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Research Article

Normal-like Curves with Respect to the Special Case of the ED-frame in Euclidean 4-space

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Abstract
The aim of this study is to present normal-like curves with respect to the special case of the ED-frame in Euclidean 4-space. Furthermore, the relationship between geodesic torsion and curvature is given so that a curve lying on an oriented surface M in 4-dimensional Euclidean space is congruent to a normal-like curve according to the special case of the ED-frame. Finally, an example of the study is presented.

Keywords and 2020 Mathematics Subject Classification
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MSC: 53A04, 53A07

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1. Introduction

Some special curves in differential geometry are quite remarkable for researchers. One of them is normal curve. After Chen [1] defined the rectifying curve in Euclidean space, Ilarslan and Nesovic [2] defined the osculating, rectifying and normal curve as a curve in Euclidean 4-space according to the definition of the rectifying curve in Euclidean 3-space, [3]. This definition given for the normal curve has been used in some studies in Minkowski space [4–6]. In addition, the generalization of normal curves to n-dimensional space was made by Bektaş [7].

Frame fields are very useful for defining curves and examining properties. One of these frame fields is the Frenet frame fields. The Frenet frame along a curve is a moving (right-handed) coordinate system determined by the tangent line and curvature [8, 9]. Another important frame field is known as Darboux frame [10]. In addition to these frame fields, a new frame fields has been introduced to the literature. This frame fields was defined by Düldül et al. and named as extended Darboux frame field (ED-frame field) [11]. In this study, we define normal-like curves with respect to the special case of the ED-frame in Euclidean 4-space and an example of the study is presented.

2. Preliminaries

Definition 1. *Let $x=\sum_{i=1}^{4} x_i e_i$, $y=\sum_{i=1}^{4} y_i e_i$, $z=\sum_{i=1}^{4} z_i e_i$ be vectors in Euclidean 4-space $\mathbb{E}^4$, where $\{e_i\}$, $1\le i\le 4$ is the standart basis vectors of $\mathbb{E}^4$. The vector product of three vectors is given by [12]*

$$x\otimes y\otimes z=\begin{vmatrix} e_1 & e_2 & e_3 & e_4\\ x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\end{vmatrix}.$$

Let $\mathscr{M}$ be an orientable hypersurface and the curve γ lies on $\mathscr{M}$. On the other hand, if the unit tangent vector field of the curve is T, the unit normal vector field of the hypersurface restricted to the curve γ is N and the unit normal field vector of $\mathscr{M}$ is $\mathscr{N}$, then T is given as $T=\gamma'(s)$ and $N(s)=\mathscr{N}(\gamma(s))$ [11].

Case 1. Let $\{N,T,\gamma''\}$ be linearly independent. In this case, the orthonormal set $\{N,T,E\}$ with

$$E=\frac{\gamma''-\left\langle\gamma'',N\right\rangle N}{\left\|\gamma''-\left\langle\gamma'',N\right\rangle N\right\|},$$

is obtained [11].

Case 2. Let $\{N,T,\gamma''\}$ be linearly dependent. In this case, the orthonormal set $\{N,T,E\}$ with

$$E=\frac{\gamma''-\left\langle\gamma'',N\right\rangle N-\left\langle\gamma''',T\right\rangle T}{\left\|\gamma''-\left\langle\gamma'',N\right\rangle N-\left\langle\gamma''',T\right\rangle T\right\|},$$

is obtained [11]. If $D=N\otimes T\otimes E$, then we obtain orthonormal frame field $\{T,E,D,N\}$ along the curve γ [11].

One can easily see that the vector fields E and D are tangent to $\mathscr{M}$. Also, $\{T,E,D\}$ spans the tangent hyperplane of the hypersurface at the point γ (s) [11].

