Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (5) (2023), 1135 – 1150
DOI : 10.15672/hujms.1127140

Research Article

On a minimal set of generators for the algebra $H^*(BE_6;\mathbb{F}_2)$ as a module over the Steenrod algebra and applications

Nguyen Khac Tin

Faculty of Applied Sciences, Ho Chi Minh City University of Technology and Education 01 Vo Van Ngan, Thu Duc, Ho Chi Minh city, Vietnam

Abstract

Let $\mathcal{P}_n \cong H^*(BE_n;\mathbb{F}_2)$ be the graded polynomial algebra over the prime field of two elements $\mathbb{F}_2$, where E_n is an elementary abelian 2-group of rank n, and BE_n is the classifying space of E_n. We study the *hit problem*, set up by Frank Peterson, of finding a minimal set of generators for the polynomial algebra $\mathcal{P}_n$, viewed as a module over the mod-2 Steenrod algebra $\mathcal{A}$. This problem remains unsolvable for $n > 4$, even with the aid of computers in the case of $n = 5$. By considering $\mathbb{F}_2$ as a trivial $\mathcal{A}$-module, then the hit problem is equivalent to the problem of finding a basis of $\mathbb{F}_2$-graded vector space $\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n$.

This paper aims to explicitly determine an admissible monomial basis of the $\mathbb{F}_2$-vector space $\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n$ in the generic degree $n(2^r - 1) + 2 \cdot 2^r$, where r is an arbitrary non-negative integer, and in the case of $n = 6$.

As an application of these results, we obtain the dimension results for the polynomial algebra $\mathcal{P}_n$ in degrees $(n-1)\cdot(2^{n+u-1} - 1) + \ell \cdot 2^{n+u}$, where u is an arbitrary non-negative integer, $\ell = 13$, and $n = 7$.

Moreover, for any integer $r > 1$, the behavior of the sixth Singer algebraic transfer in degree $6(2^r - 1) + 2 \cdot 2^r$ is also discussed at the end of this paper. Here, the Singer algebraic transfer is a homomorphism from the homology of the Steenrod algebra to the subspace of $\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n$ consisting of all the $GL_n(\mathbb{F}_2)$-invariant classes. It is a useful tool in describing the homology groups of the Steenrod algebra, $\mathrm{Tor}^{\mathcal{A}}_{n,n+*}(\mathbb{F}_2, \mathbb{F}_2)$.

Mathematics Subject Classification (2020). 55S10, 55S05, 55T15, 16W50

Keywords. Steenrod algebra, polynomial algebra, hit problem, graded rings

1. Introduction

Let X be a topological space. Cohomology operations are generated by the natural transformations of degree i which are so-called Steenrod squares

$$Sq^i : H^*(X,\mathbb{F}_2) \longrightarrow H^{*+i}(X,\mathbb{F}_2),$$

where $H^*(X,\mathbb{F}_2)$ is the singular cohomology of X with coefficients in the two-element field $\mathbb{F}_2$, and i is arbitrary non-negative integers. In 1952, Serre established the structure of the

Email address: tinnk@hcmute.edu.vn
Received: 07.07.2022; Accepted: 27.12.2022

set of all cohomology operations. Serre [18] proved that the Steenrod squares generate all stable cohomology operations with the usual addition and the composition of maps. The algebra of stable cohomology operations with coefficients in $\mathbb{F}_2$ is known as the modulo 2 Steenrod algebra, $\mathcal{A}$.

Furthermore, the Steenrod algebra is able to be defined algebraically as a quotient algebra of $\mathbb{F}_2$-free graded associative algebra generated by the symbols Sq^i of degree i, where i is a non-negative integer, by the two-sided ideal generated by the relation $Sq^0 = 1$ and the Adem's relations

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j, \quad \text{for } 0 < a < 2b$$

(see Chapter 1 of [21]).

Calculations of various homotopy groups of spheres by Jean-Pierre Serre were among the first Steenrod algebra applications. Milnor proved in [9] that the dual of the mod 2 Steenrod algebra is a polynomial algebra and that the mod 2 Steenrod algebra admits a Hopf algebra's structure. As a result, an object that was previously considered intractable suddenly became considerably easier to control. Recently, the mod 2 Steenrod algebra and the mod 2 dual Steenrod algebra as a subalgebra of the mod 2 dual Leibniz-Hopf algebra have been studied by many authors (see Crossley-Turgay [4], Crossley [5], Turgay-Kaji [27] and others).

Let E_n be an elementary abelian 2-group of rank n. We will denote by BE_n the classifying space of E_n. It may be thought of as the product of n copies of real project space $\mathbb{R}P^\infty$. Then, based on the Künneth formula for cohomology, one gets an isomorphism of $\mathbb{F}_2$-algebras

$$\mathcal{P}_n := H^*(BE_n; \mathbb{F}_2) \cong \mathbb{F}_2[x_1] \otimes_{\mathbb{F}_2} \ldots \otimes_{\mathbb{F}_2} \mathbb{F}_2[x_n] \cong \mathbb{F}_2[x_1, x_2, \ldots, x_n],$$

where $x_i \in H^1(BE_n; \mathbb{F}_2)$ for every i.

As is well-known, $\mathcal{P}_n$ is a module over the mod-2 Steenrod algebra $\mathcal{A}$. The action of $\mathcal{A}$ on $\mathcal{P}_n$ is determined by the formula

$$Sq^k(x_j) = \begin{cases} x_j, & k = 0, \\ x_j^2, & k = 1, \\ 0, & k > 1, \end{cases}$$

and the Cartan formula $Sq^k(uv) = \sum_{i=0}^k Sq^i(u) Sq^{k-i}(v)$, where $u, v \in \mathcal{P}_n$ (see Steenrod-Epstein [21], and Turgay [26]).

The *Peterson hit problem* is to find a minimal generating set for $\mathcal{P}_n$ regarded as a module over the mod-2 Steenrod algebra. The hit problem is analogous to the problem of finding a basis for the $\mathbb{F}_2$-graded vector space $\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n$ if we treat $\mathbb{F}_2$ as a trivial $\mathcal{A}$-module.

This issue has first been studied by Peterson [14], Singer [19], Wood [34], and Priddy [16], who shows its relationship to several classical problems in cobordism theory, modular respresentation theory, Adams spectral sequence for the stable homotopy of spheres, stable homotopy type of the classifying space of finite groups. Then, this issue and its applications were investigated by Silverman [20], Repka-Selick [17], Janfada-Wood [6], Nam [13], Sum [22,24], Mothebe-Kaelo-Ramatebele [12], Sum-Tin [25], Walker-Wood [33], the present writer [29,30] and others.

Let $\alpha(d)$ be the number of digits 1 in the binary expansion of a natural integer d. Consider the function $\mu : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N} \cup \{0\}$ which is defined as follows:

$$\mu(0) = 0, \text{ and } \mu(d) = \min\{m \in \mathbb{N} \ : \ \alpha(d+m) \leqslant m\}.$$

In [14], Peterson hypothesized that as a module over the Steenrod algebra $\mathcal{A}$, $\mathcal{P}_n$ is generated by monomials of degree d obeying the inequality $\alpha(d+n) \leqslant n$, and proved it for $n \leqslant 2$. And then, Wood [34] demonstrates this in general. This is a fantastic tool for figuring out $\mathcal{A}$-generators for $\mathcal{P}_n$.

Kameko's squaring operation

$$\widetilde{Sq}^0_* := (\widetilde{Sq}^0_*)_{(n;n+2d)} : (\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_{n+2d} \to (\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n)_d,$$

which is induced by an $\mathbb{F}_2$-linear map $S_n : \mathcal{P}_n \to \mathcal{P}_n$, given by

$$S_n(x) = \begin{cases} y, & \text{if } x = x_1x_2\dots x_n y^2 \\ 0, & \text{otherwise} \end{cases}$$

for any monomial $x \in \mathcal{P}_n$, is one of the most important tools in the analysis of the hit problem. Clearly, $(\widetilde{Sq}^0_*)_{(n;n+2d)}$ is an $\mathbb{F}_2$-epimorphism.

From the results of Kameko [7], Sum [24], and Wood [34], the hit problem is reduced to the case of degree d of the form $d = r(2^t - 1) + 2^t m$, where r, m, t are non-negative intergers such that $0 \leqslant \mu(m) < r \leqslant n$.

