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Research Article

On the study of the stress-strength reliability in Weibull-F Models

Zohreh Pakdaman*, Reza Alizadeh Noughabi

Department of Statistics, University of Hormozgan, P. O. Box 3995, Hormozgan 7916193145, Iran

Abstract

In this paper, the problem of inferencing on the stress-strength reliability under Weibull-F Models when the stress and strength systems belong to the different families of distributions from the Weibull-F Model is considered. Some stochastic comparisons between the survival distribution functions of this model are obtained. Also, the asymptotic and several bootstrap confidence intervals of stress-strength reliability are studied. The efficiency of asymptotic and bootstrap confidence intervals are analyzed by simulation. The numerical example based on real-life data is displayed as an illustration.

Mathematics Subject Classification (2020). 62N05, 60-XX

Keywords. Stress-strength reliability, stochastic order, bootstrap confidence interval

1. Introduction

Industrial, engineering and mechanical systems during the period of their operation are usually under the different controllable and uncontrollable stresses such as temperature, humidity, material quality and system configuration, and so on. Therefore, so many of the systems are a type of stress-strength model and the study of their features is important. In the stress-strength system, $R = P(X < Y)$ is a measure of assurance of the component performance with the random strength Y when it is subjected to the random stress X. In a stress-strength model, the system fails if and only if, at any time, the applied stress is greater than its strength. It is worth noting that the stress-strength reliability is used not only in reliability but also in other sciences as a measure to compare two populations. For example, in medicine, if the random variables X and Y be the number of cancer patients treated with two different chemotherapeutic methods, then R is a comparison between the above two methods to find a more effective method of treatment. Also, in engineering , if the random variables X and Y are the strength of two materials in an engineering design, then R is the probability that the strength of X is less than the strength of Y. The parametric and non-parametric inferences on R for several specific distributions of random variables X and Y under different sampling schemes have been studied by various authors. We refer the readers to [5, 7, 11, 12, 14–17, 19–21, 24]. In this paper, we study the stress-strength reliability in Weibull-F family of distributions. In many real examples,

*Corresponding Author.
Email addresses: zpakdaman@hormozgan.ac.ir (Z. Pakdaman), reza.alizadehn@gmail.com (R. Alizadeh Noughabi)
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some generalizations of Weibull distribution are more desired as their hazard rate function can take any form for different values of parameters. Zografos and Balakrishnan [25] developed a convertible and transformative gamma-G class of distributions based on Stacys generalized gamma distribution and record value theory. Recently, Bourguignon et al. [8] presented the Weibull-F family of distributions affected by the Zografos-Balakrishnan-F class. For any parent continuous distribution F, the corresponding Weibull-F distribution characterized by the notation of W-$F(\alpha,\beta,F)$ with the cumulative distribution function (CDF) of

$$G(y;\alpha,\beta)=1-e^{-\alpha(\frac{F(y)}{\bar{F}(y)})^{\beta}}, \tag{1.1}$$

and the probability density function (PDF)

$$g(y;\alpha,\beta)=\alpha\beta\frac{f(y)}{\bar{F}^{2}(y)}(\frac{F(y)}{\bar{F}(y)})^{\beta-1}e^{-\alpha(\frac{F(y)}{\bar{F}(y)})^{\beta}}, \tag{1.2}$$

where $F(y)$ indicates the baseline distribution function and $\alpha,\beta>0$ are positive parameters. Intuitively, let X be a random lifetime of a component with cdf F. The odds ratio of the component with the lifetime X which will be failed at time y is $\frac{F(y)}{\bar{F}(y)}$. Assume that the variability of this odds of failure is represented by the random variable Y and assume that it follows the Weibull model with scale parameter α and shape parameter β such as Equation (1.1). It is worth noting that the CDF of Weibull-F distribution in Equation (1.1) can be regarded as a distorted distribution.

Equation (1.1) contains an inclusive family of continuous distribution functions. Each of the Weibull-F distributions can be obtained from a specified baseline CDF $F(y)$, e.g., Weibull-Frechet is obtained by taking $F(y)$ as the CDF of the Frechet distribution. The Weibull-Gompertz, Weibull-Log-logistic, Weibull-Pareto and Weibull- Normal distributions can be obtained similarly by taking $F(y)$ as the CDF of the Gompertz, Log-logistic, Pareto and normal distributions, respectively.

The aim of this paper is to compare two Weibull-F distributions with parameters (α_1,β_1) and (α_2,β_2). More specifically, we are interested in estimating $R=P(X<Y)$, where X and Y are two random variables with Weibull-F distributions with parameters (α_1,β_1) and (α_2,β_2), respectively. The outline of this paper is as follows. In Section 2, some useful definitions and some stochastic orderings between the stress and strength distributions in Weibull-F model are discussed. Moreover, we derive the expression for $R=P(X<Y)$ and develop a procedure for estimating R in Section 3. Furthermore, we obtain the maximum likelihood estimates (MLE) of the parameters and their asymptotic variance-covariance matrix in Section 3. Section 4 provides various bootstrap confidence intervals for R. In Section 5, simulation studies are carried out to evaluate the performance of both asymptotic and bootstrap confidence intervals for R. Also, a numerical example based on real-life data is provided in Section 5. Finally, the conclusions are given in Section 6.

2. Some fundamental basic definitions and primary results

This section devoted to the review of some notes about the stochastic orders. Consider two univariate random variables of X and Y that their following characteristics are respectively termed as: CDFs F and G, survival functions $\bar{F}(=1-F)$ and $\bar{G}(=1-G)$, PDFs f and g, hazard rate functions $h_f(=f/\bar{F})$ and $h_g(=g/\bar{G})$ and reversed hazard rate functions $\tilde{r}_F(=f/F)$ and $\tilde{r}_G(=g/G)$. Denote by G^{-1} the corresponding quantile function, defined by $G^{-1}(u)=\inf\{x:G(x)\geq u\},\ 0\leq u\leq 1$. Note that, the stochastic orders are introduced for the sake of comparing the magnitudes of two random variables. More details of stochastic orders can be found in [18].

Definition 2.1. The vector X is said to be smaller than vector Y in the

(i) usual stochastic order denoted by $X \leq_{st} Y$ if $\bar{F}(t) \leq \bar{G}(t)$ for all t.

(ii) hazard rate order denoted by $X \leq_{hr} Y$ if $\bar{G}(t)/\bar{F}(t)$ increases in t. If X and Y are absolutely continuous, then $X \leq_{hr} Y$ is equivalent to $h_F(t) \geq h_G(t)$ for all t.

(iii) reversed hazard rate order denoted by $X \leq_{rhr} Y$ if $G(t)/F(t)$ increases in t. If X and Y are absolutely continuous, then $X \leq_{rhr} Y$ is equivalent to $\tilde{r}_F(t) \leq \tilde{r}_G(t)$ for all t.

(iv) likelihood ratio order denoted by $X \leq_{lr} Y$ if $g(t)/f(t)$ is increasing in t for which the ratio is well defined.

(v) mean residual life ordering denoted by $X \leq_{MRL} Y$ if $\int_x^\infty \bar{G}(u)du / \int_x^\infty \bar{F}(u)du$ is increasing in x.

(vi) convex transform order (denoted by $X \leqslant_c Y$) if $G^{-1}(F(x))$ is convex in x on the support of F.

(vii) star order (denoted by $X \leqslant_* Y$) if $\dfrac{G^{-1}(F(x))}{x}$ increases in $x > 0$.

(viii) supper-additive order (denoted by $X \leqslant_{su} Y$) if $G^{-1}(F(x+y)) \geq G^{-1}(F(x)) + G^{-1}(F(y)), \forall\ x \geq\ 0, y \geq\ 0$.

(ix) dispersive order (denoted by $X \leqslant_{disp} Y$) if and only if $G^{-1}(F(x)) - x$ increases in x.

2.1. Model description

Let X and Y be random variables satisfying Weibull-F model (1.1) with parameters (α_1, β_1) and (α_2, β_2), respectively, that is

$$X \sim \bar{G}_1(x; \alpha_1, \beta_1) = e^{-\alpha_1 (\frac{F(x)}{\bar{F}(x)})^{\beta_1}}, \qquad Y \sim \bar{G}_2(y; \alpha_2, \beta_2) = e^{-\alpha_2 (\frac{F(y)}{\bar{F}(y)})^{\beta_2}}, \tag{2.1}$$

where $F(.)$ is given parametric CDF with known parameter λ. Consider a system with random strength Y subjected to a random stress X. Furthermore, assume that the random variables X and Y are independent. The stress-strength reliability of the aforementioned system is defined by

$$\begin{aligned} R &= P(X < Y) \\ &= \int_0^\infty P(Y > X | X = x) g_1(x; \alpha_1, \beta_1) dx \\ &= \int_0^1 e^{-\alpha_2 \left(-\frac{1}{\alpha_1} \ln(1-u)\right)^\gamma} du, \end{aligned} \tag{2.2}$$

where $\gamma = \frac{\beta_2}{\beta_1}$. Note that, under the identical baseline distribution function of stress and strength random variables, the expression for R does not involve the parameter of the baseline distribution $F(.)$. If the baseline distribution functions of stress and strength random variables are non-identical, then the expression for R depends on the parameter of baseline distributions. For some selected values of parameters, the figures of R as function of γ, α_1, β_1, α_2, and β_2 are depicted in figures 1, 2, 3 and 4, respectively. From these figures, we observe that the stress-strength reliability is very sensitive with respect to the parameters of model. Also, we see that the stress-strength reliability is increasing in α_1 and is decreasing in α_2. Some properties of the stress-strength reliability function in (2.2) are summarized in the following results.

