) \cap (X \setminus A) \notin I$ for each $U \in \tau(x)$. Therefore, $cl(U) \cap (X \setminus A) \neq \varnothing$ and then $cl(U) \nsubseteq A$ for each $U \in \tau(x)$. In this case, $x \notin int_\theta(A)$. Consequently, $int_\theta(A) \subseteq \Psi_\Gamma(A)$. □

Theorem 3.3. *Let (X, τ, I) be an ideal topological space. If $A \in \tau_\theta$, then $A \in \Psi_\Gamma(X, \tau, I)$.*

Proof. If $A \in \tau_\theta$, then $A \subseteq \Psi_\Gamma(A)$ by the Corollary 4.3 in [1]. Since $\Psi_\Gamma(A) \subseteq cl(\Psi_\Gamma(A))$, we have $A \subseteq cl(\Psi_\Gamma(A))$. Consequently, A is a $\Psi_\Gamma - C$ set. □

Remark 3.4. In an ideal topological space, an open set may not be a $\Psi_\Gamma - C$ set.

Example 3.5. Let $X = \{a, b, c, d\}$, $\tau = \{\varnothing, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\varnothing, \{a\}\}$. If $A = \{a\}$, then $cl(\Psi_\Gamma(A)) = \varnothing$. So, A is an open set but it is not a $\Psi_\Gamma - C$ set.

Remark 3.6. In an ideal topological space, $\Psi_\Gamma - C$ set may not be θ-open and open.

Example 3.7. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}$ and $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. If $A = \{a\}$, then $\Psi_\Gamma(A) = \{a, c\}$ and so $cl(\Psi_\Gamma(A)) = \{a, b, c\}$. Therefore, A is a $\Psi_\Gamma - C$ set but it is neither θ-open nor open.

Remark 3.8. In an ideal topological space, θ-closed (closed, $\theta^I - closed$) sets may not be $\Psi_\Gamma - C$ set.

Example 3.9. In the ideal topological space $(\mathbb{R}, \tau_D, I_f)$, where I_f is the ideal of finite subsets of $\mathbb{R}$ (the set of all real numbers) and τ_D is the usual topology on $\mathbb{R}$. A subset $A = [0, 1] \cup \{2\}$ is a θ-closed set and so it is both closed and $\theta^I - closed$. But, A is not a $\Psi_\Gamma - C$ set.

Theorem 3.10. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If $X \setminus A$ is $\theta^I - closed$, then A is a $\Psi_\Gamma - C$ set.*

Proof. In an ideal topological space (X, τ, I), let $X \setminus A$ be a $\theta^I - closed$ set for $A \subseteq X$. Then $\Gamma(X \setminus A) \subseteq X \setminus A$ and so $A \subseteq X \setminus \Gamma(X \setminus A) = \Psi_\Gamma(A) \subseteq cl(\Psi_\Gamma(A))$. Consequently, A is a $\Psi_\Gamma - C$ set. □

Remark 3.11. The reverse of the above theorem may not be true in general.

Example 3.12. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\emptyset\}$. Take $A = \{c, d\}$. Although the set A is a $\Psi_\Gamma - C$ set, $X \setminus A$ is not $\theta^I - closed$.

Theorem 3.13. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If A is a $\Psi_\Gamma - C$ set and $\Psi_\Gamma(A)$ is closed, then $X \setminus A$ is θ^I-closed.*

Proof. Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Assume that A is a $\Psi_\Gamma - C$ set and $\Psi_\Gamma(A)$ is closed. Then $A \subseteq cl(\Psi_\Gamma(A)) = \Psi_\Gamma(A) = X \setminus \Gamma(X \setminus A)$. It implies that $\Gamma(X \setminus A) \subseteq X \setminus A$ and so $X \setminus A$ is θ^I-closed. □

Theorem 3.14. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If $X \setminus A$ is a $\Psi_\Gamma - C$ set and $\Gamma(A)$ is an open set, then A is θ^I-closed.*