Now, the tensor product $\mathcal{A}\mathcal{P}_n := \mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n$ was entirely calculated for $n \leqslant 4$ (see Peterson [14] for $n = 1, 2$, see Kameko [7] for $n = 3$, see Sum [24] for $n = 4$), but it remains unresolved for $n \geqslant 5$, even with the aid of computers in the case of $n = 5$. Recently, in the case of n=5, and in some degrees, this problem was studied by many authors (see Mong-Sum [11], Phuc [15], Tin [31] and others).

In the present paper, we explicitly determine an admissible monomial basis of the $\mathbb{F}_2$-vector space $\mathcal{A}\mathcal{P}_6$ in the generic degree $6(2^s - 1) + 2 \cdot 2^s$, with s an arbitrary non-negative integer. The MAGMA computer algebra [35] was used to double-check these results.

As an application of the above results, we obtain the dimension results for the polynomial algebra $\mathcal{P}_n$ in degree $d = (n-1) \cdot (2^{n+u-1} - 1) + \ell \cdot 2^{n+u}$, where u is an arbitrary non-negative integer, $\ell = 13$, and $n = 7$.

One of the primary applications of the hit problem is in surveying a homomorphism proposed by Singer [19], which is from the homology of the Steenrod algebra to the subspace of $\mathcal{A}\mathcal{P}_n$ consisting of all the $GL_n(\mathbb{F}_2)$-invariant classes. Here, $GL_n(\mathbb{F}_2)$ is the general linear group over the field $\mathbb{F}_2$.

Recall that the general linear group $GL_n(\mathbb{F}_2)$ acts naturally on $\mathcal{P}_n$ by matrix substitution. Due to the fact that the two actions of $\mathcal{A}$ and $GL_n(\mathbb{F}_2)$ upon $\mathcal{P}_n$ commute with each other, there is an inherited action of $GL_n(\mathbb{F}_2)$ on $\mathcal{A}\mathcal{P}_n$. At the conclusion of this article, the behavior of the sixth Singer algebraic transfer in degree $2^{s+3} - 6$ is also discussed.

Next, in Section 2, we recall some auxiliary information on admissible monomials in $\mathcal{P}_n$. The main results are presented in Section 3. Finally, in the appendix, we provide an algorithm in MAGMA [35] to verify the dimension result of the main results of this paper.

2. Preliminaries

In this section, we review some important facts from Kameko [7], Singer [19], and Sum [24], which will be used in the following section.

We will denote by $\mathbb{N}_n = \{1, 2, \dots, n\}$ and

$$X_{\mathbb{J}} = X_{\{j_1, j_2, \dots, j_s\}} = \prod_{j \in \mathbb{N}_n \setminus \mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \dots, j_s\} \subset \mathbb{N}_n,$$

In particular, $X_{\mathbb{N}_n} = 1$, $X_{\emptyset} = x_1x_2\dots x_n$, $X_j = x_1 \dots \hat{x}_j \dots x_n$, $1 \leqslant j \leqslant n$, and $X := X_n \in \mathcal{P}_{n-1}$.

Let $\alpha_t(d)$ be the t-th coefficient in dyadic expansion of d. Then, $d = \sum_{t \geqslant 0} \alpha_t(d).2^t$ where $\alpha_t(d) \in \{0, 1\}$. Let $x = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \in \mathcal{P}_n$. Denote $\nu_j(x) = a_j, 1 \leqslant j \leqslant n$. Set

$$\mathbb{J}_t(x) = \{j \in \mathbb{N}_n : \alpha_t(\nu_j(x)) = 0\},$$

for $t \geqslant 0$. Then, we have $x = \prod_{t \geqslant 0} X_{\mathbb{J}_t(x)}^{2^t}$.

Definition 2.1 (**Weight vector-Exponent vector).** For a monomial x in $\mathcal{P}_n$, define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots), \quad \text{and} \quad \sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_n(x)),$$

where $\omega_i(x) = \sum_{1\leqslant j\leqslant n} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)},\ i \geqslant 1$. The sequences $\omega(x)$ and $\sigma(x)$ are respectively called the weight vector and the exponent vector of x.

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of non-negative integers. The sequence ω is called the weight vector if $\omega_i = 0$ for $i \gg 0$. Then, we define $\deg \omega = \sum_{i>0} 2^{i-1}\omega_i$. Denote by $\mathcal{P}_n(\omega)$ the subspace of $\mathcal{P}_n$ spanned by all monomials y such that $\deg y = \deg \omega$, $\omega(y) \leqslant \omega$, and by $\mathcal{P}_n^-(\omega)$ the subspace of $\mathcal{P}_n$ spanned by all monomials $y \in \mathcal{P}_n(\omega)$ such that $\omega(y) < \omega$.

Definition 2.2 (**Equivalence relations on $\mathcal{P}_n$).** Let $\mathcal{A}^+$ be an ideal of $\mathcal{A}$ generated by all Steenrod squares of positive degrees, and $u,\ v$ two polynomials of the same degree in $\mathcal{P}_n$. We define the equivalence relations " $\equiv$ " and " $\equiv_\omega$ " on $\mathcal{P}_n$ by stating that

(i) $u \equiv v$ if and only if $u - v \in \mathcal{A}^+\mathcal{P}_n$.

(ii) $u \equiv_\omega v$ if and only if $u, v \in \mathcal{P}_n(\omega)$ and $u - v \in (\mathcal{A}^+\mathcal{P}_n \cap \mathcal{P}_n(\omega) + \mathcal{P}_n^-(\omega))$.

Then, we have an $\mathbb{F}_2$-qoutient space of $\mathcal{P}_n$ by the equivalence relation " $\equiv_\omega$ " as follows:

$$\mathcal{A}\mathcal{P}_n(\omega) = \mathcal{P}_n(\omega)/((\mathcal{A}^+\mathcal{P}_n \cap \mathcal{P}_n(\omega)) + \mathcal{P}_n^-(\omega)).$$

If a polynomial u in $\mathcal{P}_n$ can be expressed as a finite sum $u = \sum_{i\geqslant 0} Sq^{2^i}(f_i)$ for suitable polynomials $f_i \in \mathcal{P}_n$, it is called it *hit*. That means u belongs to $\mathcal{A}^+\mathcal{P}_n$.

Let $u \in \mathcal{P}_n$, and let ω be a weight vector. We denote by $[u]$ the class in $\mathcal{A}\mathcal{P}_n$ represented by u. If u belongs to $\mathcal{P}_n(\omega)$, then denote by $[u]_\omega$ the class in $\mathcal{A}\mathcal{P}_n(\omega)$ represented by u.

Definition 2.3 (**Linear order on $\mathcal{P}_n$).** Let $u,\ v$ be monomials of the same degree in $\mathcal{P}_n$. We say that $u < v$ if one of the following holds:

(i) $\omega(u) < \omega(v)$;

(ii) $\omega(u) = \omega(v)$, and $\sigma(u) < \sigma(v)$.

Definition 2.4 (**Admissible monomial-Inadmissible monomial).** Let u be a monomial in $\mathcal{P}_n$. The monomial u is said to be inadmissible if there exist monomials $v_1, v_2, \ldots, v_m$ such that $v_i < u$ for $i = 1, 2, \ldots, m$ and $u - \sum_{i=1}^m v_i \in \mathcal{A}^+\mathcal{P}_n$. If u is not inadmissible monomial, we say it is admissible.

For instance, since $x_1^2x_2 = Sq^1(x_1x_2) + x_1x_2^2$, and $\sigma(x_1x_2^2) < \sigma(x_1^2x_2)$, it follows that $x_1^2x_2$ is inadmissible in $\mathcal{P}_2$. Moreover, the monomials $x_1^3; x_2^3$ are admissible in $\mathcal{P}_2$.

It is crucial to note that the set of all admissible monomials of degree d in $\mathcal{P}_n$ is a minimal set of $\mathcal{A}$-generators for $\mathcal{P}_n$ in degree d. And therefore, $(\mathcal{A}\mathcal{P}_n)_d$ is an $\mathbb{F}_2$-vector space with a basis consisting of all the classes represent by the elements in $(\mathcal{P}_n)_d$.

Definition 2.5 (**Strictly inadmissible monomial).** Let u be a monomial in $\mathcal{P}_n$. We say u is strictly inadmissible if there exist monomials $v_1, v_2, \ldots, v_m$ in $\mathcal{P}_n$ such that $v_j < u$, for $j = 1, 2, \ldots, m$ and $u = \sum_{j=1}^m v_j + \sum_{i=1}^{2^s-1} Sq^i(f_i)$ with $s = \max\{k : \omega_k(u) > 0\}$, $f_i \in \mathcal{P}_n$.