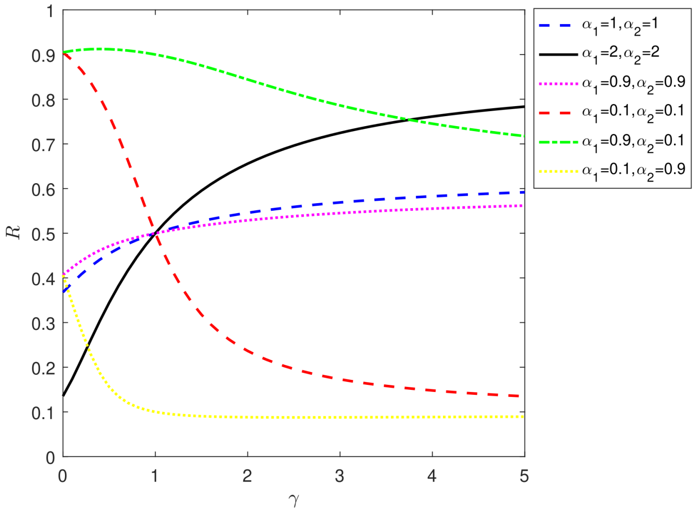

Figure 1. Plot of R versus γ in Equation (2.2).

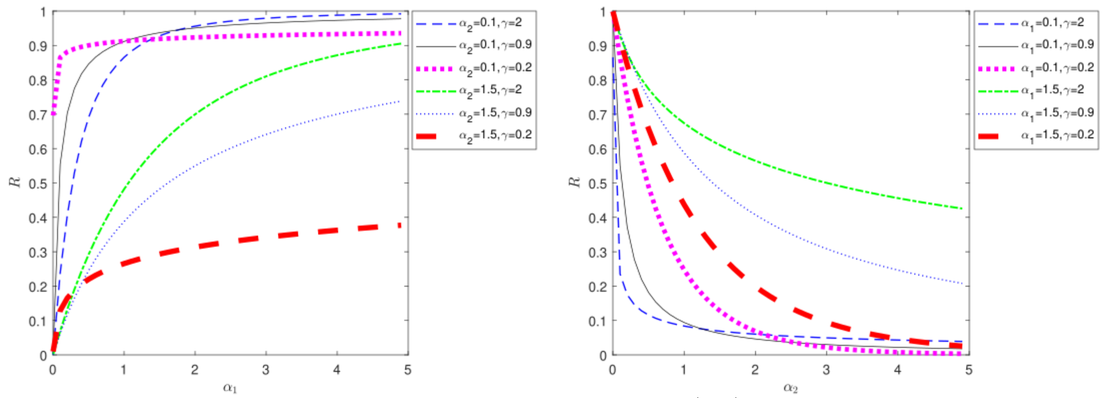

Figure 2. Plot of R versus α_1 and α_2 in Equation (2.2) in left and right panel, respectively.

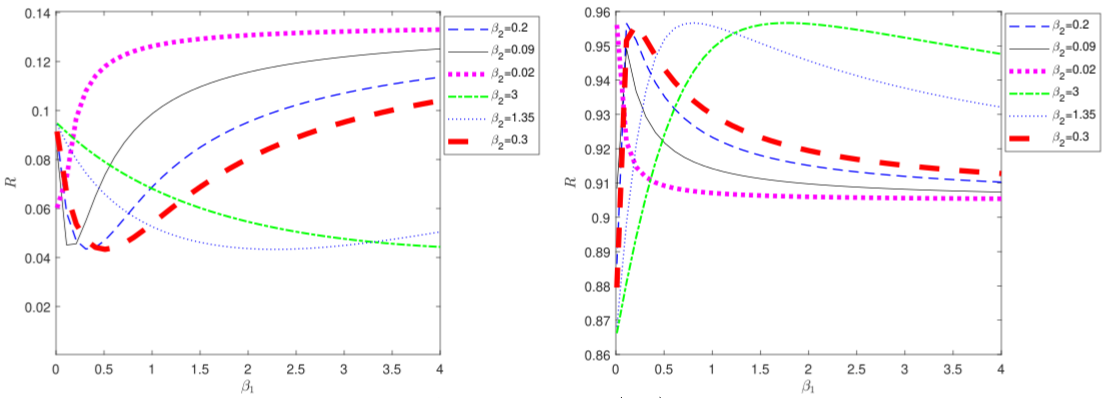

Figure 3. Plot of R versus β_1 in Equation (2.2). Left panel for $\alpha_1 = 0.1$ and $\alpha_2 = 2$ and right panel for $\alpha_1 = 2$ and $\alpha_2 = 0.1$.

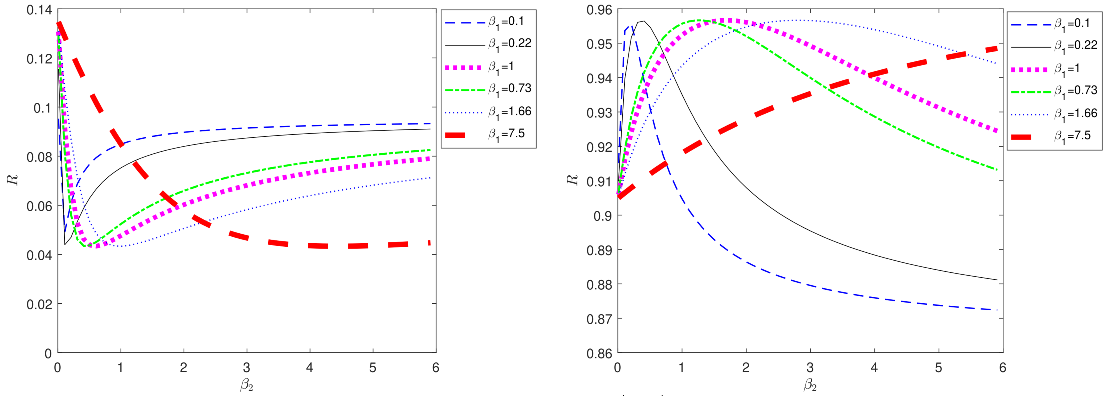

Figure 4. Plot of R versus β_2 in Equation (2.2). Left panel for $\alpha_1 = 0.1$ and $\alpha_2 = 2$ and right panel for $\alpha_1 = 2$ and $\alpha_2 = 0.1$

Result 2.2. *The stress-strength reliability function in* (2.2) *based on Weibull-F model has the following properties.*

i) *R is a continuous function of γ over $(0, \infty)$ and for $\gamma = 1$, we have $R = \frac{\alpha_1}{\alpha_1+\alpha_2}$.*
ii) *If $\alpha_1 = \alpha_2 = \alpha$ and $\alpha > 1$ then R is monotonically increasing in γ.*

In the following, for convenience study of stochastic orderings with respect to the parameters of the stress-strength model, a particular model is investigated. In this model, the baseline distribution function of the strength random variable is identical to the distribution function of random stress.

2.2. Particular model

Consider a strength system with lifetime $Y \sim W - F(\alpha, \beta, F)$ subjected to a random stress X with the CDF F. Then stress-strength reliability of the aforementioned system is given by

$$R = P(X < Y) = \int_0^1 e^{-\alpha(\frac{u}{1-u})^\beta} du. \tag{2.3}$$

The stress-strength models in generalized Weibull-G family of distributions have been investigated by researchers such as [1,2,13]. We now study the stochastic ordering between Y and X. Using relation (1.2), we have

$$\frac{g(y;\alpha,\beta)}{f(x)} = \alpha\beta \frac{F^{\beta-1}(x)}{\bar{F}^{\beta+1}(x)} e^{-\alpha(\frac{F(x)}{\bar{F}(x)})^\beta} \tag{2.4}$$

This gives

$$\frac{d}{dx}\left[\frac{g(y;\alpha,\beta)}{f(x)}\right] = -\frac{1}{F(x)\bar{F}(x)}\left(\alpha\beta(\frac{F(x)}{\bar{F}(x)})^\beta - 2F(x) - \beta + 1\right) g(y;\alpha,\beta) \tag{2.5}$$

Therefore, $\beta = 1$ and $\alpha \geq 2$ implies $Y \leq_{lr} X$.
Based on the already stated, the following results are obtained.

Result 2.3. *Suppose model (2.3) holds. Then, if $\beta = 1$ and $\alpha \geq 2$,*

i) $Y \leq_{hr} X$;
ii) $Y \leq_{rh} X$;
iii) $Y \leq_{st} X$;
iv) $Y \leq_{MRL} X$.

Next, we investigate the stochastic orderings with respect to the parameters β and α. Suppose Y_1 and Y_2 are two random variables satisfying the Weibull-F model (1.1), with parameters (α_1, β_1) and (α_2, β_2), respectively. Then

$$\frac{g_1(x;\alpha_1,\beta_1)}{g_2(x;\alpha_2,\beta_2)} = \frac{\alpha_1\beta_1}{\alpha_2\beta_2}(\frac{F(x)}{\bar{F}(x)})^{\beta_1-\beta_2} e^{-\alpha_1(\frac{F(x)}{\bar{F}(x)})^{\beta_1}+\alpha_2(\frac{F(x)}{\bar{F}(x)})^{\beta_2}} \tag{2.6}$$

and

$$\frac{d}{dx}\left[\frac{g_1(x;\alpha_1,\beta_1)}{g_2(x;\alpha_2,\beta_2)}\right] = \frac{\alpha_1\beta_1}{\alpha_2\beta_2}(\frac{F(x)}{\bar{F}(x)})^{\beta_1-\beta_2-1}\frac{f(x)}{\bar{F}(x)} e^{-\alpha_1(\frac{F(x)}{\bar{F}(x)})^{\beta_1}+\alpha_2(\frac{F(x)}{\bar{F}(x)})^{\beta_2}} \times\left(\alpha_2\beta_2(\frac{F(x)}{\bar{F}(x)})^{\beta_2} - \alpha_1\beta_1(\frac{F(x)}{\bar{F}(x)})^{\beta_1} - \beta_2 + \beta_1\right) \tag{2.7}$$

Thus $\beta_1 = \beta_2 = \beta$ and $\alpha_1 > \alpha_2$ implies $Y_1 \leq_{lr} Y_2$.
By the previously mentioned, the following results are achieved.