Let κ_n be the normal curvature of the hypersurface in the direction of the tangent vector T, κ_g^i and τ_g^i be the geodesic curvature and the geodesic torsion of order i (i=1,2), respectively, [11]. The derivative equations for Case 1 and Case 2

$$\begin{bmatrix} T' \\ E' \\ D' \\ N' \end{bmatrix}=\begin{bmatrix} 0 & \kappa_g^1 & 0 & \kappa_n \\ -\kappa_g^1 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & \tau_g^2 \\ -\kappa_n & -\tau_g^1 & -\tau_g^2 & 0 \end{bmatrix}\begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix}, \tag{1}$$

and

$$\begin{bmatrix} T' \\ E' \\ D' \\ N' \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & \kappa_n \\ 0 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & 0 \\ -\kappa_n & -\tau_g^1 & 0 & 0 \end{bmatrix}\begin{bmatrix} T \\ E \\ D \\ N \end{bmatrix}. \tag{2}$$

On the other hand, for Case 1 and Case 2 the following statements hold, respectively [11]:

$$\left\langle T',D\right\rangle=\left\langle D',T\right\rangle=0,$$

$$\left\langle T',E\right\rangle=\kappa_g^1,\left\langle T',N\right\rangle=\kappa_n,\left\langle E',T\right\rangle=-\kappa_g^1,\left\langle E',D\right\rangle=\kappa_g^2,\left\langle E',N\right\rangle=\tau_g^1,\left\langle D',N\right\rangle=\tau_g^2,$$

$$\left\langle N',T\right\rangle=-\kappa_n,\left\langle N',E\right\rangle=\tau_g^1,\left\langle N',D\right\rangle=-\tau_g^2, \tag{3}$$

$$\left\langle T',E\right\rangle=\left\langle T',D\right\rangle=\left\langle E',T\right\rangle=\left\langle D',T\right\rangle=\left\langle D',N\right\rangle=\left\langle N',D\right\rangle=0,$$

$$\left\langle T',N\right\rangle=\kappa_n,\left\langle E',N\right\rangle=\tau_g^1,\left\langle E',D\right\rangle=\kappa_g^2,\left\langle D',E\right\rangle=-\kappa_g^2,\left\langle N',T\right\rangle=-\kappa_n,\left\langle N',E\right\rangle=-\tau_g^1. \tag{4}$$

3. Normal-like curves with respect to the special case of the ED-frame in Euclidean 4-space

In this section, we define the normal curves according to the Case 2 ED-frame in Euclidean 4-space. And then, we find the relationship between the curvatures for any unit speed curve which lies on the orientable hypersurface $\mathscr{M}$ to be congruent to this normal curves in $\mathbb{E}^4$.

Definition 2. *Let $\alpha : I \subset \mathbb{R} \to \mathscr{M}$ be a unit speed curve on an oriented hypersurface $\mathscr{M}$ in Euclidean 4-space and $\{T,E,D,N\}$ denote the ED-frame field of $\alpha(s)$. Then we define the normal curve according to the ED-frame in the Euclidean space $\mathbb{E}^4$ as a curve whose position vector always lies in the orthogonal complement $T^{\perp}$ of tangent vector field T, and we express it with*

$$\alpha(s) = \lambda(s)E(s) + \mu_1(s)D(s) + \mu_2(s)N(s) \tag{5}$$

for some differentiable functions $\lambda(s), \mu_1(s)$ and $\mu_2(s)$ of $s \in I \subset \mathbb{R}$.

Theorem 3. *Let $\alpha : I \subset \mathbb{R} \to \mathscr{M}$ be a unit speed curve on an oriented hypersurface $\mathscr{M}$ in Euclidean 4-space and $\{T,E,D,N\}$ denote the Case 2 ED-frame field of $\alpha(s)$. Then $\alpha(s)$ is congruent to a normal curve if and only if*

$$\left\{\frac{1}{\kappa_g^2(s)}\left[\frac{\tau_g^1(s)}{\kappa_n(s)} + \left(\frac{1}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)'\right)'\right]\right\}' + \frac{\kappa_g^2(s)}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)' = 0$$

where $\kappa_n, \kappa_g^1, \kappa_g^2$ and $\tau_g^1 \neq 0$.