Observe that if u is strictly inadmissible monomial, then it is inadmissible monomial, as defined by the definitions 2.4, and 2.5. In general, the inverse is not true.

For example, the monomial $x_1x_2^2x_3^2x_4^2x_5^2x_6$ is inadmissible, but it is not strictly inadmissible in $\mathcal{P}_6$.

Theorem 2.6 (**Kameko [7], Sum [24]).** *Let u, v, w be monomials in $\mathcal{P}_n$ such that $\omega_t(u) = 0$ for $t > k > 0$, $\omega_r(w) \neq 0$ and $\omega_t(w) = 0$ for $t > r > 0$. Then,*

(i) *uw^{2^k} is inadmissible if w is inadmissible.*
(ii) *wv^{2^r} is strictly inadmissible if w is strictly inadmissible.*

Definition 2.7 (Minimal spike monomial). Let $z = x_1^{d_1}x_2^{d_2}\ldots x_n^{d_n}$ in $\mathcal{P}_n$. The monomial z is called a spike if $d_j = 2^{t_j}-1$ for t_j a non-negative integer and $j = 1, 2, \ldots, n$. Moreover, z is called the minimal spike, if it is a spike such that $t_1 > t_2 > \ldots > t_{r-1} \geqslant t_r > 0$ and $t_j = 0$ for $j > r$.

The following is a Singer's criterion on the hit monomials in $\mathcal{P}_n$.

Theorem 2.8 (Singer [19]). *Assume that z is the minimal spike of degree d in $\mathcal{P}_n$, and $u \in (\mathcal{P}_n)_d$ satisfying the condition $\mu(d) \leqslant n$. If $\omega(u) < \omega(z)$, then u is hit.*

From now on, let us denote by $\mathcal{C}_n^{\otimes}(d)$ the set of all admissible monomials of degree d in $\mathcal{P}_n$. The cardinal of a set U is denoted by $|U|$. The $\mathcal{A}$-submodules of $\mathcal{P}_n$ that spanned all the monomials $x_1^{d_1}x_2^{d_2}\ldots x_n^{d_n}$ such that $d_1\ldots d_n = 0$, and $d_1\ldots d_n > 0$, respectively, will be denoted by $\mathcal{P}_n^0$ and $\mathcal{P}_n^+$. It is easy to check that $\mathcal{P}_n^0$ and $\mathcal{P}_n^+$ are $\mathcal{A}$-submodules of $\mathcal{P}_n$.

Moreover, we have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces:

$$\mathcal{A}\mathcal{P}_n = \mathcal{A}\mathcal{P}_n^0 \oplus \mathcal{A}\mathcal{P}_n^+$$

where $\mathcal{A}\mathcal{P}_n^0 := \mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n^0$, $\mathcal{A}\mathcal{P}_n^+ := \mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_n^+$.

3. Main Results

First, we explicitly determine an admissible monomial basis of the $\mathbb{F}_2$-vector space $\mathcal{A}\mathcal{P}_6$ in the generic degree $m_s := 6(2^s-1) + 2\cdot 2^s$, with s an arbitrary non-negative integer.

For $s = 0$, then $m_0 = 2$. It is easy to see that the set $\{[x_ix_j] : 1 \leqslant i < j \leqslant 6\}$ is a basis of $\mathbb{F}_2$-vector space $(\mathcal{A}\mathcal{P}_6)_{m_0}$. Consequently, $|\mathcal{C}_6^{\otimes}(2)| = 15$.

For $s = 1$, then $m_1 = 6(2^1-1) + 2\cdot 2^1$. Set $\widetilde{\omega_1} := (2, 4)$, $\widetilde{\omega_2} := (4, 1, 1)$, and $\widetilde{\omega_3} := (4, 3)$. Then, the $\mathbb{F}_2$-vector space $(\mathcal{A}\mathcal{P}_6)_{6(2^1-1)+2\cdot 2^1}$ is determined as follows:

Since Kameko's homomorphism $(\widetilde{Sq^0_*})_{(6;10)}$ is an $\mathbb{F}_2$-epimorphism, it follows that

$$(\mathcal{A}\mathcal{P}_6)_{m_1} \cong (\mathcal{A}\mathcal{P}_6^0)_{m_1} \bigoplus (\mathrm{Ker}(\widetilde{Sq^0_*})_{(6;10)} \cap (\mathcal{A}\mathcal{P}_6^+)_{m_1}) \bigoplus \mathrm{Im}(\widetilde{Sq^0_*})_{(6;10)}$$

Consider the homomorphism $\mathcal{T}_t : \mathcal{P}_{n-1} \to \mathcal{P}_n$, for $1 \leqslant t \leqslant n$ by substituting:

$$\mathcal{T}_t(x_k) = \begin{cases} x_k, & \text{if } 1 \leqslant k \leqslant t-1, \\ x_{k+1}, & \text{if } t \leqslant k < n. \end{cases}$$

We have the following theorem.

Theorem 3.1. *Let us denote by $\mathcal{D}_0^{\otimes 6} := \{b :\ b \in \bigcup_{t=1}^6 \mathcal{T}_t(\mathcal{C}_5^{\otimes}(6(2^1-1)+2\cdot 2^1))\}$, and set $\mathcal{D}_{Im}^{\otimes 6} := \{[c] :\ c = \Phi_6(u),\ for\ all\ \ u \in \mathcal{C}_5^{\otimes}(6(2^0-1)+2\cdot 2^0)\}$, where the map $\Phi_n : \mathcal{P}_n \to \mathcal{P}_n$ is a homomorphism determined by $\Phi_n(x) = \prod_{i=1}^n x_ix^2,\ \ x \in \mathcal{P}_n$. Then*

(i) *We have $|\mathcal{D}_0^{\otimes 6}| = 880$, and the set $\{[b] :\ b \in \mathcal{D}_0^{\otimes 6}\}$ is a basis of the $\mathbb{F}_2$-vector space $(\mathcal{A}\mathcal{P}_6^0)_{6(2^1-1)+2\cdot 2^1}$. This implies that $(\mathcal{A}\mathcal{P}_6^0)_{6(2^1-1)+2\cdot 2^1}$ has dimension 880.*

(ii) *The space $\mathrm{Im}(\widetilde{Sq^0_*})_{(6;10)}$ is isomorphic to a subspace of $(\mathcal{A}\mathcal{P}_6)_{10}$ generated by all the classes $[c]$ of $\mathcal{D}_{Im}^{\otimes 6}$. Consequently, $|\mathcal{D}_{Im}^{\otimes 6}| = \dim(\mathrm{Im}(\widetilde{Sq^0_*})_{(6;10)}) = 15$.*

Proof. We begin by proving Part (i) of the above theorem. Recall that Tin [29] showed that the space $(\mathcal{A}\mathcal{P}_5)_{10}$ is an $\mathbb{F}_2$-vector space of dimension 280 with a basis consisting of all the classes represented by the monomials $a_j,\ 1 \leqslant j \leqslant 280$. Consequently, $|\mathcal{C}_5^{\otimes}(10)| = 280$.

Using the above result, an easy computation shows that

$$\left|\bigcup_{t=1}^{6} \mathcal{T}_t(\mathcal{C}_5^{\otimes}(10))\right| = \left|\bigcup_{t=1}^{6} \mathcal{T}_t(a_j), 1 \leqslant j \leqslant 280\right| = 880,$$

and the set $\{b : b \in \bigcup_{t=1}^{6} \mathcal{T}_t(a_j), 1 \leqslant j \leqslant 280\}$ is a minimal set of generators for $\mathcal{A}$-modules $\mathcal{P}_6^0$ in degree ten. Hence, $\dim(\mathcal{A}\mathcal{P}_6^0)_{10} = 880$.