Proof. Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Assume that $X \setminus A$ is a $\Psi_\Gamma - C$ set and $\Gamma(A)$ is an open set. Then $X \setminus A \subseteq cl(\Psi_\Gamma(X \setminus A)) = cl(X \setminus \Gamma(A)) = X \setminus \Gamma(A)$. It implies that $\Gamma(A) \subseteq A$ and so A is a θ^I-closed set. □

Corollary 3.15. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If $X \setminus A$ is a $\Psi_\Gamma - C$ set and $\Gamma(A)$ is an open set, then A is an R_Γ-perfect set.*

Proof. The proof is obvious by the Theorem 2.17 in [18]. □

Remark 3.16. In an ideal topological space, an I_Γ-perfect set may not be a $\Psi_\Gamma - C$ set. Similarly, a $\Psi_\Gamma - C$ set may not be an I_Γ-perfect set.

Example 3.17. In the ideal topological space $(\mathbb{R}, \tau_D, I = \{\emptyset\})$, the subset $A = [0, 1] \cup \{2\}$ is an I_Γ-perfect set, but it is not a $\Psi_\Gamma - C$ set. The set $B = (0, 1)$ is a $\Psi_\Gamma - C$ set, but B is not an I_Γ-perfect set.

Theorem 3.18. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If $X \setminus A$ is I_Γ-perfect, then A is a $\Psi_\Gamma - C$ set.*

Proof. Let $X \setminus A$ be an I_Γ-perfect subset of X in an ideal topological space (X, τ, I). Then $\Gamma(X \setminus A) = X \setminus A$ and $A = X \setminus \Gamma(X \setminus A) = \Psi_\Gamma(A) \subseteq cl(\Psi_\Gamma(A))$. As a result, A is a $\Psi_\Gamma - C$ set. □

Remark 3.19. The reverse of the above theorem may not be true in general.

Example 3.20. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\emptyset, \{a\}\}$. In the ideal topological space (X, τ, I), the set $A = \{c, d\}$ is a $\Psi_\Gamma - C$ set, but $X \setminus A$ is not I_Γ-perfect.

Theorem 3.21. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. $X \setminus A$ is I_Γ-dense iff $\Psi_\Gamma(A) = \emptyset$.*

Proof. Let (X, τ, I) be an ideal topological space and $A \subseteq X$. $\Psi_\Gamma(A) = \emptyset \Leftrightarrow X \setminus \Gamma(X \setminus A) = \emptyset \Leftrightarrow \Gamma(X \setminus A) = X \Leftrightarrow X \setminus A$ is I_Γ-dense. □

Remark 3.22. In an ideal topological space, an I_Γ-dense set may not be a $\Psi_\Gamma - C$ set. Similarly, a $\Psi_\Gamma - C$ set may not be an I_Γ-dense set.

Example 3.23. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\emptyset\}$. In the ideal topological space (X, τ, I), an empty set is a $\Psi_\Gamma - C$ set, but it is not an I_Γ-dense set. The set $\{c\}$ is I_Γ-dense, but it is not a $\Psi_\Gamma - C$ set.

Theorem 3.24. *Let (X, τ, I) be an ideal topological space and $cl(\tau) \cap I = \{\emptyset\}$. The empty set is the only one $\Psi_\Gamma - C$ set in the ideal.*

Proof. Let (X, τ, I) be an ideal topological space and $cl(\tau) \cap I = \{\emptyset\}$. Assume that $A \in I$. Since $cl(\tau) \cap I = \{\emptyset\}$, $\Psi_\Gamma(A) = \emptyset$ by the Theorem 2.1 and so $cl(\Psi_\Gamma(A)) = \emptyset$. If A is a $\Psi_\Gamma - C$ set, A must be an empty set. □

Corollary 3.25. *Let (X, τ, I) be an ideal topological space where $cl(\tau) \cap I = \{\emptyset\}$. If $A \in I$ or $X \setminus A$ is I_Γ-dense, then $\emptyset \neq A \notin \Psi_\Gamma(X, \tau, I)$.*

Proof. Let (X, τ, I) be an ideal topological space where $cl(\tau) \cap I = \{\emptyset\}$. If $A \in I$, we know that $\emptyset \neq A \notin \Psi_\Gamma(X, \tau, I)$ by the above theorem. If $X \setminus A$ is I_Γ-dense, then $\Psi_\Gamma(A) = \emptyset$ and so $cl(\Psi_\Gamma(A)) = \emptyset$. In this situation, an empty set is the only one $\Psi_\Gamma - C$ set. □

Remark 3.26. In an ideal topological space, a Γ-dense in itself set may not be a $\Psi_\Gamma - C$ set. Similarly, a $\Psi_\Gamma - C$ set may not be Γ-dense in itself.