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathscr{M}$ be a unit speed curve on an oriented hypersurface $\mathscr{M}$ in Euclidean 4-space and $\{T,E,D,N\}$ denote the ED-frame field of $\alpha(s)$. If the derivative of both sides of equation (5) with respect to s is taken and the derivative equation (2) is applied, the following expression for Case 2 is obtained that

$$\alpha'(s) = (-\mu_2(s)\kappa_n(s))T(s) + \left(\lambda'(s) - \mu_1(s)\kappa_g^2(s) - \mu_2(s)\tau_g^1(s)\right)E(s)$$
$$+ \left(\mu_1'(s) + \lambda(s)\kappa_g^2(s)\right)D(s) + \left(\mu_2'(s) + \lambda(s)\tau_g^1(s)\right)N(s).$$

We know that $\alpha'(s) = T(s)$. So, using the equality of both sides, we get the following expressions for the coefficients of $T(s)$, $E(s)$, $D(s)$ and $N(s)$

Case 2

$$-\mu_2(s)\kappa_n(s) = 1, \tag{6}$$

$$\lambda'(s) - \mu_1(s)\kappa_g^2(s) - \mu_2(s)\tau_g^1(s) = 0, \tag{7}$$

$$\mu_1'(s) + \lambda(s)\kappa_g^2(s) = 0, \tag{8}$$

$$\mu_2'(s) + \lambda(s)\tau_g^1(s) = 0. \tag{9}$$

From (6), we can find the following coefficient function:

$$\mu_2(s) = -\frac{1}{\kappa_n(s)}. \tag{10}$$

When the coefficient function (10) is used in equation (9), the other coefficient function is as follows:

$$\lambda(s) = \frac{1}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)'. \tag{11}$$

The coefficient function $\mu_1(s)$ is given similarly with the help of the related coefficient functions

$$\mu_1(s) = \frac{1}{\kappa_g^2(s)}\left[\frac{\tau_g^1(s)}{\kappa_n(s)} + \left(\frac{1}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)'\right)'\right]. \tag{12}$$

Substituting (11) and (12) into (8), we get the following relations:

$$\left\{\frac{1}{\kappa_g^2(s)}\left[\frac{\tau_g^1(s)}{\kappa_n(s)}+\left(\frac{1}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)'\right)'\right]\right\}'+\frac{\kappa_g^2(s)}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)'=0 \tag{13}$$

where $\kappa_n, \kappa_g^1, \kappa_g^2$ and $\tau_g^1 \neq 0$.

Conversely, consider an arbitrary unit speed curve on an oriented hypersurface $\mathscr{M}$ in Euclidean 4-space for which the curvature functions satisfy the relations (13) and (11). Then, we consider the vector $X \in \mathbb{E}^4$ defined by

$$X(s)=\alpha(s)-\frac{1}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)'E(s)-\frac{1}{\kappa_g^2(s)}\left[\frac{\tau_g^1(s)}{\kappa_n(s)}+\left(\frac{1}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)'\right)'\right]D(s)+\frac{1}{\kappa_n(s)}N(s).$$

It can be seen that $X(s)=0$ through the relations (2), (13). Thus, X is a constant vector. This implies that α is congruent to an normal curve. ■

Theorem 4. *Let $\alpha : I \subset \mathbb{R} \to \mathscr{M}$ be a unit speed curve on an oriented hypersurface $\mathscr{M}$ in Euclidean 4-space and $\{T,E,D,N\}$ denote the Case 2 ED-frame field of $\alpha(s)$. Then α is a normal curve if and only if*

$$\langle \alpha(s), E(s)\rangle=\frac{1}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)', \langle \alpha(s), N(s)\rangle=-\frac{1}{\kappa_n(s)}, \tau_g^1(s), \kappa_n(s)\neq 0. \tag{14}$$

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathscr{M}$ be a unit speed curve on an oriented hypersurface $\mathscr{M}$ in Euclidean 4-space and $\{T,E,D,N\}$ denote the Case 2 ED-frame field of $\alpha(s)$. Substituting (10), (11) and (12) into (5), we get

$$\alpha(s)=\frac{1}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)'E(s)+\frac{1}{\kappa_g^2(s)}+\frac{\tau_g^1(s)}{\kappa_n(s)}+\left(\frac{1}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)'\right)'D(s)-\frac{1}{\kappa_n(s)}N(s).$$

When the inner product of both sides of the equation with respect to $E(s)$ and $N(s)$, respectively, is taken, the expressions $\langle \alpha(s), E(s)\rangle=\frac{1}{\tau_g^1(s)}\left(\frac{1}{\kappa_n(s)}\right)'$ and $\langle \alpha(s), N(s)\rangle=-\frac{1}{\kappa_n(s)}$, $\tau_g^1(s)$, $\kappa_n(s)\neq 0$ are found.