The proof of Part (ii) of the above theorem is straightforward. Indeed, since $\Phi_6 : \mathcal{P}_6 \to \mathcal{P}_6$ is the homomorphism defined by $\Phi_6(x) = \prod_{i=1}^{6} x_i x^2$ for $x \in \mathcal{P}_6$, one obtains

$$\left|\{[c] :\ c = \Phi_6(x),\ \text{for all } x \in \mathcal{C}_5^{\otimes}(5(2^0-1)+2\cdot 2^0)\}\right| = \left|\mathcal{D}_{Im}^{\otimes 6}\right| = 15.$$

Moreover, combining the above results with the fact that $(\widetilde{Sq}_*^0)_{(6;10)}$ is an epimorphism, the space $\mathrm{Im}(\widetilde{Sq}_*^0)_{(6;10)}$ turns out to be isomorphic to a subspace of $(\mathcal{A}\mathcal{P}_6)_{10}$ generated by all the classes $[c]$ of $\mathcal{D}_{Im}^{\otimes 6}$. The theorem is proved. $\square$

Theorem 3.2. *Let $\mathcal{C}_6^{\otimes >}(\omega)$ be the set of all admissible monomials in $\mathcal{P}_6^+(\omega)$, and set $\mathcal{A}\mathcal{P}_6^+(\omega) := \mathcal{A}\mathcal{P}_6(\omega) \cap \mathcal{A}\mathcal{P}_6^+$. Then, the space $\mathrm{Ker}(\widetilde{Sq}_*^0)_{(6;10)} \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^1-1)+2\cdot 2^1}$ is 50-dimensional. Moreover, we have*

$$\mathrm{Ker}(\widetilde{Sq}_*^0)_{(6;10)} \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^1-1)+2\cdot 2^1} \cong \bigoplus_{k=1}^{3} \mathcal{A}\mathcal{P}_6^+(\widetilde{\omega_k}),$$

and

$$\dim\left(\mathcal{A}\mathcal{P}_6^+(\widetilde{\omega_k})\right) = \left|\mathcal{C}_6^{\otimes >}(\widetilde{\omega_k})\right| = \begin{cases} 4, & \text{if } k = 1,\\ 10, & \text{if } k = 2,\\ 36, & \text{if } k = 3.\end{cases}$$

Proof. The idea of the proof of the above theorem is to explicitly determine an admissible monomial basis of the $\mathbb{F}_2$-vector space $\mathrm{Ker}(\widetilde{Sq}_*^0)_{(6;210)} \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^1-1)+2\cdot 2^1}$.

For a weight vector ω of degree d. We set $\mathcal{C}_6^{\otimes}(\omega) := \mathcal{C}_6^{\otimes}(d) \cap \mathcal{P}_6(\omega)$. It is easy to see that $\mathcal{C}_6^{\otimes}(d) = \bigcup_{\deg \omega = d} \mathcal{C}_6^{\otimes}(\omega)$. Putting

$$Q\mathcal{P}_6^{\omega} := \langle \{[x] \in \mathcal{A}\mathcal{P}_6 : x \text{ is admissible and } \omega(x) = \omega\}\rangle.$$

It is straightforward to check that the map $\mathcal{A}\mathcal{P}_6(\omega) \longrightarrow Q\mathcal{P}_6^{\omega},\ \ [x]_\omega \longrightarrow [x]$ is an isomorphism of $\mathbb{F}_2$-vector spaces. And therefore, $Q\mathcal{P}_6^{\omega} \subset \mathcal{A}\mathcal{P}_6$ can be used to identify the vector space $\mathcal{A}\mathcal{P}_6(\omega)$. From this, one obtains

$$(\mathcal{A}\mathcal{P}_6)_d = \bigoplus_{\deg \omega = d} Q\mathcal{P}_6^{\omega} \cong \bigoplus_{\deg \omega = d} \mathcal{A}\mathcal{P}_6(\omega).$$

From this, it follows that $(\mathcal{A}\mathcal{P}_6^+)_{6(2^1-1)+2\cdot 2^1} \cong \bigoplus_{\deg \omega = 10} \mathcal{A}\mathcal{P}_6^+(\omega)$.

Suppose that x is an admissible monomial of degree ten in $\mathcal{P}_6^+$ such that $[x]$ belongs to $\mathrm{Ker}(\widetilde{Sq}_*^0)_{(6;10)}$. Observe that $z = x_1^7 x_2^3$ is the minimal spike of degree 10 in $\mathcal{P}_6$ and $\omega(z) = (2,2,1)$. Using Theorem 2.8, we obtain $\omega_1(x) \geqslant \omega_1(z) = 2$. Since the degree of (x) is even, one get $\omega_1(x) = 2$, or $\omega_1(x) = 4$, or $\omega_1(x) = 6$.

Since $\omega_1(x) = 6$, $x = X_{\emptyset} y^2$ with y a monomial of degree two in $\mathcal{P}_6$. Since x is admissible, by Theorem 2.6, it shows that y is also admissible, and $[y] \neq 0$. Hence, $[y] = (\widetilde{Sq}_*^0)_{(6;10)}([x]) \neq 0$. This contradicts the fact that $[x] \in \mathrm{Ker}(\widetilde{Sq}_*^0)_{(6;10)}$.

Since $\omega_1(x) = 4$, $x = X_{\{i,j\}} u^2$, with u an admissible monomial of degree three in $\mathcal{P}_6$, and $1 \leqslant i < j \leqslant 6$. It is easy to see that either $\omega(u) = (3,0)$ or $\omega(u) = (1,1)$. Hence, $\omega(x) = (4,3,0)$ or $\omega(x) = (4,1,1)$.

Since $\omega_1(x) = 2$, $x = x_i x_j v^2$, with v an admissible monomial of degree four in $\mathcal{P}_6$ and $1 \leqslant i < j \leqslant 6$. By Theorem 2.6, and $x \in \mathcal{P}_6^+$ it implies that $\omega(x) = (2,4)$.

From the above results, one obtains

$$(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;10)} \cap (\mathcal{A}\mathcal{P}_6^+)_{10}) = \bigoplus_{k=1}^{3} \mathcal{A}\mathcal{P}_6^+(\widetilde{\omega_{(k)}}).$$

Now, we explicitly determining all admissible monomials in $\mathcal{P}_6^+(\widetilde{\omega_{(k)}})$, where $k \in \{1,2,3\}$. The proof is divided into three parts:

Case 1. Consider the weight vector $\omega = \widetilde{\omega_1}$. Assume that x is an admissible monomial in $\mathcal{P}_6$ such that $\omega(x) = \widetilde{\omega_1}$, then $x = x_ix_jy^2$, where $y \in \mathcal{C}_6^{\otimes}(4)$, and $1 \leqslant i < j \leqslant 6$. We set $\mathcal{D}_6^1 := \{x_ix_j.y^2 : \omega(y) = (2,4),\ 1 \leqslant i < j \leqslant 6\} \ \cap\ \mathcal{P}_6^+$. It is easy to see that $\mathrm{Span}\{\mathcal{D}_6^1\} = \mathcal{P}_6^+(\widetilde{\omega_1})$, and if $u \in \mathcal{D}_6^1$ then $u = x_ix_jx_k^2x_\ell^2x_m^2x_n^2$, where (i,j,k,ℓ,m,n) is an arbitrary permutation of $(1,2,3,4,5,6)$. Using the Cartan formula, we have

$$\mathcal{Y} = x_1x_2^2x_3^2x_4^2x_5^2x_6 = Sq^4(x_1x_2x_3x_4x_5x_6) + Sq^1(\mathcal{Y}_1) + \text{smaller than},$$

where $\mathcal{Y}_1 = x_1^2x_2^2x_3^2x_4x_5x_6 + x_1^2x_2^2x_3x_4^2x_5x_6 + x_1^2x_2x_3x_4x_5^2x_6^2 + x_1^2x_2x_3^2x_4^2x_5x_6$. From this, the monomial $\mathcal{Y}$ is inadmissible.

Clearly, every monomial $x_1^2x_jx_kx_\ell^2x_m^2x_n^2$ is an inadmissible (more precisely by Sq^1), with (j,k,ℓ,m,n) an arbitrary permutation of $(2,3,4,5,6)$.

From the aforementioned results, $\mathcal{P}_6^+(\widetilde{\omega_1})$ is generated by the following four elements:

$$c_1 = x_1x_2^2x_3^2x_4^2x_5x_6^2, c_2 = x_1x_2^2x_3^2x_4x_5^2x_6^2, c_3 = x_1x_2^2x_3x_4^2x_5^2x_6^2, c_4 = x_1x_2x_3^2x_4^2x_5^2x_6^2.$$

We then show that the vectors $[c_i], 1 \leqslant i \leqslant 4$, are linearly independent in $\mathcal{A}\mathcal{P}_6$. Assume that there is a linear relation

$$\mathcal{S}_1 = \sum_{1\leqslant i\leqslant 4} \gamma_ic_i \equiv 0, \quad \text{with } \gamma_i \in \mathbb{F}_2. \tag{3.1}$$

For $1 \leqslant k < j \leqslant 6$, consider the homomorphism $\varphi_{(k;j)} : \mathcal{P}_6 \to \mathcal{P}_5$ by substituting:

$$\varphi_{(k;j)}(x_m) = \begin{cases} x_m, & \text{if } 1 \leqslant m \leqslant k-1,\\ x_{j-1}, & \text{if } m = k,\\ x_{m-1}, & \text{if } k < m \leqslant 6.\end{cases}$$

It is simple to figure out if these homomorphisms are $\mathcal{A}$-modules homomorphisms.