Example 3.27. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = P(X)$. The set $A = \{a\}$ is a $\Psi_\Gamma - C$ set, but it is not Γ-dense in itself.

Example 3.28. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\emptyset, \{a\}\}$. In the ideal topological space (X, τ, I), the set $A = \{c\}$ is Γ-dense in itself, but it is not a $\Psi_\Gamma - C$ set.

Theorem 3.29. *Let (X, τ, I) be an ideal topological space and $cl(\tau) \cap I = \{\emptyset\}$. Then, every $\Psi_\Gamma - C$ set is Γ-dense in itself.*

Proof. Let (X, τ, I) be an ideal topological space where $cl(\tau) \cap I = \{\emptyset\}$ and $A \subseteq X$. If A is a $\Psi_\Gamma - C$ set, then $A \subseteq cl(\Psi_\Gamma(A))$. Since $cl(\tau) \cap I = \{\emptyset\}$, $\Psi_\Gamma(A) \subseteq \Gamma(A)$ by the Theorem 2.1 and so $A \subseteq cl(\Psi_\Gamma(A)) \subseteq cl(\Gamma(A))$. We know that $\Gamma(A)$ is closed by the Theorem 2.6 in [1]. Therefore, $A \subseteq \Gamma(A)$ and A is Γ-dense in itself. □

Remark 3.30. In an ideal topological space, $\Psi_\Gamma - C$ sets may not be L_Γ-perfect.

Example 3.31. In the ideal topological space $(\mathbb{R}, P(X), I_f)$, $\mathbb{R}$ is not L_Γ-perfect, but it is a $\Psi_\Gamma - C$ set.

Corollary 3.32. *Let (X, τ, I) be an ideal topological space where $cl(\tau) \cap I = \{\emptyset\}$. Then, every $\Psi_\Gamma - C$ set is L_Γ-perfect.*

Proof. Let (X, τ, I) be an ideal topological space where $cl(\tau) \cap I = \{\emptyset\}$. By the above theorem, every $\Psi_\Gamma - C$ set is Γ-dense in itself. By the Theorem 2.20 in [18] every Γ-dense in itself set is L_Γ-perfect. Consequently, every $\Psi_\Gamma - C$ set is L_Γ-perfect. □

Theorem 3.33. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If A is a σ_0-open set, then A is a $\Psi_\Gamma - C$ set.*

Proof. Let (X, τ, I) be an ideal topological space and A be a σ_0-open subset of X. Then, $A \subseteq int(cl(\Psi_\Gamma(A)))$ and so $A \subseteq cl(\Psi_\Gamma(A))$. Consequently, A is a $\Psi_\Gamma - C$ set. □

Corollary 3.34. *In an ideal topological space (X, τ, I), every σ-open set is a σ_0-open set [1]. By the above theorem, we can say that every σ-open set is a $\Psi_\Gamma - C$ set.*

Remark 3.35. In an ideal topological space (X, τ, I), a $\Psi_\Gamma - C$ set may not be a σ-open set and a σ_0-open set.

Example 3.36. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\emptyset\}$. In the ideal topological space (X, τ, I), the set $A = \{c, d\}$ is a $\Psi_\Gamma - C$ set, but A is neither a σ-open set nor a σ_0-open set.

