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Research Article

Fixed points of generalized hybrid mappings in 2-uniformly convex hyperbolic metric spaces

Safeer Hussain KHAN[1*], Amna KALSOOM[2], Maliha RASHID[2], Lariab SHAHID[2]

[1]Government College University, Abdus Salam School of Mathematical Sciences, Lahore, Pakistan
[2]Department of Mathematics & Statistics, International Islamic University, H-10, Islamabad, Pakistan

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ABSTRACT

In this manuscript, we introduce generalized hybrid mappings in hyperbolic metric spaces. We first show that the set of fixed points of such mappings is closed and convex. We then prove the existence of fixed point of these mappings and finally approximate the fixed point using Mann iterative process. We have also provided an example of generalized hybrid mappings in the setting of -trees for the first time in the literature.

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INTRODUCTION

Fixed point theorems for several types of mappings have already been proven and developed in different spaces by mathematicians. For instance, Takahashi [1] have contributed a lot to the study of nonexpansive mappings, nonspreading mappings [2] and hybrid mappings [3]. Takahashi et al. [4] then introduced a broader class of mappings known as generalized hybrid mappings which contains all the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings. They proved fixed point results for such mappings in Hilbert spaces. The idea of generalized hybrid mappings has then been extended to (α, β)-generalized hybrid mappings [5], $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mappings [6] and normal generalized hybrid mappings [7]. Lin et al. [8] extended the idea of generalized hybrid mappings to *CAT* (0) spaces and approximated fixed points results through iterations.

In the present article, we extend the idea of generalized hybrid mappings for 2-uniformly convex hyperbolic metric spaces. We first show that the fixed point set of these mappings is closed and convex. Secondly, we prove the existence of fixed points of these mappings. Thirdly, we approximate these fixed points using Mann iteration process. Finally, we provide a non-trivial example of generalized hybrid mappings in the setting of -trees.

***Corresponding author.**
*E-mail address: safeerhussain5@yahoo.com; safeer@sms.edu.pk

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PRELIMINARIES

Following concepts will be needed in the sequel.

Definition 1: [9] Let (M, d) be a metric space. Suppose that there exists a family F of metric segments such that any two points x, y in M are end points of a unique metric segment $[x,y] \in F([x,y]$ is an isometric image of the real line interval $[0,d(x,y)])$. Denote the unique point z of $[x, y]$ by $\beta x \oplus (1-\beta) y$, which satisfies

$$d(x,z) = (1-\beta)d(x,y) \text{ and } d(z,y) = \beta d(x,y)$$

where $\beta \in [0,1]$.

Such metric spaces are known as convex metric spaces [10].

Moreover, if we have

$$d(\alpha p \oplus (1-\alpha)x, \alpha q \oplus (1-\alpha)y) \le \alpha d(p,q) + (1-\alpha)d(x,y)$$

for all $p,q,x,y \in M$ and $\alpha \in [0,1]$, then M is said to be a hyperbolic metric space [11]

A subset C of a hyperbolic metric space M is convex if $[x,y] \subset C$ whenever x, y are in C.

Definition 2: [9] Let (M, d) be a hyperbolic metric space. M is uniformly convex if for any $a \in M$ for every $r > 0$ and for each $\varepsilon > 0$,

$$\delta(r,\varepsilon) = inf\left\{ \begin{array}{l} 1 - \frac{1}{r}d(\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right); d(x,a) \\ \le r, d(y,a) \le r, d(x,y) \ge r\varepsilon \end{array} \right\} > 0.$$

where infimum is taken all over $x, y \in M$.

From now onwards, we assume that M is a hyperbolic metric space.

Theorem 1: [11] Let (M, d) be uniformly convex. Fix $a \in M$. For each $r > 0$, $\varepsilon > 0$, denote

$$\Psi(r,\varepsilon) = inf\left\{ \frac{1}{2}d^2(a,x) + \frac{1}{2}d^2(a,y) - d^2\left(a, \frac{1}{2}x \oplus \frac{1}{2}y\right) \right\},$$

where infimum is taken all over $x, y \in M$ such that $d(a, x) \le r$, $d(a, y) \le r$ and $d(x, y) \le r\varepsilon$. Then $\Psi(r, \varepsilon) > 0$ for any $r > 0$ and for each $\varepsilon > 0$. Moreover for a fixed $r > 0$, we have

(i) $\Psi(r, 0) = 0$;

(ii) $\Psi(r, \varepsilon)$ is nondecreasing function of ε;

(iii) If $\lim_{n\to\infty} \Psi(r,t_n) = 0$, then $\lim_{n\to\infty} t_n = 0$.