Using the result in Tin [29], acting the homomorphisms $\varphi_{(5;6)}$ on both sides of (3.1), one obtains $\gamma_2 = \gamma_4 = 0$.

So, the relation (3.1) becomes

$$\mathcal{S}_1 = \gamma_1c_1 + \gamma_3c_3 \equiv 0, \tag{3.2}$$

Similarly, acting the homomorphisms $\varphi_{(2;5)}$ on both sides of (3.2), we obtains $\gamma_1 = \gamma_3 = 0$.

Hence, the set $\{[c_i] : 1 \leqslant i \leqslant 4\}$ is a basis of the $\mathbb{F}_2$-vector space $\mathcal{A}\mathcal{P}_6^+(\widetilde{\omega_1})$.

Case 2. Consider the weight vector $\omega = \widetilde{\omega_2}$. Based on similar arguments, we also see that $\mathrm{Span}\{\mathcal{D}_6^2\} = \mathcal{P}_6^+(\widetilde{\omega_2})$, where

$$\mathcal{D}_6^2 := \{x_ix_jx_kx_\ell.u^2 : \omega(u) = (4,1,1),\ 1 \leqslant i < j < k < \ell \leqslant 6\} \ \cap\ \mathcal{P}_6^+.$$

Observe, if $u \in \mathcal{D}_6^2$ then $u = x_ix_jx_kx_\ell x_m^2x_n^4$, where (i,j,k,ℓ,m,n) is an arbitrary permutation of $(1,2,3,4,5,6)$. It is clear that the monomials $x_1^2x_jx_kx_\ell x_mx_n^4$ and $x_1^4x_jx_kx_\ell x_mx_n^2$ are inadmissible by Sq^1, with (j,k,ℓ,m,n) an arbitrary permutation of $(2,3,4,5,6)$.

On the one hand, for any $1 < m < n \leqslant 6$, we have

$$x_ix_jx_kx_\ell x_m^4x_n^2 = Sq^2(x_ix_jx_kx_\ell x_m^2x_n^2) + Sq^1(x_i^2x_jx_kx_\ell x_m^2x_n^2) + \text{smaller than},$$

where (i,j,k,ℓ,m,n) is a permutation of $(1,2,3,4,5,6)$. It shows that the monomials $x_ix_jx_kx_\ell x_m^4x_n^2$ are inadmissible for $m<n$.

According to the aforementioned results, $\mathcal{P}_6^+(\widetilde{\omega_2})$ is generated by 10 elements c_i, with $5\leqslant i\leqslant 14$ as follows:

5. $x_1x_2x_3x_4x_5^2x_6^4$	6. $x_1x_2x_3x_4^2x_5x_6^4$	7. $x_1x_2x_3x_4^2x_5^4x_6$	8. $x_1x_2x_3^2x_4x_5x_6^4$
9. $x_1x_2x_3^2x_4x_5^4x_6$	10. $x_1x_2x_3^2x_4^4x_5x_6$	11. $x_1x_2^2x_3x_4x_5x_6^4$	12. $x_1x_2^2x_3x_4x_5^4x_6$
13. $x_1x_2^2x_3x_4^4x_5x_6$	14. $x_1x_2^2x_3^4x_4x_5x_6$		

We then show that the vectors $[c_i], 5\leqslant i\leqslant 14$, are linearly independent in $\mathcal{A}\mathcal{P}_6$. Assume that there is a linear relation

$$\mathcal{S}_2=\sum_{1\leqslant i\leqslant 10}\gamma_ic_{i+4}\equiv 0,\quad \text{with } \gamma_i\in\mathbb{F}_2. \tag{3.3}$$

We recall from [29] the finding that $(\mathcal{A}\mathcal{P}_5^+)_{10}$ is an $\mathbb{F}_2$-vector space of dimension 50 with a basis consisting of all the classes represented by the following admissible monomials:

$s_1=x_1x_2x_3^2x_4^2x_5^4$	$s_2=x_1x_2x_3^2x_4^4x_5^2$	$s_3=x_1x_2^2x_3x_4^2x_5^4$	$s_4=x_1x_2^2x_3x_4^4x_5^2$
$s_5=x_1x_2^2x_3^4x_4x_5^2$	$s_6=x_1x_2^2x_3^2x_4^2x_5^3$	$s_7=x_1x_2^2x_3^2x_4^3x_5^2$	$s_8=x_1x_2^2x_3^3x_4^2x_5^2$
$s_9=x_1x_2^3x_3^2x_4^2x_5^2$	$s_{10}=x_1^3x_2x_3^2x_4^2x_5^2$	$s_{11}=x_1x_2x_3x_4x_5^6$	$s_{12}=x_1x_2x_3x_4^2x_5^5$
$s_{13}=x_1x_2x_3x_4^3x_5^4$	$s_{14}=x_1x_2x_3x_4^6x_5$	$s_{15}=x_1x_2x_3^2x_4x_5^5$	$s_{16}=x_1x_2x_3^2x_4^5x_5$
$s_{17}=x_1x_2x_3^3x_4x_5^4$	$s_{18}=x_1x_2x_3^3x_4^4x_5$	$s_{19}=x_1x_2x_3^6x_4x_5$	$s_{20}=x_1x_2^2x_3x_4x_5^5$
$s_{21}=x_1x_2^2x_3x_4^5x_5$	$s_{22}=x_1x_2^2x_3^5x_4x_5$	$s_{23}=x_1x_2^3x_3x_4x_5^4$	$s_{24}=x_1x_2^3x_3x_4^4x_5$
$s_{25}=x_1x_2^3x_3^4x_4x_5$	$s_{26}=x_1x_2^6x_3x_4x_5$	$s_{27}=x_1^3x_2x_3x_4x_5^4$	$s_{28}=x_1^3x_2x_3x_4^4x_5$
$s_{29}=x_1^3x_2x_3^4x_4x_5$	$s_{30}=x_1^3x_2^4x_3x_4x_5$	$s_{31}=x_1x_2x_3^2x_4^3x_5^3$	$s_{32}=x_1x_2x_3^3x_4^2x_5^3$
$s_{33}=x_1x_2x_3^3x_4^3x_5^2$	$s_{34}=x_1x_2^2x_3x_4^3x_5^3$	$s_{35}=x_1x_2^2x_3^3x_4x_5^3$	$s_{36}=x_1x_2^2x_3^3x_4^3x_5$
$s_{37}=x_1x_2^3x_3x_4^2x_5^3$	$s_{38}=x_1x_2^3x_3x_4^3x_5^2$	$s_{39}=x_1x_2^3x_3^2x_4x_5^3$	$s_{40}=x_1x_2^3x_3^2x_4^3x_5$
$s_{41}=x_1x_2^3x_3^3x_4x_5^2$	$s_{42}=x_1x_2^3x_3^3x_4^2x_5$	$s_{43}=x_1^3x_2x_3x_4^2x_5^3$	$s_{44}=x_1^3x_2x_3x_4^3x_5^2$
$s_{45}=x_1^3x_2x_3^2x_4x_5^3$	$s_{46}=x_1^3x_2x_3^2x_4^3x_5$	$s_{47}=x_1^3x_2x_3^3x_4x_5^2$	$s_{48}=x_1^3x_2x_3^3x_4^2x_5$
$s_{49}=x_1^3x_2^3x_3x_4x_5^2$	$s_{50}=x_1^3x_2^3x_3x_4^2x_5$		

By direct computations, apply the homomorphism $\varphi_{(3;5)}$ to (3.3), one obtains

$$\begin{aligned}\varphi_{(3;5)}(\mathcal{S}_2)\equiv{}&\gamma_1s_{13}+\gamma_2s_1+\gamma_3s_{16}+\gamma_4s_{13}+\gamma_5s_{14}+\gamma_6(s_4+s_7)\\&+\gamma_7s_3+\gamma_8s_{21}+\gamma_9(s_1+s_2+s_5)+\gamma_{10}s_{21}\equiv 0.\end{aligned}$$

From this, it implies $\gamma_1=\gamma_2=\gamma_3=\gamma_4=\gamma_5=\gamma_6=\gamma_7=\gamma_9=0$.

Using the above result, acting the homomorphism $\varphi_{(4;6)}$ on both sides of (3.3), one gets $\gamma_8=\gamma_{10}=0$.