Theorem 3.37. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If A is a $\Psi_\Gamma - C$ set, then A is a $\Psi - C$ set.*

Proof. Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Assume that A is a $\Psi_\Gamma - C$ set. Then, $A \subseteq cl(\Psi_\Gamma(A))$. Since $A^* \subseteq \Gamma(A)$ by the Lemma 2.2 in [1], $\Psi_\Gamma(A) \subseteq \Psi(A)$ and thus $cl(\Psi_\Gamma(A)) \subseteq cl(\Psi(A))$. Therefore, we have $A \subseteq cl(\Psi(A))$. As a result, A is a $\Psi - C$ set. □

Remark 3.38. In an ideal topological space, the inverse of the above theorem may not be true.

Example 3.39. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\emptyset, \{a\}\}$. In the ideal topological space (X, τ, I), the set $A = \{a\}$ is a $\Psi - C$ set, but it is not a $\Psi_\Gamma - C$ set.

Family of closed-discrete subsets I_{cd}, family of relatively compact subsets I_k, family of nowhere dense subsets I_n and family of meager subsets I_m are an ideal on X for a topological space (X, τ).

Theorem 3.40 ([16]). *In an ideal topological space (X, τ, I), each of the following conditions implies, the local function and the local closure function are equivalent.*
(1) τ has a clopen base β.
(2) τ is T_3.
(3) $I = I_{cd}$.
(4) $I = I_k$.
(5) $I_n \subseteq I$.
(6) $I = I_m$.

Theorem 3.41 ([17]). *In an ideal topological space (X, τ, I), each of the following conditions implies, the local function and the local closure function are equivalent.*
(1) τ has a clopen base β.
(2) τ is T_3.
(3) $I = I_{cd}$.
(4) $I = I_k$.
(5) $I_n \subseteq I$.
(6) $I = I_m$.
(7) Every open set is a preclosed set in (X, τ).
(8) Every open set is a closed set in (X, τ).
(9) Every open set is a g-closed set in (X, τ).
(10) Every preopen set is a closed set in (X, τ).

Corollary 3.42. *By the above theorem, each of the above conditions (1)-(10) implies A is a $\Psi_\Gamma - C$ set iff A is a $\Psi - C$ set for $A \subseteq X$.*

Theorem 3.43. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If A is a regular θ-closed set, then A is a $\Psi_\Gamma - C$ set.*

Proof. Let A be a regular θ-closed subset of X in an ideal topological space (X, τ, I). Then, A is equivalent to $cl_\theta(int_\theta(A))$. Let an element x of X be not in $cl(\Psi_\Gamma(A))$. Then, there exists $U \in \tau(x)$ with $U \cap \Psi_\Gamma(A) = \varnothing$. Namely, $U \cap (X \setminus \Gamma(X \setminus A)) = \varnothing$. Therefore, $x \in U \subseteq \Gamma(X \setminus A)$. Since $\Gamma(X \setminus A)$ is closed, $x \in cl(U) \subseteq cl(\Gamma(X \setminus A)) = \Gamma(X \setminus A) \subseteq cl_\theta(X \setminus A)$ by the Theorem 2.6 in [1]. Thus, $x \in int_\theta(cl_\theta(X \setminus A))$ and $x \notin X \setminus int_\theta(cl_\theta(X \setminus A)) = cl_\theta(int_\theta(A)) = A$. Consequently, $A \subseteq cl(\Psi_\Gamma(A))$ and so A is a $\Psi_\Gamma - C$ set. □

Remark 3.44. The inverse of the above theorem may not be true in general.

Example 3.45. Let $X = \{a, b, c, d\}, \tau = \{\varnothing, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\varnothing, \{a\}\}$. In the ideal topological space (X, τ, I), the set $A = \{b, c\}$ is a $\Psi_\Gamma - C$ set, but it is not regular θ-closed.