Result 2.4. *Let Y_1 and Y_2 be the two random variables with parameters (α_1, β_1) and (α_2, β_2) satisfying the Weibull-F model* (1.1)*. Then, if $\beta_1 = \beta_2 = \beta$ and $\alpha_1 > \alpha_2$,*

i) $Y_1 \leq_{hr} Y_2$;
ii) $Y_1 \leq_{rh} Y_2$;
iii) $Y_1 \leq_{st} Y_2$;
iv) $Y_1 \leq_{MRL} Y_2$.

The next result shows that some stochastic orders preserved under the aforementioned models.

Result 2.5. *Let X_0 and Y_0 be two non-negative random variables with CDF F_0 and G_0, respectively. Also, assume that X and Y be the two random variables with parameters (α, β) satisfying the Weibull-F_0 model and Weibull-G_0 model, respectively. Denote the CDFs of X and Y by $G_{F_0}(x)$ and $G_{G_0}(y)$, respectively. Then we have*

i) $X_0 \leqslant_c Y_0$ *if, and only if,* $X \leqslant_c Y$.
ii) $X_0 \leqslant_* Y_0$ *if, and only if,* $X \leqslant_* Y$.
iii) $X_0 \leqslant_{su} Y_0$ *if, and only if,* $X \leqslant_{su} Y$.
iv) $X_0 \leqslant_{disp} Y_0$ *if, and only if,* $X \leqslant_{disp} Y$.

Proof. By assumptions, we have

$${G_{G_0}}^{-1}(G_{F_0}(x)) = G_0^{-1}\Big(\frac{\big(-\frac{1}{\alpha}\ln(1-G_{F_0}(x))\big)^{\frac{1}{\beta}}}{1+\big(-\frac{1}{\alpha}\ln(1-G_{F_0}(x))\big)^{\frac{1}{\beta}}}\Big) = G_0^{-1}(F_0(x)). \tag{2.8}$$

The results follow from (2.8) and parts **(vi)** to **(ix)** of Definition 1. □

In the following, we examine Result 2.4 via a numerical example. Consider random variables $Y_1 \sim W - F(2, 2, 1 - e^{2y})$ and $Y_2 \sim W - F(1.5, 2, 1 - e^{2y})$ with CDFs $G_1(y)$ and $G_2(y)$, respectively. Let $k_1(y) = \frac{\bar{G}_1(y)}{\bar{G}_2(y)}$, $k_2(y) = \frac{G_1(y)}{G_2(y)}$, and $k_3(y) = \frac{\int_x^\infty \bar{G}_1(u)du}{\int_x^\infty \bar{G}_2(u)du}$. The stochastic comparisons between random variables Y_1 and Y_1 are demonstrated in Figure 5.

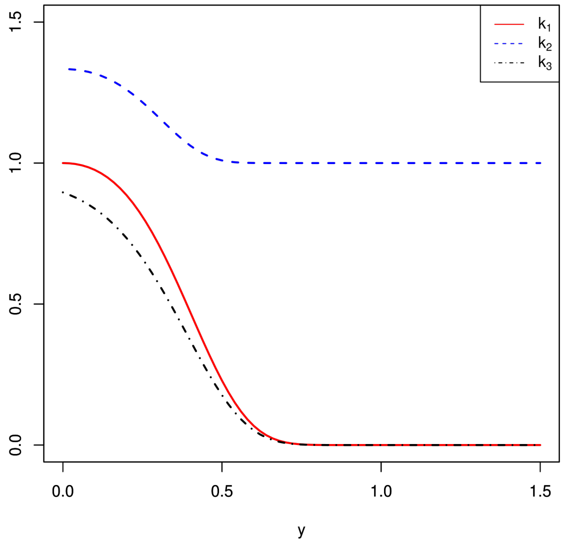

Figure 5. Plot of functions $k_1(y)$, $k_2(y)$, and $k_3(y)$.

3. ML estimation of the stress-strength reliability in Weibull-F models

Here, we will find the ML estimation of R based on the exponential distribution. We assume that F is exponential distribution with known mean $\frac{1}{\lambda}$. Therefore, the two density functions associated with (2.1) are given, respectively, by

$$X \sim g_1(x;\alpha_1,\beta_1) = \alpha_1\beta_1\lambda e^{\beta_1\lambda x}(1-e^{-\lambda x})^{\beta_1-1}e^{-\alpha_1(\frac{1-e^{-\lambda x}}{e^{-\lambda x}})^{\beta_1}}, \tag{3.1}$$

and

$$Y \sim g_2(y;\alpha_2,\beta_2) = \alpha_2\beta_2\lambda e^{\beta_2\lambda y}(1-e^{-\lambda y})^{\beta_2-1}e^{-\alpha_2(\frac{1-e^{-\lambda y}}{e^{-\lambda y}})^{\beta_2}}. \tag{3.2}$$

Let $X_1,\dots,X_{n_1}$ be a random sample of size n_1 from X with PDF in (3.1) and $Y_1,\dots,Y_{n_2}$ be a random sample of size n_2 distributed as Y with PDF in (3.2). Then, the likelihood function of the observed sample is easily provided by

$$\begin{aligned} L(\alpha_1,\beta_1,\alpha_2,\beta_2) &= (\alpha_1\beta_1)^{n_1}(\alpha_2\beta_2)^{n_2}\lambda^{n_1+n_2}e^{\lambda(\beta_1\sum_{i=1}^{n_1}x_i+\beta_2\sum_{j=1}^{n_2}y_j)}\prod_{i=1}^{n_1}(1-e^{-\lambda x_i})^{\beta_1-1} \\ &\times \prod_{j=1}^{n_2}(1-e^{-\lambda y_j})^{\beta_2-1}e^{-\alpha_1\sum_{i=1}^{n_1}\left(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\right)^{\beta_1}}e^{-\alpha_2\sum_{j=1}^{n_2}\left(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\right)^{\beta_2}}. \end{aligned} \tag{3.3}$$

The corresponding loglikelihood function is

$$\begin{aligned} \ell(\alpha_1,\beta_1,\alpha_2,\beta_2) &= n_1(\ln\alpha_1+\ln\beta_1)+n_2(\ln\alpha_2+\ln\beta_2)+(n_1+n_2)\ln\lambda \\ &+ \lambda(\beta_1\sum_{i=1}^{n_1}x_i+\beta_2\sum_{j=1}^{n_2}y_j) \\ &+ (\beta_1-1)\sum_{i=1}^{n_1}\ln(1-e^{-\lambda x_i})+(\beta_2-1)\sum_{j=1}^{n_2}\ln(1-e^{-\lambda y_j}) \\ &- \alpha_1\sum_{i=1}^{n_1}(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}})^{\beta_1}-\alpha_2\sum_{j=1}^{n_2}(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}})^{\beta_2}. \end{aligned} \tag{3.4}$$

Taking the first order partial derivatives with respect to α_1, β_1, α_2 and β_2 and setting them to zero, we get the following system of score equations

$$\begin{cases} \frac{\partial l(\theta)}{\partial \alpha_1} = \frac{n_1}{\alpha_1} - \sum\limits_{i=1}^{n_1} \left(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\right)^{\beta_1} = 0 \\ \frac{\partial l(\theta)}{\partial \alpha_2} = \frac{n_2}{\alpha_2} - \sum\limits_{j=1}^{n_2} \left(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\right)^{\beta_2} = 0 \\ \frac{\partial l(\theta)}{\partial \beta_1} = \frac{n_1}{\beta_1} + \lambda \sum\limits_{i=1}^{n_1} x_i + \sum\limits_{i=1}^{n_1} \ln\left(1-e^{-\lambda x_i}\right) - \alpha_1 \sum\limits_{i=1}^{n_1} \left(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\right)^{\beta_1} \ln\left(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\right) = 0 \\ \frac{\partial l(\theta)}{\partial \beta_2} = \frac{n_2}{\beta_2} + \lambda \sum\limits_{j=1}^{n_2} y_j + \sum\limits_{j=1}^{n_2} \ln\left(1-e^{-\lambda y_j}\right) - \alpha_2 \sum\limits_{i=1}^{n_2} \left(\frac{1-e^{-\lambda y_j i}}{e^{-\lambda y_j}}\right)^{\beta_2} \ln\left(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\right) = 0. \end{cases} \tag{3.5}$$

The MLEs of α_1, β_1, α_2 and β_2 denoted by $\hat{\alpha}_1$, $\hat{\beta}_1$, $\hat{\alpha}_2$ and $\hat{\beta}_2$, are the solutions to the above system of score equations which maximize the likelihood function (3.3). From (3.5), we attain

$$\hat{\alpha}_1(\beta_1) = \frac{n_1}{\sum\limits_{i=1}^{n_1} \left(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\right)^{\beta_1}}, \tag{3.6}$$

and

$$\hat{\alpha}_2(\beta_2) = \frac{n_2}{\sum\limits_{j=1}^{n_2} \left(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\right)^{\beta_2}}. \tag{3.7}$$

By substituting (3.6) and (3.7) in (3.5) we have

$$\begin{cases} \frac{\partial l(\theta)}{\partial \beta_1} = \frac{n_1}{\beta_1} + \lambda \sum\limits_{i=1}^{n_1} x_i + \sum\limits_{i=1}^{n_1} \ln\left(1-e^{-\lambda x_i}\right) \\ \quad - \frac{n_1}{\sum\limits_{i=1}^{n_1} \left(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\right)^{\beta_1}} \sum\limits_{i=1}^{n_1} \left(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\right)^{\beta_1} \ln\left(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\right) = 0 \\ \frac{\partial l(\theta)}{\partial \beta_2} = \frac{n_2}{\beta_2} + \lambda \sum\limits_{j=1}^{n_2} y_j + \sum\limits_{j=1}^{n_2} \ln\left(1-e^{-\lambda y_j}\right) \\ \quad - \frac{n_2}{\sum\limits_{j=1}^{n_2} \left(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\right)^{\beta_2}} \sum\limits_{i=1}^{n_2} \left(\frac{1-e^{-\lambda y_j i}}{e^{-\lambda y_j}}\right)^{\beta_2} \ln\left(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\right) = 0. \end{cases} \tag{3.8}$$

The existence and uniqueness of solutions for (3.8) are shown in Appendix.