Conversely the statement (14) holds. Differentiating equation $\langle \alpha(s), N(s)\rangle=-\frac{1}{\kappa_n(s)}$ with respect to s and by applying (2), we find $\langle \alpha(s), T(s)\rangle = 0$ which means that α is a normal curve. ■

Example 5. *Let $\alpha(s)=\left(\frac{2s}{\sqrt{11}}, sin\left(\sqrt{\frac{2}{11}}s\right), \sqrt{\frac{5}{11}}s, -cos\left(\sqrt{\frac{2}{11}}s\right)\right)$ be a unit speed curve on an hypersurface $\mathscr{M} \ldots\ x_2^2+x_4^2=1$ in Euclidean 4-space. The unit normal vector of $\mathscr{M}$ along α is $N(s)=(0,x_2,0,x_4)$. If we calculate the unit tangent vector field we can find as follows:*

$$T(s)=\left(\frac{2}{\sqrt{11}}, \sqrt{\frac{2}{11}}cos\left(\sqrt{\frac{2}{11}}s\right), \sqrt{\frac{5}{11}}, \sqrt{\frac{2}{11}}sin\left(\sqrt{\frac{2}{11}}s\right)\right).$$

The derivative of $T(s)$ is given by

$$T'(s)=\alpha''(s)=\left(0,-\frac{2}{11}sin\left(\sqrt{\frac{2}{11}}s\right),0,\frac{2}{11}cos\left(\sqrt{\frac{2}{11}}s\right)\right).$$

Considering $T'(s)$ and $N(s)=\mathscr{N}(\alpha(s))=(0, sin\left(\sqrt{\frac{2}{11}}s\right), 0, -cos\left(\sqrt{\frac{2}{11}}s\right))$, ($T'(s)=-\frac{2}{11}N(s)$) they appear to be linearly dependent. Therefore, Case 2 applies. Thus, we get

$$E(s)=\left(\frac{4}{3\sqrt{22}}, \frac{-3}{\sqrt{11}}cos\left(\sqrt{\frac{2}{11}}s\right), \frac{2\sqrt{5}}{3\sqrt{22}}, \frac{-3}{\sqrt{11}}sin\left(\sqrt{\frac{2}{11}}s\right)\right),$$

and

$$D(s) = \left(\frac{-\sqrt{5}}{3}, 0, \frac{2}{3}, 0 \right).$$

If we use the equation (4) we get

$$\left\langle T', N \right\rangle = \kappa_n = -\frac{2}{11}, \left\langle E', N \right\rangle = \tau_g^1 = \frac{3\sqrt{2}}{11}, \left\langle E', D \right\rangle = \kappa_g^2 = 0.$$

According to the results obtained from here, it is concluded that the given unit speed curve cannot be congruent to a normal curve, since $\kappa_g^2 = 0$.

On the other hand, the normal like curve is obtained as follows:

$$\alpha(s) = \lambda(s) \left(\frac{4}{3\sqrt{22}}, \frac{-3}{\sqrt{11}} cos\left(\sqrt{\frac{2}{11}} s \right), \frac{2\sqrt{5}}{3\sqrt{22}}, \frac{-3}{\sqrt{11}} sin\left(\sqrt{\frac{2}{11}} s \right) \right) + \mu_1(s) \left(\frac{-\sqrt{5}}{3}, 0, \frac{2}{3}, 0 \right)$$
$$+ \mu_2(s) \left(0, sin\left(\sqrt{\frac{2}{11}} s \right), 0, -cos\left(\sqrt{\frac{2}{11}} s \right) \right)$$

or

$$\alpha(s) = \left(\frac{4}{3\sqrt{22}} \lambda(s) - \frac{\sqrt{5}}{3} \mu_1(s), -\frac{3}{11} \lambda(s) cos\left(\sqrt{\frac{2}{11}} s \right), \frac{2\sqrt{5}}{3\sqrt{22}} \lambda(s) + \frac{2}{3} \mu_1(s), \frac{-3}{\sqrt{11}} sin\left(\sqrt{\frac{2}{11}} s \right) \lambda(s) - \mu_1(s) cos\left(\sqrt{\frac{2}{11}} s \right) \right)$$

for some differentiable functions $\lambda(s), \mu_1(s)$ *and* $\mu_2(s)$ *of* $s \in I \subset \mathbb{R}$.

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