Theorem 3.46. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If A is semi θ-open, then A is a $\Psi_\Gamma - C$ set.*

Proof. Let A be a semi θ-open subset of X in an ideal topological space (X, τ, I). Then, $A \subseteq cl_\theta(int_\theta(A))$. Suppose that an element x of X is not in $cl(\Psi_\Gamma(A))$. Then, there exists $U \in \tau(x)$ such that $U \cap \Psi_\Gamma(A) = \varnothing$. Therefore, $x \in U \subseteq X \setminus \Psi_\Gamma(A) = \Gamma(X \setminus A)$. Since $\Gamma(X \setminus A)$ is closed, we can say that $x \in cl(U) \subseteq \Gamma(X \setminus A)$ and so $x \in int_\theta(\Gamma(X \setminus A))$. As $\Gamma(X \setminus A) \subseteq cl_\theta(X \setminus A)$, $x \in int_\theta(\Gamma(X \setminus A)) \subseteq int_\theta(cl_\theta(X \setminus A))$. Thus, $x \notin X \setminus int_\theta(cl_\theta(X \setminus A)) = cl_\theta(int_\theta(A))$. Since A is semi θ-open, $x \notin A$ and so $A \subseteq cl(\Psi_\Gamma(A))$. Consequently, A is a $\Psi_\Gamma - C$ set. □

Remark 3.47. In an ideal topological space, a $\Psi_\Gamma - C$ set may not be a semi θ-open set.

Example 3.48. Let $X = \{a, b, c, d\}, \tau = \{\varnothing, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\varnothing, \{a\}\}$. In the ideal topological space (X, τ, I), the set $A = \{b, c\}$ is a $\Psi_\Gamma - C$ set, but it is not semi θ-open.

Theorem 3.49. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If A is a θ-semiopen set, then A is a semi θ-open set.*

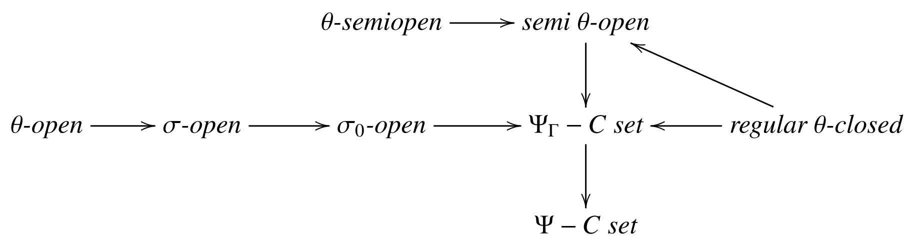
Proof. Let A be a θ-semiopen subset of X. Then, $A \subseteq cl(int_\theta(A))$ by the Lemma 1.1 in [4]. Since $cl(int_\theta(A)) \subseteq cl_\theta(int_\theta(A))$, we have $A \subseteq cl_\theta(int_\theta(A))$. Thus, A is a semi θ-open set. □

Corollary 3.50. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If A is a θ-semiopen set, then A is a $\Psi_\Gamma - C$ set.*

Remark 3.51. In an ideal topological space (X, τ, I), a $\Psi_\Gamma - C$ set may not be a θ-semiopen set.

Example 3.52. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}$ and $I = \{\emptyset, \{c\}, \{b\}, \{b, c\}\}$. If $A = \{a\}$, then A is a $\Psi_\Gamma - C$ set, but it is not a θ-semiopen set.

Corollary 3.53. *The following diagram is obtained from Theorem 3.33, Theorem 3.37, Theorem 3.43, Theorem 3.46, Theorem 3.49, Proposition 2.5 in [2], Theorem 4.2 in [1] and Corollary 4.3 in [1].*



Corollary 3.54. *In an ideal topological space (X, τ, I) where $cl(\tau) \cap I = \{\emptyset\}$, $\tau_\theta \subseteq SO_{\theta s}(X, \tau) \subseteq \Psi_\Gamma(X, \tau, I)$.*

Proof. It is obvious by the Remark 1.1 in [4] and Corollary 3.50. □

4. Further Properties

Remark 4.1. In an ideal topological space, subsets of $\Psi_\Gamma - C$ sets may not be a $\Psi_\Gamma - C$ set.

Example 4.2. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\emptyset\}$. If $A = \{c, d\}$, then A is a $\Psi_\Gamma - C$ set but $B = \{c\}$ is not a $\Psi_\Gamma - C$ set.