The concept of p-uniform convexity was used extensively by Xu [12]. Its nonlinear version for $p = 2$ has been introduced by Khamsi and Khan [11] using the above function Ψ as follows.

Definition 3: The space (M, d) *is* 2-uniformly convex if

$$c_M = inf\left\{ \frac{\Psi(r,\varepsilon)}{r^2\varepsilon^2} \ : \ r,\varepsilon > 0 \right\}.$$

From the definition of C_M, the following inequality is obtained:

$$d^2\left(a, \frac{1}{2}x \oplus \frac{1}{2}y\right) + c_M d^2(x,y) \le \frac{1}{2}d^2(a,x) + \frac{1}{2}d^2(a,y),$$

for any $a, x, y \in M$.

Theorem 2: [9] Assume that (M, d) *is* 2-uniformly convex. Then for any $\alpha \in [0,1]$, there exists $C_M > 0$ such that

$$d^2(a, \alpha x \oplus (1-\alpha)y) + C_M min(\alpha^2, 1-\alpha^2)) \\ d^2(x,y) \le \alpha d^2(a,x) + (1-\alpha)d^2(a,y)$$

for any $a, x, y \in M$.

Theorem 3: [13] Let (M, d) be complete and uniformly convex and C be a nonempty, closed, bounded and convex subset of M. Let τ be a type defined on C. *i.e.,* $\tau: M \to R_+$, if there exists $\{x_n\}$ in C such that $\tau(x) = \lim_{n\to\infty} d(x, x_n)$. Then any minimizing sequence of τ is convergent and its limit is independent of the minimizing sequence.

Lin [8] gave the definition of generalized hybrid mappings in CAT(0) spaces. We now extend the idea of generalized hybrid mapping to hyperbolic metric spaces as follows.

Definition 4: Let C be nonempty subset of a 2-uniformly convex hyperbolic metric space M. We say that $T: C \to M$ is a generalized hybrid mapping if there are functions a_1, a_2, a_3, k_1, k_2: $C \to [0,1)$ such that

i) $$d^2(Tx,Ty) \le \left[\begin{array}{c} a_1(x)d^2(x,y) + a_2(x)d^2(Tx,y) \\ + a_3(x)d^2(Ty,x) + k_1(x)d^2(Tx,x) \\ + k_2(x)d^2(Ty,y) \end{array} \right]$$

for all $x, y \in C$.

ii) $a_1(x) + a_2(x) + a_3(x) \le 1$ *for all* $x \in C$.

iii) $2k_1(x) < 1 - a_2(x)$ *and* $k_2(x) < 1 - a_3(x)$ *for all* $x \in C$.

And here we provide a nontrivial example of generalized hybrid mappings in the setting of $\mathbb{R}$-trees. Moreover, we show that a generalized hybrid mapping is not necessarily a nonexpansive mapping.

Example 1: Consider $\mathbb{R}^2$ with usual Euclidean metric $d(.,.)$ and $\|\cdot\|$ defined by

$$d(x,y) = \|x-y\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Define the radial metric d_r by

$$d_r(x,y) = \begin{cases} d(x,y), & \text{if } y = tx \text{ for some } t \in \mathbb{R} \\ d(x,0) + d(y,0), & \text{otherwise.} \end{cases}$$

Then $X := (\mathbb{R}^2, d_r)$ is an $\mathbb{R}$-tree with radial meter d_r [14]. Consider

$$C=\left\{\begin{array}{l}(\alpha,0):\\ \alpha\in[0,2]\cup\left[4,5\frac{1}{2}\right]\end{array}\right\}\cup\left\{\begin{array}{l}(0,\alpha):\\ \alpha\in[0,2]\cup\left[4,5\frac{1}{2}\right]\end{array}\right\}\subset\mathbb{R}^2.$$

$$a_1(x)=a_2(x)=a_3(x)=\frac{1}{5},\ k_1(x)=\frac{1}{4},k_2(x)=\frac{3}{4}.$$

Define a mapping $T: C\to C$ by

$$T(\alpha,0)=\begin{cases}(0,0) & \alpha\in[0,2]\\ (0,\alpha) & \alpha\in\left[4,5\frac{1}{2}\right]\end{cases} \text{ and}$$

$$T(0,\alpha)=\begin{cases}(0,0) & \alpha\in[0,2]\\ (\alpha,0) & \alpha\in\left[4,5\frac{1}{2}\right]\end{cases}.$$

Then we show that T is a generalized hybrid mapping.