Hence, the set $\{[c_i]:5\leqslant i\leqslant 14\}$ is a basis of the $\mathbb{F}_2$-vector space $\mathcal{A}\mathcal{P}_6^+(\widetilde{\omega_2})$.

Case 3. Consider the weight vector $\omega=\widetilde{\omega_3}$. Let us denote by

$$\mathcal{D}_6^3:=\{x_ix_jx_kx_\ell.v^2:\omega(v)=(4,3),\ 1\leqslant i<j<k<\ell\leqslant 6\}\ \cap\ \mathcal{P}_6^+.$$

We also have $\mathrm{Span}\{\mathcal{D}_6^3\}=\mathcal{P}_6^+(\widetilde{\omega_3})$, and if $x\in\mathcal{D}_6^3$ then $x=x_ix_jx_kx_\ell^2x_m^2x_n^3$, with (i,j,k,ℓ,m,n) an arbitrary permutation of $(1,2,3,4,5,6)$.

The monomials $x_1^2x_jx_kx_\ell x_m^2x_n^3$ and $x_1^3x_2^2x_kx_\ell x_mx_n^2$ are clearly inadmissible by Sq^1. And therefore, $\mathcal{P}_6^+(\widetilde{\omega_3})$ is generated by 36 elements c_i, with $15\leqslant i\leqslant 50$ as follows:

15. $x_1^3x_2^1x_3^1x_4^1x_5^2x_6^2$	16. $x_1^1x_2^3x_3^1x_4^1x_5^2x_6^2$	17. $x_1^1x_2^1x_3^3x_4^1x_5^2x_6^2$	18. $x_1^1x_2^1x_3^1x_4^3x_5^2x_6^2$
19. $x_1^3x_2^1x_3^1x_4^2x_5^1x_6^2$	20. $x_1^1x_2^3x_3^1x_4^2x_5^1x_6^2$	21. $x_1^1x_2^1x_3^3x_4^2x_5^1x_6^2$	22. $x_1^1x_2^1x_3^1x_4^2x_5^3x_6^2$
23. $x_1^3x_2^1x_3^1x_4^2x_5^2x_6^1$	24. $x_1^1x_2^3x_3^1x_4^2x_5^2x_6^1$	25. $x_1^1x_2^1x_3^3x_4^2x_5^2x_6^1$	26. $x_1^1x_2^1x_3^1x_4^2x_5^2x_6^3$
27. $x_1^3x_2^1x_3^2x_4^1x_5^1x_6^2$	28. $x_1^1x_2^3x_3^2x_4^1x_5^1x_6^2$	29. $x_1^1x_2^1x_3^2x_4^3x_5^1x_6^2$	30. $x_1^1x_2^1x_3^2x_4^1x_5^3x_6^2$
31. $x_1^3x_2^1x_3^2x_4^1x_5^2x_6^1$	32. $x_1^1x_2^3x_3^2x_4^1x_5^2x_6^1$	33. $x_1^1x_2^1x_3^2x_4^3x_5^2x_6^1$	34. $x_1^1x_2^1x_3^2x_4^1x_5^2x_6^3$
35. $x_1^3x_2^1x_3^2x_4^2x_5^1x_6^1$	36. $x_1^1x_2^3x_3^2x_4^2x_5^1x_6^1$	37. $x_1^1x_2^1x_3^2x_4^2x_5^3x_6^1$	38. $x_1^1x_2^1x_3^2x_4^2x_5^1x_6^3$
39. $x_1^1x_2^2x_3^3x_4^1x_5^1x_6^2$	40. $x_1^1x_2^2x_3^1x_4^3x_5^1x_6^2$	41. $x_1^1x_2^2x_3^1x_4^1x_5^3x_6^2$	42. $x_1^1x_2^2x_3^3x_4^1x_5^2x_6^1$
43. $x_1^1x_2^2x_3^1x_4^3x_5^2x_6^1$	44. $x_1^1x_2^2x_3^1x_4^1x_5^2x_6^3$	45. $x_1^1x_2^2x_3^3x_4^2x_5^1x_6^1$	46. $x_1^1x_2^2x_3^1x_4^2x_5^3x_6^1$
47. $x_1^1x_2^2x_3^1x_4^2x_5^1x_6^3$	48. $x_1^1x_2^2x_3^2x_4^3x_5^1x_6^1$	49. $x_1^1x_2^2x_3^2x_4^1x_5^3x_6^1$	50. $x_1^1x_2^2x_3^2x_4^1x_5^1x_6^3$

Assume that there is a linear relation

$$\mathcal{S}_3 = \sum_{1\leqslant i\leqslant 36}\gamma_i c_{i+14} \equiv 0, \quad \text{with } \gamma_i \in \mathbb{F}_2. \tag{3.4}$$

Using the same method as above, we explicitly compute $\varphi_{(k;j)}(\mathcal{S}_3)$ in terms of $s_i, 1 \leqslant i \leqslant 50$ in $\mathcal{P}_5(\mathrm{mod}(\mathcal{A}^+\mathcal{P}_5))$.

From the relations $\varphi_{(k;j)}(\mathcal{S}_3) \equiv 0$ with $1 \leqslant k < j \leqslant 6$, one gets $\gamma_i = 0$ for all $15 \leqslant i \leqslant 50$. That means, the vectors $[c_i], 15 \leqslant i \leqslant 50$, are linearly independent in $\mathcal{A}\mathcal{P}_6$.

Hence, $\{[c_i],\ 15 \leqslant i \leqslant 50\}$ is a basis of $\mathcal{A}\mathcal{P}_6^+(\widetilde{\omega_3})$. The theorem is proved. □

From the above results, we obtain the following corollary.

Corollary 3.3. *There exist exactly* 945 *admissible monomials of degree ten in* $\mathcal{P}_6$. *This implies that the space* $(\mathcal{A}\mathcal{P}_6)_{6(2^1-1)+2\cdot 2^1}$ *has dimension* 945.

It should be noted that Mothebe-Kaelo-Ramatebele [12] utilized a different method to verify the dimension result of the vector space $(\mathcal{A}\mathcal{P}_6)_{6(2^1-1)+2\cdot 2^1}$.

Next, we consider the degree $m_s := 6(2^s-1)+2\cdot 2^s$, for any $s \geqslant 2$. For $s = 2$, we have $m_2 = 6(2^2-1)+2\cdot 2^2$. Since $(\widetilde{Sq}^0_*)_{(6;26)}$ is a $\mathbb{F}_2$-epimorphism, it shows that

$$(\mathcal{A}\mathcal{P}_6)_{m_2} \cong (\mathcal{A}\mathcal{P}_6^0)_{m_2} \bigoplus (\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;26)} \cap (\mathcal{A}\mathcal{P}_6^+)_{m_2}) \bigoplus \mathrm{Im}(\widetilde{Sq}^0_*)_{(6;26)}$$

Then, the $\mathbb{F}_2$-vector space $(\mathcal{A}\mathcal{P}_6)_{6(2^2-1)+2\cdot 2^2}$ is explicitly determined by the following theorem.

Theorem 3.4. *Let us denote by* $\mathcal{A}\mathcal{P}_n^+(\omega) = \mathcal{A}\mathcal{P}_n(\omega) \cap \mathcal{A}\mathcal{P}_n^+$, $\widetilde{\omega_{[1]}} := (4,3,4)$, $\widetilde{\omega_{[2]}} := (4,3,2,1)$, $\widetilde{\omega_{[3]}} := (4,5,3)$, $\widetilde{\omega_{[4]}} := (4,5,1,1)$. *Then*

(i) *The space* $\mathrm{Im}(\widetilde{Sq}^0_*)_{(6;26)}$ *is isomorphic to a subspace of* $(\mathcal{A}\mathcal{P}_6)_{6(2^2-1)+2\cdot 2^2}$ *generated by all the classes represented by the admissible monomials of the form* $\Phi_6(u)$ *for all* $u \in \mathcal{C}_6^{\otimes}(6(2^1-1)+2\cdot 2^1)$. *Consequently,* $\dim \mathrm{Im}(\widetilde{Sq}^0_*)_{(6;26)} = 945$.

(ii) *The set* $\{[e_i] :\ e_i \in \bigcup_{t=1}^6 \mathcal{T}_t(\mathcal{C}_5^{\otimes}(26)), 1 \leqslant i \leqslant 5184\}$ *is a basis of the* $\mathbb{F}_2$*-vector space* $(\mathcal{A}\mathcal{P}_6^0)_{6(2^2-1)+2\cdot 2^2}$. *This implies that* $(\mathcal{A}\mathcal{P}_6^0)_{6(2^2-1)+2\cdot 2^2}$ *has dimension* 5184.