Theorem 4.3. *Let (X, τ, I) be an ideal topological space and $A \in I$. If a set A is a $\Psi_\Gamma - C$ set, every subset of A is also a $\Psi_\Gamma - C$ set.*

Proof. Let $A \in I$ in an ideal topological space (X, τ, I). Assume that A is a $\Psi_\Gamma - C$ set and $B \subseteq A$. By the heredity, $B \in I$. Then, we can say that $\Gamma(X) = \Gamma(X \backslash B) = \Gamma(X \backslash A)$ from the Corollary 2.10 in [1]. Therefore, $cl(\Psi_\Gamma(A)) = cl(\Psi_\Gamma(B))$. Since a set A is $\Psi_\Gamma - C$ set, $B \subseteq A \subseteq cl(\Psi_\Gamma(A)) = cl(\Psi_\Gamma(B))$. Consequently, B is a $\Psi_\Gamma - C$ set. □

Remark 4.4. In an ideal topological space, an element of ideal may not be a $\Psi_\Gamma - C$ set.

Example 4.5. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\emptyset, \{a\}\}$. In the ideal topological space (X, τ, I), the set $A = \{a\}$ is not a $\Psi_\Gamma - C$ set.

Theorem 4.6. *Let $\{A_\alpha : \alpha \in \triangle\}$ be a collection of nonempty $\Psi_\Gamma - C$ sets in an ideal topological space (X, τ, I). Then, $\cup A_\alpha \in \Psi_\Gamma(X, \tau, I)$.*

Proof. Let $\{A_\alpha : \alpha \in \triangle\}$ be a collection of nonempty $\Psi_\Gamma - C$ sets in an ideal topological space (X, τ, I). Then, $A_\alpha \subseteq cl(\Psi_\Gamma(A_\alpha))$ and so $A_\alpha \subseteq cl(\Psi_\Gamma(A_\alpha)) \subseteq cl(\Psi_\Gamma(\cup A_\alpha))$ for each $\alpha \in \triangle$ by the Theorem 4.2 in [1]. It implies that $\cup A_\alpha \subseteq cl(\Psi_\Gamma(\cup A_\alpha))$. This means that $\cup A_\alpha$ is a $\Psi_\Gamma - C$ set and then $\cup A_\alpha \in \Psi_\Gamma(X, \tau, I)$. □

Remark 4.7. In an ideal topological space, the intersection of two $\Psi_\Gamma - C$ sets may not be a $\Psi_\Gamma - C$ set.

Example 4.8. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\emptyset, \{a\}\}$. Let $A = \{b, c\}$ and $B = \{c, d\}$. Then, A and B are $\Psi_\Gamma - C$ sets. But their intersection $\{c\}$ is not a $\Psi_\Gamma - C$ set.

Proposition 4.9. *Let (X, τ) be a topological space. If $I = \{\emptyset\}$, in an ideal topological space (X, τ, I), $\Gamma(A) = cl_\theta(A)$ for each $A \subseteq X$. Then, $cl(\Psi_\Gamma(A)) = cl(X \setminus \Gamma(X \setminus A)) = cl(X \setminus cl_\theta(X \setminus A)) = cl(int_\theta(A))$. Therefore, A is a $\Psi_\Gamma - C$ set iff $A \subseteq cl(int_\theta(A))$. If $I = P(X)$, then $\Gamma(A) = \emptyset$ for each $A \subseteq X$. Therefore, $cl(\Psi_\Gamma(A)) = cl(X \setminus \Gamma(X \setminus A)) = cl(X \setminus \emptyset) = cl(X) = X$ and $A \subseteq cl(\Psi_\Gamma(A))$ for each $A \subseteq X$. Consequently, every subset A of X is a $\Psi_\Gamma - C$ set.*

Theorem 4.10. *Let (X, τ, I) be an ideal topological space and A be a nonempty subset of X. If there exists $U \in \tau(x)$ with $cl(U) \setminus A \in I$ for each $x \in A$, then A is a $\Psi_\Gamma - C$ set.*

Proof. Let (X, τ, I) be an ideal topological space and A be a nonempty subset of X. Assume that there exists $U \in \tau(x)$ with $cl(U) \setminus A \in I$ for each $x \in A$. Then, $cl(U) \cap (X \setminus A) \in I$ and so $x \notin \Gamma(X \setminus A)$. Therefore, $x \in X \setminus \Gamma(X \setminus A)$. It implies that $A \subseteq X \setminus \Gamma(X \setminus A) \subseteq cl(X \setminus \Gamma(X \setminus A)) = cl(\Psi_\Gamma(A))$. Finally, A is a $\Psi_\Gamma - C$ set. □

Remark 4.11. The inverse of the above theorem is not true.