Case 1: $x = (\alpha, 0)$, $y = (\beta, 0)$.

(i) *If* $\alpha, \beta \in [0,2]$, *then* $Tx = (0,0)$, $Ty = (0,0)$. This gives

$$d_r^2(Tx,Ty)=d_r^2((0,0),(0,0))=0$$

and hence,

$$d_r^2(Tx,Ty)\le\frac{1}{5}d^2(x,y)+\frac{1}{5}d_r^2(Tx,y)+\frac{1}{5}d_r^2(Ty,x)+\frac{1}{4}d_r^2(Tx,x)+\frac{3}{4}d_r^2(Ty,y)$$

is true as

$$0\le\frac{(\alpha-\beta)^2}{5}+\frac{\alpha^2}{5}+\frac{\beta^2}{5}+\frac{\alpha^2}{4}+\frac{3\alpha^2}{4}.$$

(ii) If $\alpha,\beta\in\left[4,5\frac{1}{2}\right]$, then $Tx = (0, \alpha)$, $Ty = (0, \beta)$ and hence

$$\begin{aligned}d_r^2(Tx,Ty)&=(\alpha-\beta)^2\\ &\le\frac{1}{5}(\alpha-\beta)^2+\frac{\beta^2+\alpha^2}{5}+\frac{\beta^2+\alpha^2}{5}+\frac{\alpha^2}{2}+\frac{3\beta^2}{2}\\ &=\frac{1}{5}d_r^2(x,y)+\frac{1}{5}d_r^2(x,y)+\frac{1}{5}d_r^2(Ty,x)+\frac{1}{4}d_r^2(Tx,x)\\ &\quad+\frac{3}{4}d_r^2(Ty,y)\end{aligned}$$

(iii) If $\alpha\in[0,2],\beta\in\left[4,5\frac{1}{2}\right]$, then $Tx = (0, 0)$, $Ty = (0, \beta)$ and hence

$$\beta^2\le\frac{(\alpha-\beta)^2}{5}+\frac{\beta^2}{5}+\frac{\alpha^2+\beta^2}{5}+\frac{\alpha^2}{4}+\frac{3\beta^2}{2}$$

(iv) If $\alpha\in\left[4,5\frac{1}{2}\right],\beta\in[0,2]$ then $Tx = (0, \alpha)$, $Ty = (0, 0)$ and hence

$$\alpha^2\le\frac{(\alpha-\beta)^2}{5}+\frac{\alpha^2}{5}+\frac{\alpha^2}{5}+\frac{\alpha^2}{2}+\frac{3}{4}\beta^2$$

Case 2: $x = (\alpha, 0)$, $y = (0, \beta)$.

(i) If $\alpha, \beta \in [0,2]$ then $Tx = (0, 0)$, $Ty = (0, 0)$ and hence

$$d_r^2(Tx,Ty)=0\le\frac{\alpha^2+\beta^2}{5}+\frac{\beta^2}{5}+\frac{\alpha^2}{5}+\frac{\alpha^2}{4}+\frac{3\beta^2}{4}.$$

(ii) If $\alpha,\beta\in\left[4,5\frac{1}{2}\right]$, then $Tx = (\alpha, 0)$, $Ty = (\beta, 0)$ and hence

$$\alpha^2+\beta^2\le\frac{\alpha^2+\beta^2}{5}+\frac{(\alpha-\beta)^2}{5}+\frac{(\beta-\alpha)^2}{5}+\frac{\alpha^2}{2}+\frac{3\beta^2}{2}.$$

(iii) If $\alpha\in[0,2],\beta\in\left[4,5\frac{1}{2}\right]$ *then* $Tx = (0, 0)$, $Ty = (t, 0)$ and hence

$$\beta^2\le\frac{\alpha^2+\beta^2}{5}+\frac{\beta^2}{5}+\frac{(\beta-\alpha)^2}{5}+\frac{\alpha^2}{4}+\frac{3\beta^2}{2}.$$

(iv) If $\alpha\in\left[4,5\frac{1}{2}\right],\beta\in[0,2]$ then $Tx = (0, \alpha)$, $Ty = (t, 0)$ and hence

Case 3: $x = (0, \alpha)$, $y = (0, \beta)$, the proof is similar to Case 2.