By solving the system of non-linear equations in (3.8), $\hat{\alpha}_1$, $\hat{\beta}_1$, $\hat{\alpha}_2$ and $\hat{\beta}_2$ will be obtained. Therefore, the MLE of R can be obtained by

$$\hat{R} = \int_0^1 e^{-\hat{\alpha}_2\left(-\frac{1}{\hat{\alpha}_1}\ln(1-u)\right)^{\frac{\hat{\beta}_2}{\hat{\beta}_1}}} du. \tag{3.9}$$

In the next subsection, we will derive some asymptotic results about $\hat{R}$.

3.1. Some asymptotic results

Let $\theta = (\alpha_1, \beta_1, \alpha_2, \beta_2)^\intercal$. The Hessian is the matrix of second derivatives of the likelihood with respect to the parameters and defined by

$$\mathbf{H}(\theta) = \frac{\partial^2 l(\theta)}{\partial\theta\partial\theta^\intercal} = \begin{pmatrix} \frac{\partial^2 l(\theta)}{\partial \alpha_1^2} & \frac{\partial^2 l(\theta)}{\partial \alpha_1 \partial \beta_1} & \frac{\partial^2 l(\theta)}{\partial \alpha_1 \partial \alpha_2} & \frac{\partial^2 l(\theta)}{\partial \alpha_1 \partial \beta_2} \\ \frac{\partial^2 l(\theta)}{\partial \beta_1 \partial \alpha_1} & \frac{\partial^2 l(\theta)}{\partial \beta_1^2} & \frac{\partial^2 l(\theta)}{\partial \beta_1 \partial \alpha_2} & \frac{\partial^2 l(\theta)}{\partial \beta_1 \partial \beta_2} \\ \frac{\partial^2 l(\theta)}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 l(\theta)}{\partial \alpha_2 \partial \beta_1} & \frac{\partial^2 l(\theta)}{\partial \alpha_2^2} & \frac{\partial^2 l(\theta)}{\partial \alpha_2 \partial \beta_2} \\ \frac{\partial^2 l(\theta)}{\partial \beta_2 \partial \alpha_1} & \frac{\partial^2 l(\theta)}{\partial \beta_2 \partial \beta_1} & \frac{\partial^2 l(\theta)}{\partial \beta_2 \partial \alpha_2} & \frac{\partial^2 l(\theta)}{\partial \beta_2^2} \end{pmatrix}, \tag{3.10}$$

where

$$\frac{\partial^2 l(\theta)}{\partial\alpha_1\partial\beta_1} = -\sum_{i=1}^{n_1}\big(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\big)^{\beta_1}\ln\big(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\big),$$

$$\frac{\partial^2 l(\theta)}{\partial\beta_1^2} = -\frac{n_1}{\beta_1^2} - \alpha_1\sum_{i=1}^{n_1}\big(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\big)^{\beta_1}\big[\ln\big(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\big)\big]^2,$$

$$\frac{\partial^2 l(\theta)}{\partial\beta_1\partial\alpha_1} = -\sum_{i=1}^{n_1}\big(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\big)^{\beta_1}\ln\big(\frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}\big),$$

$$\frac{\partial^2 l(\theta)}{\partial\alpha_2\partial\beta_2} = -\sum_{j=1}^{n_2}\big(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\big)^{\beta_2}\ln\big(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\big),$$

$$\frac{\partial^2 l(\theta)}{\partial\beta_2^2} = -\frac{n_2}{\beta_2^2} - \alpha_2\sum_{j=1}^{n_2}\big(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\big)^{\beta_2}\big[\ln\big(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\big)\big]^2,$$

$$\frac{\partial^2 l(\theta)}{\partial\beta_2\partial\alpha_2} = -\sum_{j=1}^{n_2}\big(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\big)^{\beta_2}\ln\big(\frac{1-e^{-\lambda y_j}}{e^{-\lambda y_j}}\big),$$

$\frac{\partial^2 l(\theta)}{\partial\alpha_1^2} = -\frac{n_1}{\alpha_1^2}$, $\frac{\partial^2 l(\theta)}{\partial\alpha_2^2} = -\frac{n_2}{\alpha_2^2}$ and $\frac{\partial^2 l(\theta)}{\partial\alpha_1\partial\alpha_2} = \frac{\partial^2 l}{\partial\alpha_2\partial\alpha_1} = \frac{\partial^2 l}{\partial\alpha_1\partial\beta_2} = \frac{\partial^2 l}{\partial\beta_2\partial\alpha_1} = \frac{\partial^2 l(\theta)}{\partial\alpha_2\partial\beta_1} = \frac{\partial^2 l(\theta)}{\partial\beta_1\partial\alpha_2} = \frac{\partial^2 l(\theta)}{\partial\beta_1\partial\beta_2} = \frac{\partial^2 l(\theta)}{\partial\beta_2\partial\beta_1} = 0$. It can be demonstrated that the likelihood function satisfies the regularity conditions prepared in Bickel and Doksum (2001, pages 384-385). The observed Fisher information matrix can be presented as

$$\mathbb{I}_n(\hat\theta) = -H(\hat\theta) = -\frac{\partial^2 l(\theta)}{\partial\theta\partial\theta^\intercal}\Big|_{\theta=\hat\theta}. \tag{3.11}$$

Let $n = n_1 + n_2$. We define the Fisher information matrix of θ based on the Weibull-F Model as follows

$$\mathbb{I}(\theta) = \lim_{\substack{n_1\to\infty\\ n_2\to\infty}} E\Big(-\frac{1}{n}\frac{\partial^2 l(\theta)}{\partial\theta\partial\theta^\intercal}\Big) = \begin{pmatrix} \mathbb{I}_{11} & \mathbb{I}_{12} & \mathbb{I}_{13} & \mathbb{I}_{14} \\ \mathbb{I}_{21} & \mathbb{I}_{22} & \mathbb{I}_{23} & \mathbb{I}_{24} \\ \mathbb{I}_{31} & \mathbb{I}_{32} & \mathbb{I}_{33} & \mathbb{I}_{34} \\ \mathbb{I}_{41} & \mathbb{I}_{42} & \mathbb{I}_{43} & \mathbb{I}_{44} \end{pmatrix}, \tag{3.12}$$

where

$$\mathbb{I}_{11} = \frac{\rho_1}{\alpha_1^2}, \quad \mathbb{I}_{12} = \rho_1\Psi(\alpha_1,\beta_1,1), \quad \mathbb{I}_{13} = \mathbb{I}_{14} = 0,$$

$$\mathbb{I}_{12} = \beta_1\rho_1\Psi(\alpha_1,\beta_1,1), \quad \mathbb{I}_{22} = \frac{\rho_1}{\beta_1^2} + \alpha_1\rho_1\Psi(\alpha_1,\beta_1,1) + \alpha_1\beta_1\rho_1\Psi(\alpha_1,\beta_1,2), \quad \mathbb{I}_{23} = \mathbb{I}_{24} = 0,$$

$$\mathbb{I}_{31} = \mathbb{I}_{32} = 0, \quad \mathbb{I}_{33} = \frac{\rho_2}{\alpha_2^2}, \quad \mathbb{I}_{34} = \rho_2\Psi(\alpha_2,\beta_2,1),$$

$$\mathbb{I}_{41} = \mathbb{I}_{42} = 0, \quad \mathbb{I}_{43} = \beta_2\rho_2\Psi(\alpha_2,\beta_2,1), \quad \mathbb{I}_{44} = \frac{\rho_2}{\beta_2^2} + \alpha_2\rho_2\Psi(\alpha_2,\beta_2,1) + \alpha_2\beta_2\rho_2\Psi(\alpha_2,\beta_2,2),$$

and

$$\Psi(\alpha,\beta,\nu) = -\frac{1}{\alpha\beta}\int_0^1 \ln(1-u)\Big(\ln\big(-\frac{1}{\alpha_1}\ln(1-u)\big)\Big)^\nu du.$$

Applying the above mentioned notations, we obtain the following asymptotic normality of the maximum likelihood estimates $\hat\theta = (\hat{\alpha_1}, \hat\beta_1, \hat{\alpha_2}, \hat\beta_2)^\intercal$, of $\theta = (\alpha_1, \beta_1, \alpha_2, \beta_2)$.

Theorem 3.1. *If model* (1.1) *holds, the MLE* $(\hat{\alpha_1}, \hat{\beta}_1, \hat{\alpha_2}, \hat{\beta}_2)^\intercal$ *of* $(\alpha_1, \beta_1, \alpha_2, \beta_2)^\intercal$ *weakly converges to the following multivariate normal distribution:*

$$\sqrt{n}\begin{pmatrix} \hat{\alpha_1} - \alpha_1 \\ \hat{\beta}_1 - \beta_1 \\ \hat{\alpha_2} - \alpha_2 \\ \hat{\beta}_2 - \beta_2 \end{pmatrix} \xrightarrow{D} \boldsymbol{N}(\mathbf{0}, \mathbb{I}^{-1}(\theta)), \tag{3.13}$$

where $\mathbb{I}^{-1}(\theta)$ *is the inverse of the Fisher information matrix* $\mathbb{I}(\theta)$.