(iii) *We have* $(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;26)} \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^2-1)+2\cdot 2^2}) \cong \bigoplus_{i=1}^4 \mathcal{A}\mathcal{P}_6^+(\widetilde{\omega_{[i]}})$. *Moreover, the vector space* $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;26)} \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^2-1)+2\cdot 2^2}$ *is* 3636*-dimensional.*

Proof. Since Kameko's homomorphism

$$(\widetilde{Sq}^0_*)_{(6;26)} : (\mathcal{A}\mathcal{P}_6)_{6(2^2-1)+2\cdot 2^2} \to (\mathcal{A}\mathcal{P}_6)_{6(2^1-1)+2\cdot 2^1}$$

is an $\mathbb{F}_2$-epimorphism, it shows that the proof of Part (i) of the above theorem is straightforward. It is an immediate consequence of Corollary 3.3. More specifically, we see that the space $\mathrm{Im}(\widetilde{Sq}^0_*)_{(6;26)}$ turns out to be isomorphic to a subspace of $(\mathcal{A}\mathcal{P}_6)_{26}$ generated by all the classes $[e]$ of $\mathcal{E}^{\otimes 6}_{Im}$, where

$$\mathcal{E}^{\otimes 6}_{Im} = \{[e] :\ e = \Phi_6(u),\ \text{for all } u \in \mathcal{C}_5^{\otimes}(5(2^1-1)+2\cdot 2^1)\}.$$

Hence, $\dim \mathrm{Im}(\widetilde{Sq}^0_*)_{(6;26)} = \left|\mathcal{E}^{\otimes 6}_{Im}\right| = 945$.

Next, we prove Part (ii) of the above theorem. Recall that, Wood-Walker [33] proved that the space $\mathcal{A}\mathcal{P}_n$ has dimension $2^{\binom{n}{2}}$ in degree $d(n) = 2^n - n - 1$. For $n = 5$, we have $d(5) = 2^5 - 5 - 1 = 26$, and therefore, $\dim(\mathcal{A}\mathcal{P}_5)_{26} = 2^{\binom{5}{2}} = 1024$.

Assume that the set $\mathcal{C}_5^{\otimes}(26) = \{u_i \in (\mathcal{P}_5)_{26} : 1 \leqslant i \leqslant 1024\}$. A simple calculation reveals that
$$\left|\left\{\bigcup_{t=1}^{6}\mathcal{T}_t(u_i) :\ 1 \leqslant i \leqslant 1024\right\}\right| = 5184,$$
and the set
$$\left\{v :\ v \in \bigcup_{t=1}^{6}\mathcal{T}_t(u_i), 1 \leqslant j \leqslant 1024\right\}$$
is a minimal set of generators for $\mathcal{A}$-modules $\mathcal{P}_6^0$ in degree twenty-six.

Therefore, $\dim(\mathcal{A}\mathcal{P}_6^0)_{6(2^2-1)+2\cdot 2^2} = 5184$. The second part has been established.

Finally, we prove Part (iii) of the theorem. For a weight vector ω of degree d, we set $\mathcal{C}_n^{\otimes}(\omega) := \mathcal{C}_n^{\otimes}(d) \cap \mathcal{P}_n(\omega)$. Then, one has $\mathcal{C}_n^{\otimes}(d) = \bigcup_{\deg \omega = d} \mathcal{C}_n^{\otimes}(\omega)$. Denote by $Q\mathcal{P}_n^{\omega}$ the subspace of $\mathcal{A}\mathcal{P}_n$ spanned by all the classes represented by the admissible monomials of weight vector ω in $\mathcal{P}_n$.

By the same arguments as in the proof of the previous theorem, the map $\mathcal{A}\mathcal{P}_n(\omega) \longrightarrow Q\mathcal{P}_n^{\omega},\ [u]_\omega \longrightarrow [u]$ is an isomorphism of $\mathbb{F}_2$-vector spaces. And therefore, $Q\mathcal{P}_n^{\omega} \subset \mathcal{A}\mathcal{P}_n$ can be used to identify the vector space $\mathcal{A}\mathcal{P}_n(\omega)$. From this, one obtains
$$(\mathcal{A}\mathcal{P}_n)_d = \bigoplus_{\deg \omega = d} Q\mathcal{P}_n^{\omega} \cong \bigoplus_{\deg \omega = d} \mathcal{A}\mathcal{P}_n(\omega).$$
Hence, it shows that $(\mathcal{A}\mathcal{P}_6^+)_{m_2} = \bigoplus_{\deg \omega = m_2} \mathcal{A}\mathcal{P}_6^+(\omega)$.

Suppose that x is an admissible monomial of degree twenty-six in $\mathcal{P}_6$ such that $[x]$ belongs to $\mathrm{Ker}(\widetilde{Sq}_*^0)_{(6;26)}$. Observe that $z = x_1^{15}x_2^7x_3^3x_4$ is the minimal spike of degree twenty-six in $\mathcal{P}_6$ and $\omega(z) = \widetilde{\omega_{[2]}}$. By Theorem 2.8, we obtain $\omega_1(x) \geqslant \omega_1(z) = 4$. Since the degree of (x) is even, one gets either $\omega_1(x) = 4$, or $\omega_1(x) = 6$.

If $\omega_1(x) = 6$ then $x = X_\emptyset y^2$, with y a monomial of degree ten in $\mathcal{P}_6$. Since x is admissible, by Theorem 2.6, it shows that y is also admissible, and $[y] \neq 0$. Hence, $[y] = (\widetilde{Sq}_*^0)_{(6;26)}([x]) \neq 0$ which contradicts the fact that $[x]$ belongs to $\mathrm{Ker}(\widetilde{Sq}_*^0)_{(6;26)}$.

Hence, $\omega_1(x) = 4$ then $x = X_{\{i,j\}}u^2$, with u an admissible monomial of degree eleven in $\mathcal{P}_6$, and $1 \leqslant i < j \leqslant 6$. Using the result in [12], one has $\omega(u) = (3,4)$ or $\omega(u) = (3,2,1)$, or $\omega(u) = (5,3)$ or $\omega(u) = (5,1,1)$. Hence, $\omega(x) = \widetilde{\omega_{[i]}}$ for all $i = 1,2,3,4$.

From the above results, one obtains
$$(\mathrm{Ker}(\widetilde{Sq}_*^0)_{(6;26)} \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^2-1)+2\cdot 2^2}) = \bigoplus_{i=1}^{4}\mathcal{A}\mathcal{P}_6^+(\widetilde{\omega_{[i]}}).$$
By the same arguments as in the proof of the previous theorem, we determine the space $\mathrm{Ker}(\widetilde{Sq}_*^0)_{(6;26)} \cap (\mathcal{A}\mathcal{P}_6^+)_{m_2}$ by explicitly determining all admissible monomials in $\mathcal{P}_6^+(\widetilde{\omega_{[i]}})$, where $i \in \{1,2,3,4\}$. However, to list all the elements of the admissible monomial basis of these subspaces is far too long and computationally very technical. The following is a sketch of its proof with the aid of computers. At the same time, we provide an algorithm in MAGMA [35] to verify the dimension result of the vector space $\mathrm{Ker}(\widetilde{Sq}_*^0)_{(6;26)} \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^2-1)+2\cdot 2^2}$ which is presented in the appendix of this article.

We will denote by $\mathcal{D}^{\otimes 6}_{6(2^2-1)+2\cdot 2^2}(\omega)$ the set of classes represented by the admissible monomials of the vector space $\mathrm{Ker}(\widetilde{Sq}_*^0)_{(6;26)} \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^2-1)+2\cdot 2^2}$. Consider the set
$$B_{26}^{\otimes 6} := \{X_{\{i,j\}} \cdot f^2 : f \in \mathcal{C}_6^{\otimes}(11), 1 \leqslant i < j \leqslant 6\}\ \cap\ \mathcal{P}_6^+.$$
Using Theorem 2.6, it shows that if u is an admissible monomial of degree twenty-six in $\mathcal{P}_6^+$ such that $(\widetilde{Sq}_*^0)_{(6;26)}([u])$ does not belong to $\mathrm{Im}(\widetilde{Sq}_*^0)_{(6;26)}$, then $u \in B_{26}^{\otimes 6}$.