Case 4: $x = (0, \alpha)$, $y = (0, \beta)$, the proof is similar to Case 1.

Hence, in all the cases, T is a generalized hybrid mapping. Moreover, generalized hybrid mapping is not necessarily a nonexpansive mapping as below.

Consider the mapping same as defined above and let

$$x=(0,5),\ y=(0,1),\ Tx=(5,0),\ Ty=(0,0).$$

Then $d_r^2(Tx,Ty)=d_r^2((5,0),(0,0))=25.$

$$d_r^2(x,y)=d_r^2((5,0),(0,1))=16\Rightarrow d_r^2(Tx,Ty)>d_r^2(x,y).$$

MAIN RESULTS

In this section, we use extended idea of generalized hybrid mapping to hyperbolic metric spaces to show that the fixed point set of these mappings is closed and convex. We also prove the existence of fixed points of these mappings and then approximate these points using Mann iterative process.

In order to achieve our first goal, we prove in the following proposition that the set of fixed points $F(T)$ of a generalized hybrid mapping T is closed and convex.

Proposition 1: Let C be a nonempty closed and convex subset of a complete 2-uniformly convex hyperbolic metric space M and $T:C \to M$ be a generalized hybrid mapping with $F(T) \neq \emptyset$ and $c_M = \left(\frac{1}{t} - 1\right) > 0$, where $t \in (0,1)$. Then $F(T)$ is a closed and convex subset of C.

Proof: If $\{x_n\}$ is a sequence in $F(T)$ and $\lim_{n\to\infty} x_n = x$, then

$$d^2(Tx,x_n) \leq d^2(Tx,Tx_n)$$
$$\leq a_1(x)d^2(x,x_n) + a_2(x)d^2(Tx,x_n) + a_3(x)d^2(Tx_n,x)$$
$$+k_1(x)d^2(Tx,x) + k_2(x)d^2(Tx_n,x_n).$$

This gives

$$(1-a_2(x))d^2(Tx,x_n) \leq a_1(x)d^2(x,x_n) + a_3(x)d^2(x_n,x)$$
$$+k_1(x)d^2(Tx,x)$$
$$\leq (a_1(x)+a_3(x))d^2(x,x_n) + k_1(x)d^2(Tx,x)$$
$$\leq (1-a_2(x))d^2(x,x_n) + k_1(x)d^2(Tx,x),$$

and so

$$d^2(Tx,x_n) \leq d^2(x,x_n) + \frac{k_1(x)}{(1-a_2(x))}d^2(Tx,x).$$

Applying limit on both sides in the above inequality, we get that

$$\left(1 - \frac{k_1}{1-a_2(x)}\right)d^2(Tx,x) = 0.$$

Since $\left(1 - \frac{k_1}{1-a_2(x)}\right) \neq 0$, therefore $d^2(Tx, x) = 0$.

Next, we show that $F(T)$ is convex. If $x, y \in F(T) \subseteq M$ and $x \neq y$ set $z = tx \oplus (1-t)y$ for $t \in [0,1]$. Note that

$$d^2(Tz,z) = d^2(Tz,tx \oplus (1-t)y) \leq td^2(Tz,x)$$
$$+(1-t)d^2(Tz,y) - c_M \min(t^2,(1-t)^2)d^2(x,y) \tag{3.1}$$

Now,

$$d^2(Tz,x) = d^2(Tz,Tx) \leq a_1(z)d^2(z,x) + a_2(z)d^2(Tz,x)$$
$$+a_3(z)d^2(Tx,z) + k_1(z)d^2(Tz,z) + k_2(z)d^2(Tx,x),$$

implies

$$(1-a_2(z))d^2(Tz,x) \leq (a_1(z)+a_3(z))d^2(z,x)$$
$$+k_1(z)d^2(Tz,z).$$

This gives

$$d^2(Tz,x) \leq d^2(z,x) + \frac{k_1(z)}{(1-a_2(z))}d^2(Tz,z).$$

Further,

$$d^2(z,x) \leq d^2(tx \oplus (1-t)y,x)$$
$$\leq td^2(x,x) + (1-t)d^2(y,x) - C_M \min(t^2,(1-t)^2)d^2(x,y)$$
$$\leq (1-t)d^2(y,x) - C_M \min(t^2,(1-t)^2)d^2(x,y).$$