Since $\mathbb{I}(\theta)$ includes integrals, one has to apply numerical procedure to evaluate these integrals in order to use this asymptotic normality. Practically, it is convenient to substitute the Fisher information matrix $\mathbb{I}(\theta)$ by

$$-\frac{1}{n}H(\hat{\theta}) = -\frac{1}{n}\frac{\partial^2 l(\theta)}{\partial\theta\partial\theta^\intercal}\Big|_{\theta=\hat{\theta}}. \tag{3.14}$$

In fact, $H(\hat{\theta})$ is a consistent estimator of $\mathbb{I}(\theta)$ since

$$\lim_{\substack{n_1\to\infty \\ n_2\to\infty}} -\frac{1}{n_1+n_2}\frac{\partial^2 l(\theta)}{\partial\theta\partial\theta^\intercal} = \lim_{\substack{n_1\to\infty \\ n_2\to\infty}} -\frac{1}{n_1+n_2}\mathbb{I}_n(\theta) = \mathbb{I}(\theta). \tag{3.15}$$

To construct the asymptotic normality of R represented in (2.2), we define

$$V(\alpha_1, \beta_1, \alpha_2, \beta_2) = \Big(\frac{\partial R}{\partial \alpha_1}, \frac{\partial R}{\partial \beta_1}, \frac{\partial R}{\partial \alpha_2}, \frac{\partial R}{\partial \beta_2}\Big), \tag{3.16}$$

where

$$\frac{\partial R}{\partial \alpha_1} = \frac{\alpha_2\beta_2}{\alpha_1\beta_1}\Psi_1(\alpha_1, \beta_1, \alpha_2, \beta_2),$$
$$\frac{\partial R}{\partial \alpha_2} = -\Psi_1(\alpha_1, \beta_1, \alpha_2, \beta_2),$$
$$\frac{\partial R}{\partial \beta_1} = -\frac{\beta_2}{\beta_1^2}\Psi_2(\alpha_1, \beta_1, \alpha_2, \beta_2),$$
$$\frac{\partial R}{\partial \beta_2} = \frac{1}{\beta_1}\Psi_2(\alpha_1, \beta_1, \alpha_2, \beta_2),$$
$$\Psi_1(\alpha_1, \beta_1, \alpha_2, \beta_2) = \int_0^1 \big(-\frac{1}{\alpha_1}\ln(1-u)\big)^\gamma e^{-\alpha_2\left(-\frac{1}{\alpha_1}\ln(1-u)\right)^\gamma} du,$$

and

$$\Psi_2(\alpha_1, \beta_1, \alpha_2, \beta_2) = -\alpha_2\int_0^1 \big(-\frac{1}{\alpha_1}\ln(1-u)\big)^\gamma \ln\big(-\frac{1}{\alpha_1}\ln(1-u)\big) e^{-\alpha_2\left(-\frac{1}{\alpha_1}\ln(1-u)\right)^\gamma} du.$$

Applying the Delta method on the MLE of R, we obtain

$$\sqrt{n}\hat{R} = \sqrt{n}R + V(\alpha_1, \beta_1, \alpha_2, \beta_2)\sqrt{n}\begin{pmatrix} \hat{\alpha_1} - \alpha_1 \\ \hat{\beta}_1 - \beta_1 \\ \hat{\alpha_2} - \alpha_2 \\ \hat{\beta}_2 - \beta_2 \end{pmatrix} + o_p(1). \tag{3.17}$$

Using the results of the Theorem 3.1, we obtain the next theorem.

Theorem 3.2. *Suppose that model* (1.1) *holds, we have*

$$\sqrt{n}\Big(\hat{R} - R\Big) \xrightarrow{D} \boldsymbol{N}\bigg(\mathbf{0}, V(\alpha_1, \beta_1, \alpha_2, \beta_2)\mathbb{I}^{-1}V^\intercal(\alpha_1, \beta_1, \alpha_2, \beta_2)\bigg), \tag{3.18}$$

where $V^\intercal(\alpha_1, \beta_1, \alpha_2, \beta_2)$ *is the transpose of* $V(\alpha_1, \beta_1, \alpha_2, \beta_2)$ *and* $\mathbb{I}^{-1}$ *is the inverse of information matrix presented in* (3.12).

To build confidence intervals for R, we apply the following consistent estimated variance

$$\widehat{Var}(\hat{R}) = \frac{1}{n} V(\hat{\alpha_1}, \hat{\beta}_1, \hat{\alpha_2}, \hat{\beta}_2) \mathbb{I}^{-1} V^{\intercal}(\hat{\alpha_1}, \hat{\beta}_1, \hat{\alpha_2}, \hat{\beta}_2). \tag{3.19}$$

Applying the consistent estimated variance provided in (3.19) besides the asymptotic normal distributions in Theorem 3.2, we able to construct inferences on R by building confidence intervals.

4. Bootstrap confidence intervals

The behaviour of conclusions on R based on the asymptotic theory expanded in Section 3 is enormously dependent on the approximation of the sampling distribution of the MLE for the parameters of interest to a normal distribution. Occasionally, such a normal approximation requires a very large sample size which might be unpractised in real world problems. In this section, we investigate two bootstrap confidence intervals, i.e., bootstrap-t and bootstrap percentile confidence intervals, which only need a feasible sample size to obtain a suitable estimate of the CDF of the original populations. The bootstrap method presented by [10], is a re-sampling procedure with enormous success in solving many complex statistical issues. In this paper, the parametric and non-parametric bootstrap methods have been applied to generate random samples and based on which to build confidence intervals for the parameters of interest. Let $X_1, \ldots, X_{n_1} \sim G_1(x; \alpha_1, \beta_1)$ and $Y_1, \ldots, Y_{n_2} \sim G_2(y; \alpha_2, \beta_2)$ be the two identically independently distributed (i.i.d.) random samples. Applying the method in Section 3, we able to gain the MLE of α_1, β_1, α_2 and β_2 indicated by $\hat{\alpha}_1$, $\hat{\beta}_1$, $\hat{\alpha}_2$ and $\hat{\beta}_2$, respectively. The i.i.d. samples $X_1^*, \ldots, X_{n_1}^* \sim G_1(x; \hat{\alpha_1}, \hat{\beta}_1)$ and $Y_1^*, \ldots, Y_{n_2}^* \sim G_2(y; \hat{\alpha_2}, \hat{\beta}_2)$ are named parametric bootstrap samples. Let $\hat{G}_{1,n_1}$ and $\hat{G}_{2,n_2}$ be the empirical CDFs determined by $X_1, \ldots, X_{n_1}$ and $Y_1, \ldots, Y_{n_2}$, respectively. The simple random samples with replacement $X_1^*, \ldots, X_{n_1}^* \sim \hat{G}_{1,n_1}$ and $Y_1^*, \ldots, Y_{n_2}^* \sim \hat{G}_{2,n_2}$ are named non-parametric bootstrap samples.

4.1. Steps for constructing bootstrap estimates of parameters

The next algorithm is applied to compute the parametric and non-parametric bootstrap estimates $\hat{\alpha}_{1,b}^*$, $\hat{\beta}_{1,b}^*$, $\hat{\alpha}_{2,b}^*$, $\hat{\beta}_{2,b}^*$ and $\hat{R}_b^*$ for $b = 1, \ldots, B$. Several bootstrap confidence intervals will be obtained based on these bootstrap estimates of parameters.

Algorithm 1. Algorithm outline to compute bootstrap estimates of parameters

(1) Choose bootstrap samples with sizes n_1 and n_2 from the equivalent bootstrap populations, i.e., $X_1^*, \ldots, X_{n_1}^* \sim G_1(x; \hat{\alpha_1}, \hat{\beta}_1)$ or $\hat{G}_{1,n_1}$ and $Y_1^*, \ldots, Y_{n_2}^* \sim G_2(y; \hat{\alpha_2}, \hat{\beta}_2)$ or $\hat{G}_{2,n_2}$, respectively.

(2) Apply the method explained in Section 3 to estimate bootstrap MLEs $\hat{\alpha}_1^*$, $\hat{\beta}_1^*$, $\hat{\alpha}_2^*$ and $\hat{\beta}_2^*$ depend on $X_1^*, \ldots, X_{n_1}^*$ and $Y_1^*, \ldots, Y_{n_2}^*$ and compute the MLEs according to the following pattern

$$\hat{R}^* = \int_0^1 e^{-\hat{\alpha}_2^* \left(-\frac{1}{\hat{\alpha}_1^*} \ln(1-u)\right)^{\frac{\hat{\beta}_2^*}{\hat{\beta}_1^*}}} du. \tag{4.1}$$

(3) Repeat steps 1 and 2, B times and save the MLEs of parameters into their equivalent sets of bootstrap estimates: $\hat{\alpha}_{1,b}^*$, $\hat{\beta}_{1,b}^*$, $\hat{\alpha}_{2,b}^*$, $\hat{\beta}_{2,b}^*$ and $\hat{R}_b^*$ for $b = 1, \ldots, B$.

4.2. Kinds of bootstrap confidence intervals

In the following, two different kinds of bootstrap confidence intervals for the parameters of interest are proposed. For the sake of simplicity of display, we reduce our writing only to R. The steps of building confidence intervals of the other of the four parameters of interest α_1, β_1, α_2, and β_2 are similar to R. Suppose that $\hat{R}_b^*$ for $b = 1, \ldots, B$ be the bootstrap estimates of R. Moreover, assume that $\hat{R}$ be the MLE obtained from the original dataset, and the confidence level is considered to be $100(1-\alpha)\%$.