Example 4.12. Let $X = \{a, b, c, d\}, \tau = \{\varnothing, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\varnothing\}$. In the ideal topological space (X, τ, I), the set $A = \{c, d\}$ is a $\Psi_\Gamma - C$ set, but for the set X which is the only one set in $\tau(c)$, $cl(X) \setminus A = X \setminus A = \{a, b\} \notin I$.

Corollary 4.13. *Let (X, τ, I) be an ideal topological space. Every subset of X is a $\Psi_\Gamma - C$ set if there exists $U \in \tau(x)$ such that $cl(U) \setminus \{x\} \in I$ for each $x \in X$.*

Proof. Let (X, τ, I) be an ideal topological space. Assume that there exists $U \in \tau(x)$ such that $cl(U) \setminus \{x\} \in I$ for each $x \in X$. Also assume that $A \subseteq X$ is nonempty. Therefore, there exists $U \in \tau(x)$ such that $cl(U) \setminus \{x\} \in I$ for each $x \in A$. Since $cl(U) \setminus A \subseteq cl(U) \setminus \{x\}$, $cl(U) \setminus A \in I$ by the heredity. As a result, there exists $U \in \tau(x)$ such that $cl(U) \setminus A \in I$ for each $x \in A$. Finally, A is a $\Psi_\Gamma - C$ set by the Theorem 4.10. □

Remark 4.14. In an ideal topological space (X, τ, I), $\varnothing$ and X are $\Psi_\Gamma - C$ sets.

Corollary 4.15. *In an ideal topological space (X, τ, I), $\Psi_\Gamma(X, \tau, I)$ forms a supratopology on X.*

Proof. It is obvious from the Theorem 4.6 and the Remark 4.14. □

Theorem 4.16. *Let (X, τ, I) be an ideal topological space and A be a $\Psi_\Gamma - C$ set. If B is a $M^\star$-open set, then $A \cap B \in \Psi_\Gamma(X, \tau, I)$.*

Proof. Let (X, τ, I) be an ideal topological space and $A, B \subseteq X$. Assume that A is a $\Psi_\Gamma - C$ set and B is an $M^\star$-open set. Then, $A \cap B \subseteq cl(\Psi_\Gamma(A)) \cap B \subseteq cl(\Psi_\Gamma(A)) \cap int(cl(int_\theta(B))) \subseteq cl(\Psi_\Gamma(A) \cap int(cl(int_\theta(B)))) = cl(int(\Psi_\Gamma(A)) \cap int(cl(int_\theta(B)))) = cl(int(\Psi_\Gamma(A) \cap cl(int_\theta(B)))) \subseteq cl(int(\Psi_\Gamma(A) \cap cl(\Psi_\Gamma(B))))$ by the Theorem 3.2. Then, $cl(int(\Psi_\Gamma(A) \cap cl(\Psi_\Gamma(B)))) \subseteq cl(\Psi_\Gamma(A) \cap cl(\Psi_\Gamma(B))) \subseteq cl(cl(\Psi_\Gamma(A) \cap \Psi_\Gamma(B))) = cl(\Psi_\Gamma(A) \cap \Psi_\Gamma(B)) = cl(\Psi_\Gamma(A \cap B))$ by the Theorem 4.2 in [1]. As a result, $A \cap B$ is a $\Psi_\Gamma - C$ set. □

Corollary 4.17. *Let (X, τ, I) be an ideal topological space and $A, B \subseteq X$. If A is a $\Psi_\Gamma - C$ set and B is a θ-open set, then $A \cap B$ is a $\Psi_\Gamma - C$ set.*

Proof. It is obvious from that every θ-open set is an $M^\star$-open set by the Lemma 2.2 in [5] and by the above theorem. □

Authors Contribution Statement”

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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