Similarly,

$$d^2(Tz,y) \leq d^2(z,y) + \frac{k_1(z)}{(1-a_2(z))}d^2(Tz,z),$$

and

$$d^2(z,y) \leq td^2(y,x) - c_M \min(t^2,(1-t)^2)d^2(x,y).$$

Substituting values back in (3.1)

$$d^2(Tz,z) \leq t(1-t)d^2(x,y) - tc_M \min(t^2,(1-t)^2)d^2(x,y)$$
$$+\frac{tk_1(z)}{(1-a_2(z))}d^2(Tz,z) + t(1-t)d^2(x,y)$$
$$-(1-t)c_M \min(t^2,(1-t)^2)d^2(x,y)$$
$$+\frac{(1-t)k_1(z)}{(1-a_2(z))}d^2(Tz,z) - c_M \min(t^2,(1-t)^2)d^2(x,y).$$

Now,

$$d^2(Tz,z) \leq 2t(1-t)d^2(x,y) + \frac{k_1(z)}{(1-a_2(z))}d^2(Tz,z)$$
$$-tc_M \min(t^2,(1-t)^2)d^2(x,y) \tag{3.2}$$
$$-(1-t)c_M \, min(t^2,(1-t)^2)d^2(x,y)$$
$$-c_M \, min(t^2,(1-t)^2)d^2(x,y).$$

Case 1: If $\min(t^2,(1-t)^2) = t^2\left(t < \frac{1}{2}\right)$ then (3.2) becomes,

$$d^2(Tz,z) \leq 2t(1-t)d^2(x,y) + \frac{k_1(z)}{(1-a_2(z))}d^2(Tz,z)$$
$$-t^3c_M d^2(x,y) - t^2(1-t)c_M d^2(x,y) - t^2 c_M d^2(x,y)$$

$$\leq 2td^2(x,y)-2t^2d^2(x,y)-2t^2c_Md^2(x,y)$$
$$+\frac{k_1(z)}{(1-a_2(z))}d^2(Tz,z).$$

From above inequality, we get

$$d^2(Tz,z)\leq 2d^2(x,y)\left[t-t^2-t^2c_M\right]$$
$$+\frac{k_1(z)}{(1-a_2(z))}d^2(Tz,z),$$

which implies

$$0\leq 2d^2(x,y)\left[t-t^2-t^2c_M\right]+\left(\frac{k_1(z)}{(1-a_2(z))}-1\right)d^2(Tz,z)$$
$$\leq 2d^2(x,y)\left[t-t^2-t^2c_M\right]+\left(\frac{1-a_2(z)}{2(1-a_2(z))}-1\right)d^2(Tz,z)$$
$$\leq 2d^2(x,y)\left[t-t^2-t^2c_M\right]-\frac{1}{2}d^2(Tz,z),$$

and hence

This implies $d^2(Tz, z) = 0$ i.e. $Tz = z$, showing that the set of fixed points is closed and convex.

Case 2: Now if $\min(t^2,(1-t)^2)=(1-t)^2,\left(t>\frac{1}{2}\right)$ then (3.2) becomes,

$$d^2(Tz,z)\leq 2t(1-t)d^2(x,y)+\frac{k_1(z)}{(1-a_2(z))}d^2(Tz,z)$$
$$-tc_M(1-t)^2d^2(x,y)-c_M(1-t)^3d^2(x,y)$$
$$-c_M(1-t)^2d^2(x,y).$$

From the above inequality, we get

$$\left(1-\frac{k_1(z)}{(1-a_2(z))}\right)d^2(Tz,z)$$
$$\leq\left[2t(1-t)-tc_M(1-t)^2-c_M(1-t)^3-c_M(1-t)^2\right]d^2(x,y)$$
$$\leq\left[2t(1-t)-2c_M(1-t)^3\right]d^2(x,y),$$

which implies

$$0\leq 2(1-t)(t+c_M(1-t)^2d^2(x,y)$$
$$+\left(-1+\frac{k_1(z)}{(1-a_2(z))}\right)d^2(Tz,z)$$
$$0\leq 2(1-t)(t+c_M(1-t)^2d^2(x,y)-\frac{1}{2}d^2(Tz,z),$$

that is, if $\frac{1}{2}<t<1$ then $c_M=-\frac{t}{(1-t)^2}$ which cannot be the case as we have $C_M > 0$.