Bootstrap-t confidence interval

The bootstrap-t confidence interval imitates the method of building standard-t confidence intervals. Two parts of the confidence interval, i.e. t-like critical value, and the standard error of $\hat{R}$, are computed from the bootstrap estimates $\hat{R}_b^*$ for $b = 1, \ldots, B$. The bootstrap standard error is determined by

$$SE^*(\hat{R}) = \sqrt{\frac{1}{B}\sum_{b=1}^{B}(\hat{R}_b^* - \overline{\hat{R}_b^*})^2},$$

where

$$\overline{\hat{R}_b^*} = \frac{1}{B}\sum_{b=1}^{B}\hat{R}_b^*.$$

To obtain the t-like critical value, stated by $\hat{t}_\alpha^*$, we first standardize $\hat{R}_b^*$ for $b = 1, \ldots, B$ by applying

$$z_b^*(R) = \frac{\hat{R}_b^* - \hat{R}}{SE^*(\hat{R})}.$$

The t-like critical value $\hat{t}_\alpha^*$ based on the bootstrap estimate is determined as

$$\frac{\#\{z_b^*(R) \le \hat{t}_\alpha^*\}}{B} = \alpha.$$

Therefore, the bootstrap-t confidence interval can be described as $\left(\hat{R} - \hat{t}_{1-\frac{\alpha}{2}}^* SE^*(\hat{R}), \hat{R} + \hat{t}_{\frac{\alpha}{2}}^* SE^*(\hat{R})\right)$, where $\hat{t}_{1-\frac{\alpha}{2}}^*$ and $\hat{t}_{\frac{\alpha}{2}}^*$ are the the $(\frac{\alpha}{2})$-th and $(1-\frac{\alpha}{2})$-th percentile values of $z_b^*(R)$, respectively.

Bootstrap percentile confidence interval.

We need to construct a confidence interval based on the bootstrap distribution. Suppose that $\hat{H}_B^*(t) = Pr(\hat{R}_B^* \le t)$ where $\hat{H}_B^*$ is bootstrap CDF of $\hat{R}_B^*$. If the bootstrap distribution achieved by Mont Carlo simulation then we have $\hat{H}_B^*(t) = \frac{\#(\hat{R}_b^* \le t)}{B}$. Efron and Tibshirani (1993) established a $100(1-\alpha)\%$ approximate bootstrap percentile confidence interval for R as $(\hat{R}^{*(\frac{\alpha}{2})}, \hat{R}^{*(1-\frac{\alpha}{2})})$, where $\hat{R}^{*(\frac{\alpha}{2})}$ be the $\frac{\alpha}{2}$-th percentile of the distribution of $\hat{R}_B^*$.

5. A simulation study

In this section, we accomplish the simulation studies on the performance of some considerable estimators of R, established in preceding sections, based on small samples. Calculation in this paper are performed using the open source statistical computer package R (v.3.5.1) on Windows platform. We apply the following Inverse Transform Algorithm to generate random samples according to model (1.1). It be known that for any continuous CDF $G(.)$ the random variable stated by $X = G^{-1}(U)$ has distribution G, where U is a uniform random variable defined on $(0, 1)$. Note that, under (1.1) and $F(x) = 1 - e^{-\lambda x}$, we have

$$G(x; \alpha, \beta) = 1 - e^{-\alpha(\frac{1-e^{-\lambda x}}{e^{-\lambda x}})^{\beta}}. \tag{5.1}$$

Thus, the random number X represented by

$$X = G^{-1}(U) = \frac{1}{\lambda}\Big(1 + (-\frac{1}{\alpha}\ln(1-U))^{\frac{1}{\beta}}\Big). \tag{5.2}$$

Let $X \sim g_1(x; \alpha_1, \beta_1)$ and $Y \sim g_2(y; \alpha_2, \beta_2)$, as determined in (3.1) and (3.2). We first simulate 1000 random samples from $g_1(x; \alpha_1, \beta_1)$ and $g_2(y; \alpha_2, \beta_2)$, respectively. For a pair of two samples from g_1 and g_2, we can accomplish the approaches prepared in Sections 3 and 4 to gain the MLE of R along with the asymptotic confidence intervals of R. Also, some plots displaying the sampling distributions of the suggested MLE of R along with serial plots of MSEs versus the number of simulations to study the stability of the simulation outcomes are provided. Due to the parameter λ does not become visible in R, we select a constant $\lambda = 2$ all over this simulation study. The sample size is one of the main factors affecting the performance of the estimators. Like always, we also want to analyze the influence of sample size on different suggested estimators of R. In 7, we have graphed the values of MSE$(\hat{R})$ versus R, for some different values of n_1 and n_2. Figure 7 shows that the estimator ML has more error when R tends to 0.5. Furthermore, for $n_1 = n_2$ it is symmetric around the point $R = 0.5$ and departures from symmetry when $n_1 < n_2$ or $n_1 > n_2$. The MSE of estimator is increasing first, then decreasing and reaches its maximum at point $R \simeq \frac{n_1}{n_1+n_2}$.

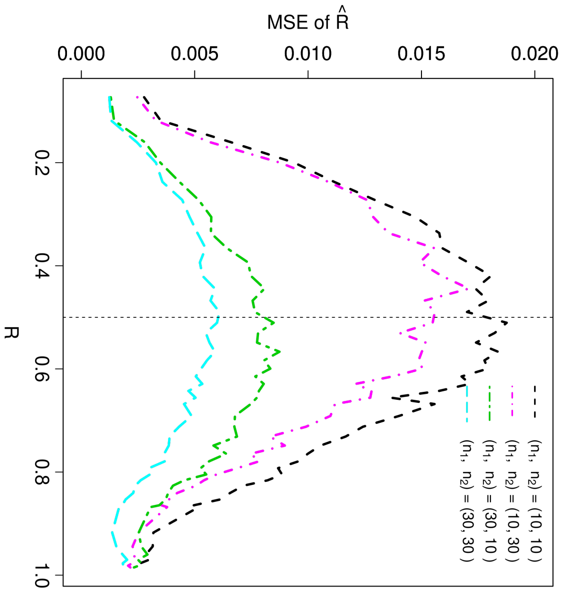

Figure 6. Plots of MSE of $\hat{R}$ versus R.

Assume that n_1 and n_2 be the sample sizes that generated from g_1 and g_2, respectively. We want to assess the performance of the suggested estimators of α_1, β_1, α_2 and β_2 with true values $\beta_1 < \beta_2$, $\beta_1 \geq \beta_2$, $\alpha_1 < \alpha_2$ and $\alpha_1 \geq \alpha_2$. To be more precise, we perform the simulation according to each of the following cases with corresponding choices of $(\alpha_1, \alpha_2, \beta_1, \beta_2, n_1, n_2) = (1, 4, 2, 5, 10, 10)$, $(1, 4, 2, 5, 10, 30)$, $(1, 4, 2, 5, 30, 10)$, (4, 1, 5, 2, 10, 10), (4, 1, 5, 2, 10, 30), (4, 1, 5, 2, 30, 10), (2, 2, 2, 2, 10, 10), (2, 2, 2, 2, 10, 30), (2, 2, 2, 2, 30, 10). From Figures 7 and 8, we can see that the simulated MSEs of $\hat{R}$ under different choices of $(\alpha_1, \alpha_2, \beta_1, \beta_2, n_1, n_2)$ become stable when the number of simulations reaches about 395. As could be expected, the MSEs showed themselves to be smaller for a bigger sample size.

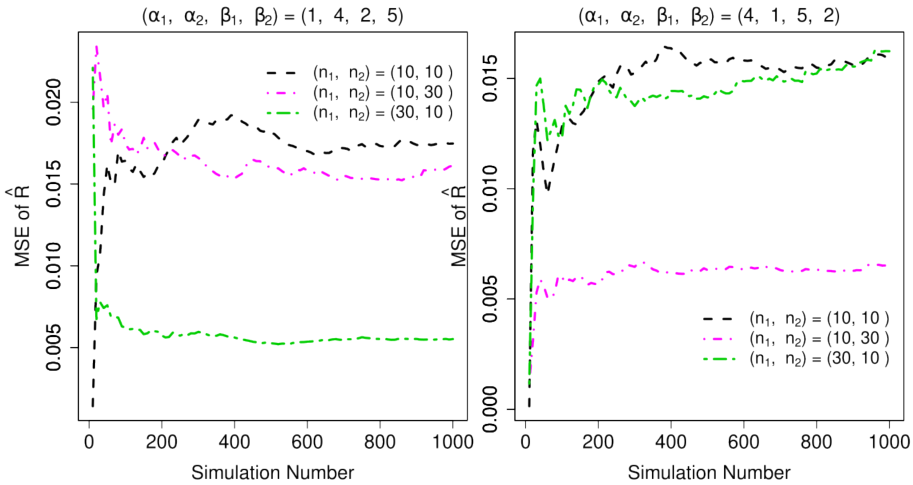

Figure 7. Plots of MSE of $\hat{R}$ versus the number of simulations.

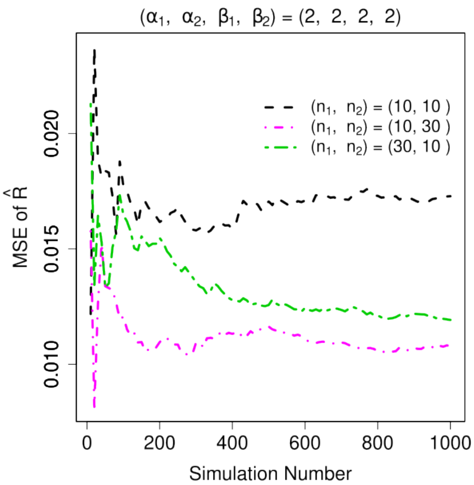

Figure 8. Plots of MSE of $\hat{R}$ versus the number of simulations.

In Figures 9 and 10, we show the sampling distributions of MLE for $\hat{R}$ based on simulation using different values of population parameters and sample sizes. It can be observed that for $(\alpha_1, \alpha_2, \beta_1, \beta_2, n_1, n_2) = (1, 4, 2, 5, 10, 10)$, $(1, 4, 2, 5, 10, 30)$, $(1, 4, 2, 5, 30, 10)$, $(4, 1, 5, 2, 10, 10)$, $(4, 1, 5, 2, 10, 30)$, $(4, 1, 5, 2, 30, 10)$ the sampling distribution of $\hat{R}$ are skewed. Also, for $(\alpha_1, \alpha_2, \beta_1, \beta_2, n_1, n_2) = (2, 2, 2, 2, 10, 10)$, $(2, 2, 2, 2, 10, 30)$, $(2, 2, 2, 2, 30, 10)$ the sampling distributions of $\hat{R}$ are approximately symmetric.