The next result is the demiclosed principle for generalized hybrid mappings.

Theorem 4: Let (M, d) be a complete and 2-uniformly convex hyperbolic space and C be a nonempty, closed, convex and bounded subset of M. Let $T{:}C \to M$ be a generalized hybrid mapping with $2c_M>\frac{k_1(x)}{1-a_2(x)}$. Let $\{x_n\}$ be a sequence in C with $x_n\to x$ and $\lim_{n\to\infty}d(x_n,Tx_n)=0$. Then $x \in C$ and $Tx = x$.

Proof: Since $x_n \to x$ so $x \in C$ and define the function $\tau(x)=\lim_{n\to\infty}d(x_n,x)$. Since, T is a generalized hybrid mapping, we compute

$$d^2(Tx,Tx_n)\leq a_1(x)d^2(x,x_n)+a_2(x)d^2(Tx,x_n)$$
$$+a_3(x)d^2(Tx_n,x)+k_1(x)d^2(Tx,x)+k_2(x)d^2(Tx_n,x_n)$$
$$\leq a_1(x)d^2(x,x_n)+a_2(x)\left(d(Tx,Tx_n)+d(Tx_n,x_n)\right)^2$$
$$+a_3(x)d^2(x_n,x)+k_1(x)d^2(Tx,x)$$
$$\leq(a_1(x)-a_3(x))d^2(x,x_n)+a_2(x)d^2(Tx,Tx_n)$$
$$+k_1(x)d^2(Tx,x)(1-a_2(x))d^2(Tx,Tx_n)$$
$$\leq(1-a_2(x))d^2(x,x_n)+k_1(x)d^2(Tx,x)$$
$$d^2(Tx,Tx_n)\leq d^2(x,x_n)+\frac{k_1(x)}{(1-a_2(x))}d^2(Tx,x)$$
$$\limsup_{n\to\infty}d^2(Tx,x_n)\leq\limsup_{n\to\infty}d^2(x,x_n)$$
$$+\frac{k_1(x)}{(1-a_2(x))}d^2(Tx,x).$$

Since M is 2-uniformly convex,

$$d^2\left(x_n,\frac{1}{2}x\oplus\frac{1}{2}Tx\right)\leq\frac{1}{2}d^2(x_n,x)$$
$$+\frac{1}{2}d^2(x_n,Tx)-c_Md^2(x,Tx)$$
$$\limsup_{n\to\infty}d^2\left(x_n,\frac{1}{2}x\oplus\frac{1}{2}Tx\right)\leq\frac{1}{2}\limsup_{n\to\infty}d^2(x_n,x)$$
$$+\frac{1}{2}\limsup_{n\to\infty}d^2(x_n,Tx)-c_Md^2(x,Tx)$$
$$\leq\frac{1}{2}\limsup_{n\to\infty}d^2(x_n,x)+\frac{1}{2}\left(\begin{array}{c}\limsup_{n\to\infty}d^2(x_n,x)+\\ \frac{k_1(x)}{(1-a_2(x))}d^2(Tx,x)\end{array}\right)$$
$$-c_Md^2(x,Tx)$$
$$\leq\limsup_{n\to\infty}d^2(x_n,x)+\frac{k_1(x)}{2(1-a_2(x))}d^2(Tx,x)$$
$$-c_Md^2(x,Tx)$$
$$\leq\limsup_{n\to\infty}d^2(x_n,x)+\left(\frac{k_1(x)}{2(1-a_2(x))}-c_M\right)d^2(x,Tx)$$

$$\left(-\frac{k_1(x)}{2(1-a_2(x))}+c_M\right)d^2(x,Tx)\le \text{limsup}_{n\to\infty}d^2(x_n,x)$$
$$-\text{limsup}_{n\to\infty}d^2\left(x_n,\frac{1}{2}x\oplus\frac{1}{2}Tx\right)$$
$$\le(\tau(x))^2-\left(\tau\left(\frac{1}{2}x\oplus\frac{1}{2}Tx\right)\right)^2\le 0.$$

Hence $Tx = x$.

Recall that Mann iterative process is defined by

$$x_{n+1}=t_nTx_n\oplus(1-t_n)x_n$$

for any $n \ge 1$ where $\{t_n\} \in (0,1)$ and $x_1 \in C$ is the initial point. Now we investigate the connection between $F(T)$ and the Mann iterative process.

Theorem 5: Let (M, d), C and T be as in Theorem 4. Let $\{x_n\}$ be defined by the Mann iterative process as given above with $\text{liminf}_{n\to\infty}t_n>0$ for all $\omega \in F(T)$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof: Note that

$$d^2(x_{n+1},\omega)\le t_nd^2(Tx_n,\omega)+(1-t_n)d^2(x_n,\omega) \quad (3.3)$$

Now,

$$d^2(Tx_n,\omega)\le d^2(Tx_n,T\omega)$$
$$\le a_1(\omega)d^2(x_n,\omega)+a_2(\omega)d^2(Tx_n,\omega)+a_3(\omega)d^2(T\omega,x_n)$$
$$+k_1(\omega)d^2(Tx_n,x_n)+k_2(\omega)d^2(T\omega,\omega)$$
$$\le d^2(x_n,\omega)+\left[\frac{k_1(\omega)}{1-a_2(\omega)}\right]d^2(Tx_n,x_n)$$
$$\le d^2(x_n,\omega)+\frac{1}{2}d^2(Tx_n,x_n).$$

Putting value of $d_2(T^n(x_n), \omega)$ back in (3.3),

$$d^2(x_{n+1},\omega)\le t_n\left[d^2(x_n,\omega)+\left[\frac{k_1(\omega)}{1-a_2(\omega)}\right]d^2(Tx_n,x_n)\right]$$
$$+(1-t_n)d^2(x_n,\omega)$$
$$d^2(x_{n+1},\omega)\le d^2(x_n,\omega)+t_n\left[\frac{k_1(\omega)}{1-a_2(\omega)}\right]d^2(Tx_n,x_n).$$

By assumption, there exists $\delta > 0$ and $N \in E$ such that

$$t_n\left[\frac{k_1(\omega)}{1-a_2(\omega)}\right]\ge\delta>0,$$

for all Without loss of generality, we may assume that,

$$t_n\left[\frac{k_1(\omega)}{1-a_2(\omega)}\right]>0$$

for all $n \in N$ And since C is bounded, so is x_n and $d(Tx_n, x_n)$. Hence $\{d^2(x_n, \omega)\}$ is decreasing and so is $\{d(x_n, \omega)\}$. Hence $\lim_{n\to\infty}d(x_n,\omega)$ exists.

Next we prove that $\lim_{n\to\infty}d^2(Tx_n,x_n)=0$.

Let $r=\lim_{n\to\infty}d^2(x_n,\omega)$. Without loss of generality, we may assume $r > 0$.

Moreover,

$$d^2(Tx_n,\omega)=d^2(Tx_n,T\omega)$$
$$\le \begin{array}{l}a_1(\omega)d^2(x_n,\omega)+a_2(\omega)d^2(Tx_n,\omega)+a_3(\omega)d^2(T\omega,x_n)\\ \qquad +k_1(\omega)d^2(Tx_n,x_n)+k_2(\omega)d^2(T\omega,\omega)\end{array}$$
$$\le d^2(x_n,\omega)+\frac{k_1(\omega)}{1-a_2(\omega)}d^2(Tx_n,x_n)$$
$$\le d^2(x_n,\omega)$$
$$\Rightarrow d^2(Tx_n,\omega)\le d^2(x_n,\omega).$$
$$\text{limsup}_{n\to\infty}d^2(Tx_n,\omega)\le\lim_{n\to\infty}\left[d^2(x_n,\omega)\right]=r.$$

Let u be a non-trivial filter over $\mathbb{N}$ Then $\lim_{u} t_n=t\in[a,b]$. Then by

$$d^2(x_{n+1},\omega)\le t_nd^2(Tx_n,\omega)$$
$$+(1-t_n)d^2(x_n,\omega)\text{ for any }n\ge1,$$

we have

$$r=\lim_{u}d^2(x_{n+1},\omega)\le t\lim_{u}d^2(Tx_n,\omega)+(1-t)r.$$

Since $t \ne 0$ we have $\lim_{u}d^2(Tx_n,\omega)\ge r.$ Hence

$$r\le\liminf_{n\to\infty}d^2(Tx_n,\omega)\le\limsup_{n\to\infty}d^2(Tx_n,\omega)\le r.$$