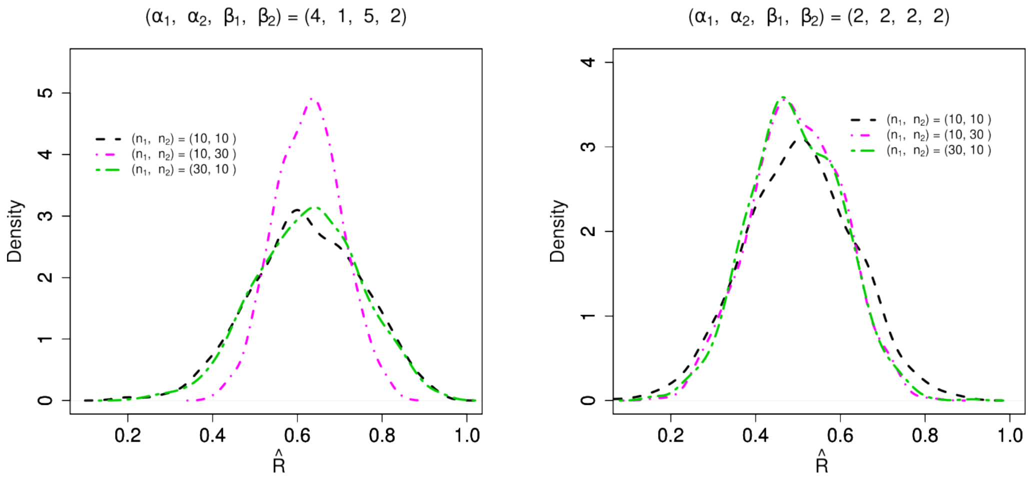

Figure 9. Sampling distribution of $\hat{R}$.

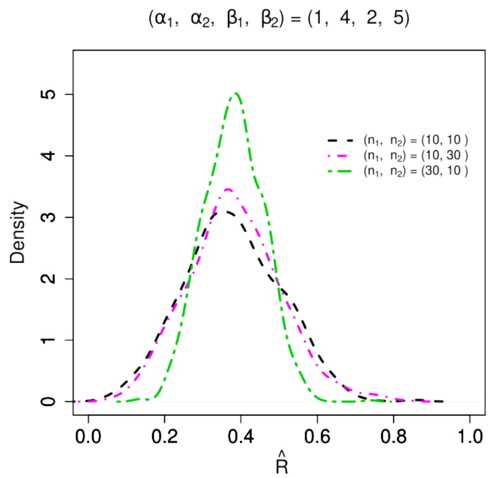

Figure 10. Sampling distribution of $\hat{R}$.

To obtain and make a comparison between the different bootstrap confidence intervals, we select 1000 parametric and non-parametric bootstrap samples using the method prepared in Section 4 from each of the simulations and find both parametric and non-parametric bootstrap-t, bootstrap-q confidence intervals. In Table 1, for all combinations of $n_1 = 10, 30$, $n_2 = 10, 30$ and $R = 0.153, 0.385, 0.500, 0.615, 0.847$, we report the average length (AL) and the coverage proportions (CP) of asymptotic, parametric and non-parametric bootstrap confidence intervals which include the true value of the corresponding parameter. In this table, CP.A. and AL.A. stand for CP and AL of asymptotic confidence intervals, respectively, P.CP.P and P.AL.P represent the CP and AL of parametric bootstrap percentile confidence intervals, respectively, P.CP.B.t and P.AL.B.t imply the CP and AL of parametric bootstrap-t confidence intervals, respectively, N-P.CP.P and N-P.AL.P represent the CP and AL of non-parametric bootstrap percentile confidence intervals, respectively, N-P.CP.B.t and N-P.AL.B.t stand for CP and AL of non-parametric bootstrap-t confidence intervals, respectively.

Table 1. The values of CP and AL of the aforementioned asymptotic, parametric and non-parametric confidence intervals for R.

R	n_1	n_2	CP.A.	AL.A.	P.CP.P	P.AL.P	P.CP.B.t	P.AL.B.t	N-P.CP.P	N-P.AL.P	N-P.CP.B.t	N-P.AL.B.t
0.153	10	10	0.998	0.294	0.997	0.299	0.942	0.209	0.997	0.284	0.938	0.277
0.153	30	10	0.831	0.344	1.000	0.386	0.875	0.267	0.990	0.376	0.873	0.220
0.153	10	30	0.991	0.299	0.986	0.288	0.960	0.271	0.999	0.285	0.967	0.184
0.153	30	30	0.956	0.241	1.000	0.176	0.955	0.154	0.995	0.172	0.950	0.156
0.385	10	10	0.996	0.331	1.000	0.323	0.962	0.290	0.997	0.323	0.967	0.243
0.385	30	10	0.899	0.292	0.988	0.293	0.912	0.243	0.980	0.308	0.910	0.241
0.385	10	30	0.955	0.357	0.995	0.373	0.962	0.264	0.992	0.378	0.967	0.222
0.385	30	30	0.970	0.272	0.988	0.264	0.912	0.241	0.990	0.265	0.933	0.206
0.500	10	10	0.899	0.368	1.000	0.429	0.888	0.451	0.990	0.401	0.990	0.483
0.500	30	10	0.998	0.363	1.000	0.464	0.938	0.496	0.997	0.462	0.967	0.499
0.500	10	30	0.899	0.362	0.988	0.375	0.900	0.415	0.990	0.389	0.899	0.470
0.500	30	30	0.999	0.393	1.000	0.399	0.938	0.412	0.994	0.376	0.933	0.451
0.615	10	10	0.995	0.369	0.988	0.243	0.960	0.378	0.983	0.270	0.967	0.381
0.615	30	10	0.994	0.324	1.000	0.276	0.952	0.392	0.990	0.342	0.967	0.386
0.615	10	30	0.995	0.331	1.000	0.244	0.950	0.343	0.994	0.348	0.967	0.348
0.615	30	30	0.998	0.346	0.988	0.219	0.965	0.370	0.980	0.259	0.967	0.379
0.847	10	10	0.998	0.286	0.100	0.252	0.938	0.269	0.999	0.241	0.933	0.257
0.847	30	10	0.991	0.277	0.978	0.223	0.968	0.225	0.980	0.221	0.967	0.267
0.847	10	30	0.899	0.239	1.000	0.232	0.962	0.246	0.999	0.259	0.958	0.272
0.847	30	30	0.994	0.219	0.995	0.179	0.955	0.166	0.990	0.178	0.957	0.250

From Table 1, by an empirical evidence, it is observed that the AL approximately reduces by raising the sample size and the maximum of the AL takes place at the middle point $R = 0.5$. Also, it appears that the CP of bootstrap percentile confidence intervals is greater than everyone else. But in the viewpoint of AL for $R < 0.5$ and $R > 0.5$, the bootstrap-t confidence intervals and the bootstrap percentile confidence intervals are appropriate, respectively. It seems that, in this simulation study, there is no meaningful difference between parametric and non-parametric confidence intervals from the point of view length and coverage proportion of the intervals.

5.1. An illustrative example

In this section, we suggest a numerical example based on a real-life dataset to illustrate the performance of the considered procedure. The datasets that have been used in this article represent the waiting times (in minutes) before customer service in two different banks. The datasets can be found in Table 11 and Table 12 of [4]. These data sets by way of Lindley distribution assessed by [4]. We are attracted to estimating the stress-strength parameter $R = P(X < Y)$ where X and Y denote the customer service time in bank I and II, respectively. First, we test to see whether the Weibull-Exponential distribution is appropriate to fit these data sets or not.

For modeling the data via the Weibull-Exponential family, we use the mpsweibullextg(...) command in the MPS R Package [22]. One of the outputs of this command is the p-value of Chi-square goodness-of-fit tests based on the maximum product spacing approach with Moran's log spacing statistic. It should be mentioned that this test is not a classical Chi-square test. For more details about this test, one can see [9]. Also, the first output of this command is the estimated parameters vector which is obtained with the maximum product spacing approach. The performance of the maximum product spacing approach is demonstrated in [22] for three sets of real data. For more details, see Page 17 of [22].

It would appear that W(1.1978,0.0738, $1-e^{-0.0103x}$) and W(1.1392,0.0624,$1-e^{-0.0117y}$) are totally good to fit data set I and data set II, respectively. For computing the p-value, we applied the command mpsweibullextg(...) in MPS R Package [23]. In this package, the significance level for the aforementioned goodness-of-fit test is reported. The corresponding p-values of aforementioned goodness-of-fit tests for bank I and II are 0.8040 and 0.8787, respectively.

Table 2. The values AL of asymptotic and various parametric and non-parametric bootstrap confidence intervals of R based on real dataset.

α	AL.A.	P .AL.Perc	P .AL.Boot.t	Non- P .AL.Perc	Non-P .AL.Boot.t
0.05	0.1691(0.5941,0.7632)	0.1380(0.5841,0.7221)	0.1871(0.5541,0.7412)	0.2229(0.5256,0.7485)	0.2900(0.5021,0.7921)
0.1	0.1506(0.5990,0.7496)	0.1262(0.5920,0.7182)	0.1808(0.5690,0.7498)	0.1902(0.5421,0.7323)	0.2575(0.5256,0.7831)

From Table 2, it appears that, there is a significant difference between bootstrap percentile, bootstrap-t, and asymptotic confidence intervals in terms of AL. The Al of the bootstrap percentile confidence interval is less than others. Therefore, the use of the bootstrap percentile confidence interval for R is recommended. Also, all confidence intervals for R don't contain the value 0.5 implying that there is a significant difference in the waiting times (in minutes) before customer service in the aforementioned two banks.