Hence $\lim_{n\to\infty}d^2(Tx_n,\omega)=r.$ Since M is 2-uniformly convex, Theorem 2 implies

$$C_M\min(t_n^2,(1-t_n)^2)d^2(Tx_n,x_n)\le t_nd^2(x_n,\omega)$$
$$+(1-t_n)d^2(Tx_n,\omega)-d^2(x_{n+1},\omega).$$

where $C_M > 0$ depends only on M. Since

$$\min(t_n^2,(1-t_n)^2)\ge\min(a^2,(1-b)^2)>0,$$

and $\lim_{n\to\infty}\left[t_nd^2(x_n,\omega)+(1-t_n)d^2(Tx_n,\omega)-d^2(x_{n+1},\omega)\right]=0$, we have $\lim_{n\to\infty}d^2(Tx_n,x_n)=0$

Now applying Theorem 4, we have $T_x = x$ and hence $\{x_n\}$ converges strongly to a fixed point of T.

Now we give an example to validate Theorem 5.

Example 2: *Let* $C=\{(0,\alpha): \quad \alpha\in[0,2]\}\subset\mathbb{R}^2$. Then C is closed, convex and bounded.

Proof. Define $T: C\to\mathbb{R}^2$ *by* $T(0,\alpha)=\left(0,\frac{\alpha}{2}\right)$. Let $(0,\alpha_1)$, $(0,\alpha_2)\in C$ and now calculating the one side of the generalized hybrid mapping, we have

$$d^2\left(T(0,\alpha_1),T(0,\alpha_2)\right)=d^2\left(\left(0,\frac{\alpha_1}{2}\right),\left(0,\frac{\alpha_2}{2}\right)\right)$$

$$=\left(\sqrt{\left(\frac{\alpha_1}{2}-\frac{\alpha_2}{2}\right)^2}\right)^2=\frac{(\alpha_1-\alpha_2)^2}{4}.$$

For the other side, we have

$$d^2\left((0,\alpha_1),(0,\alpha_2)\right)=(\alpha_1-\alpha_2)^2,$$

$$d^2\left(T(0,\alpha_1),(0,\alpha_2)\right)=d^2\left(\left(0,\frac{\alpha_1}{2}\right),(0,\alpha_2)\right)=\frac{(\alpha_1-2\alpha_2)^2}{4}$$

$$d^2\left(T(0,\alpha_2),(0,\alpha_1)\right)=\frac{(\alpha_2-2\alpha_1)^2}{4}$$

$$d^2\left(T(0,\alpha_1),(0,\alpha_1)\right)=d^2\left(\left(0,\frac{\alpha_1}{2}\right),(0,\alpha_1)\right)=\frac{\alpha_1^2}{4}$$

$$d^2\left(T(0,\alpha_2),(0,\alpha_2)\right)=d^2\left(\left(0,\frac{\alpha_1}{2}\right),(0,\alpha_1)\right)=\frac{\alpha_2^2}{4}$$

and letting $a_1=a_2=a_3=\frac{1}{5}, k_1=\frac{1}{4}, k_2=\frac{3}{4}$, we can see that T is a generalized hybrid mapping.

Let $\{(0,\alpha_n)\}_{n\in N}$ be a sequence in C with $(0,\alpha_n)\to(x,y)$ and $\alpha_n=\frac{1}{n}$, then $\lim_{n\to\infty} d\left((0,\alpha_n),\left(0,\frac{\alpha_n}{2}\right)\right)=\lim_{n\to\infty}\frac{\alpha_n}{2}=0.$

Then we have $(0,0)=(0,\alpha)\in C$ and $T(0,0)=\left(0,\frac{0}{2}\right)=(0,0)$.

CONCLUSION

We introduce generalized hybrid mappings in hyperbolic metric spaces. We first show that the set of fixed points of such mappings is closed and convex. We then prove the existence of fixed point of these mappings and finally approximate the fixed point using Mann iterative process. We have also provided an example of generalized hybrid mappings in the setting of -trees for the first time in the literature.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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