6. Summary and conclusion

In this paper, the stress-strength reliability R associated with the two populations of Weibull-F Models is investigated. First of all by using the properties of the populations parameters and functional form of Weibull-F Models the stochastic orders between random variables X and Y are assessed and the exact expression for $R = P(Y > X)$ is obtained. The asymptotic distribution of the MLE of R is determined based on exponential baseline distribution functions. Furthermore, the parametric and non-parametric bootstrap-t and bootstrap quantile confidence intervals for R are proposed. Their performances on some sample sizes with respect to AL and CP are analyzed by using a simulation study. In the view point of CP, the bootstrap percentile confidence is suitable among the other confidence intervals in the simulation study. A numerical example based on real-life data was taken to demonstrate the performance of the recommended approaches. In the real-life data, we suggest the use of bootstrap percentile confidence interval for R which is also confirmed the results of the simulation study. In the case that the baseline distribution functions of the stress and strength random variables are different, the calculations related to the computations of the estimators become longer and more complicated. On the other hand, the likelihood function has more parameters, which makes solving this system of equations more difficult than the case considered in the article. But in the case of different baseline distribution functions, the solution of the problem is the same as the one described in this article, with the difference that it has longer and more complicated calculations.

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References

[1] F.G. Akgul, B. Senoglu and S. Acitas, *Interval estimation of the system reliability for Weibull distribution based on ranked set sampling data*, Hacettepe J. Math. Stat. **47** (5), 1404–1416, 2018.

[2] M. Alizadeh, M. Rasekhi, H.M. Yousof and G.G. Hamedani, *The transmuted Weibull-G family of distributions*, Hacettepe J. Math. Stat. **47** (6), 1671–1689, 2018.

[3] A.M. Almarashi, A. Algarni and M. Nassar,*On estimation procedures of stress-strength reliability for Weibull distribution with application*, PLoS ONE **15** (8), 2020.

[4] D.K. Al-Mutairi, M.E. Ghitany and D. Kundu, *Inferences on stress-strength reliability from Lindley distributions*, Commun. Stat. - Theory Methods **42** (8), 1443–1463, 2013.

[5] X. Bai, Y. Shi, Y. Liu and B. Liu, *Reliability estimation of stress-strength model using finite mixture distributions under progressively interval censoring*, J. Comput. Appl. Math. **348**, 509–524, 2019.

[6] N. Balakrishnan and M. Kateri,*On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data*, Stat. Probab. Lett. **78** (17), 2971–2975, 2008.

[7] M. Basikhasteh, F. Lak and M. Afshari, *Bayesian estimation of stress-strength reliability for two-parameter bathtub-shaped lifetime distribution based on maximum ranked set sampling with unequal samples*, J Stat Comput Simul **90** (16), 2975–2990, 2020.

[8] M. Bourguignon, R.B. Silva and G.M. Cordeiro, *The Weibull-G family of probability distributions*, Data Sci. J. **12** (1), 53–68, 2014.

[9] R.C.H. Cheng and M.A. Stephens,*A goodness-of-fit test using Morans statistic with estimated parameters*, Biometrika **76** (2), 385–392, 1989.

[10] B. Efron, *Bootstrap methods: Another look at the jackknife*, Ann. Stat. **7** (1), 1–26, 1979.

[11] A.S. Hassan, H.F. Nagy, H.Z. Muhammed and M.S. Saad, *Estimation of multicomponent stress-strength reliability following Weibull distribution based on upper record values*, J. Taibah Univ. Sci. **14** (1), 244–253, 2020.

[12] J.K. Jose, *Estimation of stress-strength reliability using discrete phase type distribution*, Commun. Stat. - Theory Methods **51** (2),368–386, 2022.

[13] M. Jovanović, B. Milošević, M. Obradović and Z. Vidović, *Inference on reliability of stress-strength model with Peng-Yan extended Weibull distributions*, Filomat **35** (6), 1927–1948, 2021.

[14] A. Khalifeh, E. Mahmoudi and A. Chaturvedi, *Sequential fixed-accuracy confidence intervals for the stress-strength reliability parameter for the exponential distribution: two-stage sampling procedure*, Comput. Stat. **35** (4), 1553–1575, 2020.

[15] S. Kotz, Y. Lumelskii and M. Pensky, *The Stress-Strength Model and its Generalizations: Theory and Applications*, World Scientific, Singapore, 2003.

[16] Z. Pakdaman and J. Ahmadi, *Point estimation of the stress-strength reliability parameter for parallel system with independent and non-identical components*, Commun. Stat. Simul. Comput. **47** (4), 1193–1203, 2018.

[17] Z. Pakdaman and J. Ahmadi, *Stress-strength reliability for $P(X_{r:n_1} \leq Y_{k:n_2})$ in the exponential case*, İstatistik Türk İstatistik Derneği Dergisi **6** (3), 92–102, 2013.

[18] M. Shaked and J. Shanthikumar, *Stochastic Orders*, Springer, New York, 2007.

[19] K.C. Siju, M. Kumar and M. Beer, *Classical and Bayesian estimation of stress-strength reliability of a component having multiple states*, Int. J. Qual. Reliab. Manag. **38** (2), 528–535, 2020.

[20] G. Srinivasa Rao, M. Aslam and O.H. Arif, *Estimation of reliability in multicomponent stress-strength based on two parameter exponentiated Weibull Distribution*, Commun. Stat. - Theory Methods **46** (15), 7495–7502, 2017.

[21] G. Srinivasa Rao, S. Mbwambo and A. Pak, *Estimation of multicomponent stress-strength reliability from exponentiated inverse Rayleigh distribution*, Int. j. stat. manag. syst. **24** (3), 499–519, 2021.

[22] M. Teimouri, MPS: an R package for modelling new families of distributions. arXiv preprint arXiv:1809.02959, 2018.

[23] M. Teimouri and S. Nadarajah, MPS: Estimating Through the Maximum Product Spacing Approach, R package version 2.3.1, URL https://CRAN.R project.org/package=MPS, 2019.

[24] L. Wang, K. Wu, Y.M. Tripathi and C. Lodhi, *Reliability analysis of multicomponent stress-strength reliability from a bathtub-shaped distribution*, Appl. Stat. **49** (1), 122–142, 2022.

[25] K. Zografos and N. Balakrishnan, *On families of beta-and generalized gamma-generated distributions and associated inference*, Stat. Methodol. **6** (4), 344–362, 2009.

Appendix

In this Appendix, the existence and uniqueness of the maximum likelihood estimator of parameters have been considered.

From Equation 3.8, we have

$$\frac{1}{\beta_1} = -\lambda\sum_{i=1}^{n_1} x_i - \sum_{i=1}^{n_1} ln\left(1-e^{-\lambda x_i}\right) + \frac{n_1\sum_{i=1}^{n_1} {A_i}^{\beta_1}\ln A_i}{\sum_{i=1}^{n_1} {A_i}^{\beta_1}}, \tag{6.1}$$

where,
$A_i = \frac{1-e^{-\lambda x_i}}{e^{-\lambda x_i}}$.

We denote the right hand side of (6.1) by $H_1(\beta_1;\boldsymbol{x})$ and show that $H_1(\beta_1;\boldsymbol{x})$ is monotone increasing function of β_1 with a finite and positive limit as $\beta_1 \to \infty$. Since $\frac{1}{\beta_1}$ is strictly decreasing with right limit $+\infty$ at 0, it would then follow that the plots of $\frac{1}{\beta_1}$ and $H_1(\beta_1;\boldsymbol{x})$ would intersect exactly once, at the MLE of β.

Therefore, we have

$$\frac{\partial H_1(\beta_1;\boldsymbol{x})}{\partial\beta_1} = \frac{\left[n_1\sum_{i=1}^{n_1} {A_i}^{\beta_1}(\ln(A_i))^2\right]\left(\sum_{i=1}^{n_1} {A_i}^{\beta_1}\right) - \left(\sum_{i=1}^{n_1} {A_i}^{\beta_1}\ln A_i\right)^2}{\left(\sum_{i=1}^{n_1} {A_i}^{\beta_1}\right)^2}.$$

Setting $a_i = A_i^{\frac{\beta_1}{2}}$ and $b_i = A_i^{\frac{\beta_1}{2}}\ln A_i$ for $i = 1\cdots n$, becomes

$$\frac{\partial H_1(\beta_1;\boldsymbol{x})}{\partial\beta_1} = \frac{\left[\sum_{i=1}^{n_1} a_i^2\right]\left(\sum_{i=1}^{n_1} b_i^2\right) - \left(\sum_{i=1}^{n_1} a_i b_i\right)^2}{\left(\sum_{i=1}^{n_1} a_i^2\right)^2}.$$

Therefore, by Cauchy-Schwarz inequality, we have $\frac{\partial H_1(\beta_1;\boldsymbol{x})}{\partial\beta_1} \geq 0$. It should be noted that there exists a finite upper limit for $H_1(\beta_1, x)$, namely,

$$\lim_{\beta_1\to+\infty} H_1(\beta_1;\boldsymbol{x}) = -\lambda\sum_{i=1}^{n_1} x_i - \sum_{i=1}^{n_1} ln\left(1-e^{-\lambda x_i}\right) + n_1\ln\left(\frac{1-e^{-\lambda x_{(n)}}}{e^{-\lambda x_{(n)}}}\right).$$

Based on what was said above, the existence of the MLE of the parameters α_1, β_1 are shown.

Similarly, we have the same approach for the existence of the MLE of the parameters α_2, β_2. For the uniqueness of the MLE of the parameters, due to the similarity of likelihood Equation (3.8) with likelihood Equation (2.1) of [6], we have the same approach which is not given here due to similarity.