By observing that each monomial $x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}x_5^{a_5}x_6^{a_6}$ corresponds to a series of numbers of the type $(a_1;a_2;a_3;a_4;a_5;a_6)$, we set up an algorithm implemented in Microsoft Excel software to eliminate the inadmissible monomials in $B_{26}^{\otimes 6}$. By direct calculations, using Theorem 2.6, we get $|\mathcal{D}^{\otimes 6}_{6(2^2-1)+2\cdot 2^2}(\omega)| = 3636$.

So, we have $\dim\big(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;26)} \cap (\mathcal{AP}_6^+)_{6(2^2-1)+2\cdot 2^2}\big) = 3636$. The theorem has been established. $\square$

From the above results, we get the following corollary.

Corollary 3.5. *There exist exactly* 9765 *admissible monomials of degree twenty-six in* $\mathcal{P}_6$. *That means,* $(\mathcal{AP}_6)_{6(2^2-1)+2\cdot 2^2}$ *has dimension* 9765.

Consider the degree $m_s := 6(2^s-1)+2\cdot 2^s$, for any $s>2$. Then, we have the following theorem.

Theorem 3.6. *The set* $\{[x] : x \in \Phi_6^{s-2}(\mathcal{C}_6^{\otimes}(26))\}$ *is a basis of the* $\mathbb{F}_2$*-vector space* $(\mathcal{AP}_6)_{6(2^s-1)+2\cdot 2^s}$ *for any* $s>2$. *Consequently,* $|\mathcal{C}_6^{\otimes}(2^{s+3}-6)| = 9765$, *for each integer* $s \geqslant 2$.

Proof. We will begin the proof of the theorem by calling a result in [28] as follows:

Let d be an arbitrary non-negative integer, and let $\zeta(u)$ be the greatest integer v such that u is divisible by 2^v. That means $u = 2^v m$, with m an odd integer. Put

$$\lambda(n,d) = \max\{0, n-\alpha(d+n)-\zeta(d+n)\}.$$

Then, Tin-Sum showed in [28] that the map

$$(\widetilde{Sq}^0_*)^{r-t} : (\mathcal{AP}_n)_{n(2^r-1)+2^r d} \longrightarrow (\mathcal{AP}_n)_{n(2^t-1)+2^t d}$$

is an isomorphism of $GL_n(\mathbb{F}_2)$-modules for every $r \geqslant t$ if and only if $t \geqslant \lambda(n,d)$.

For $n=6$, $m_s = 6(2^s-1)+2\cdot 2^s$, and $d=2$ then $\alpha(d+n)=\alpha(8)=1$, and one has $\zeta(d+n)=\zeta(2^3\cdot 1)=3$. And therefore $\lambda(n,d)=2$. Using the above result, we get an isomorphism of $\mathbb{F}_2$-vector space:

$$(\mathcal{AP}_6)_{6(2^r-1)+2\cdot 2^r} \cong (\mathcal{AP}_6)_{6(2^2-1)+2\cdot 2^2}$$

for all $r \geqslant 2$.

Hence, the set $\{[x] : x \in \Phi_6^{s-2}(\mathcal{C}_6^{\otimes}(6(2^2-1)+2\cdot 2^2))\}$ is a basis of the $\mathbb{F}_2$-vector space $(\mathcal{AP}_6)_{6(2^s-1)+2\cdot 2^s}$ for any $s>2$.

Moreover, $\dim(\mathcal{AP}_6)_{6(2^s-1)+2\cdot 2^s} = |\mathcal{C}_6^{\otimes}(6(2^2-1)+2\cdot 2^2)| = 9765$, for any $s \geqslant 2$. The theorem has been established. $\square$

Now, we describe the dimension result for the polynomial algebra by studying the $\mathbb{F}_2$-vector space $\mathcal{AP}_n$ in the generic degree $(n-1)(2^{n+u-1}-1)+\ell 2^{n+u}$, where u is an arbitrary non-negative integer, $\ell = 13$, and $n=7$.

As is well known, after explicitly determining $\mathcal{AP}_4$, Sum [24] has established an inductive formula by n for the dimension of the vector space $(\mathcal{AP}_n)_d$, where d is of general degree (see Theorem 1.3 in [24]). As a result of combining this result in Sum [24] with the results above, we get the following.

Theorem 3.7. *For any integer* $r>0$, *there exist exactly* 1240155 *admissible monomials of degree* $d_r = 6(2^{r+5}-1)+13\cdot 2^{r+6}$ *in* $\mathcal{P}_7$. *This implies that the space* $(\mathcal{AP}_7)_{d_r}$ *has dimension* 1240155, *for all* $r \geqslant 1$.

Proof. Consider the degree $d = (n-1)(2^s-1)+2^s t$, where s and t are positive integers such that $1 \leqslant n-3 \leqslant \mu(t) \leqslant n-2$. If $s \geqslant n-1$, then we have an inductive formula by n for the dimension of the vector space $(\mathcal{AP}_n)_d$, which was shown in Sum [24] as follows:

$$\dim(\mathcal{A}\mathcal{P}_n)_d = (2^n - 1)\dim(\mathcal{A}\mathcal{P}_{n-1})_t.$$

We can easily observe that for $n = 6$, and $t = 26$, then we have

$$\mu(t) = \mu(26) = 4 = \alpha(t + \mu(t)) = \alpha(70) = n - 2.$$

Hence, using the above result, one gets

$$|\mathcal{C}_6^{\otimes}(6(2^s - 1) + 13\cdot 2^{s+1})| = (2^7 - 1)\cdot|\mathcal{C}_6^{\otimes}(26)| = 1240155,$$

for any integer $s \geqslant 6$.

So, there exist exactly 1240155 admissible monomials of degree $6.(2^r - 1) + 13\cdot 2^{r+1}$ in $\mathcal{P}_7$, for any integer $r > 5$. Consequently, $\dim(\mathcal{A}\mathcal{P}_7)_{6.(2^{r+5}-1)+13\cdot 2^{r+6}} = 1240155$, for all $r \geqslant 1$. The theorem is proved. $\square$

One of the key applications of the hit problem is in surveying a homomorphism proposed by Singer. It is a useful tool for describing the cohomology groups of the Steenrod algebra, $\mathrm{Ext}_{\mathcal{A}}^{n,n+*}(\mathbb{F}_2, \mathbb{F}_2)$. We will start by recalling the definition of the Singer algebraic transfer, which was shown in [19] as follows:

Consider the graded polynomial ring $\mathcal{P}_1 = \mathbb{F}_2[x_1]$ with $\deg(x_1) = 1$. Thus, the canonical $\mathcal{A}$-action on $\mathcal{P}_1$ is extended to an $\mathcal{A}$-action on the ring of finite Laurent series $\mathbb{F}_2[x_1, x_1^{-1}]$. Hence, there exists an $\mathcal{A}$-submodule $\widetilde{\mathcal{P}_1}$ of $\mathbb{F}_2[x_1, x_1^{-1}]$, spanned by all powers x_1^i with $i \geqslant -1$. The inclusion $\mathcal{P}_1 \subset \widetilde{\mathcal{P}_1}$ gives rise to a short exact sequence of $\mathcal{A}$-modules:

$$0 \to \mathcal{P}_1 \xrightarrow{\pi} \widetilde{\mathcal{P}_1} \xrightarrow{\sigma} \sum{}^{-1}\mathbb{F}_2 \to 0,$$

where π is the inclusion, and σ is given by $\sigma(x_1^i) = 0$ if $i \neq -1$ and $\sigma(x_1^{-1}) = 1$.

Writing e_1 for the corresponding element in $\mathrm{Ext}_{\mathcal{A}}^1(\sum^{-1}\mathbb{F}_2, \mathcal{P}_1)$. Based on the cross, the Yoneda and the cap products, Singer set

$$e_n = (e_1 \times \mathcal{P}_{n-1}) \circ (e_1 \times \mathcal{P}_{n-2}) \circ \cdots \circ (e_1 \times \mathcal{P}_1) \circ e_1 \in \mathrm{Ext}_{\mathcal{A}}^n(\sum{}^{-n}\mathbb{F}_2, \mathcal{P}_n).$$

Then, Singer defined the following homomorphism

$$\widetilde{\varphi_n} : \mathrm{Tor}_n^{\mathcal{A}}(\mathbb{F}_2, \sum{}^{-1}\mathbb{F}_2) \longrightarrow \mathrm{Tor}_0^{\mathcal{A}}(\mathbb{F}_2, \mathcal{P}_n) = \mathcal{A}\mathcal{P}_n$$

by $\widetilde{\varphi_n}(z) = e_n \cap z$. Its image is a submodule of $(\mathcal{A}\mathcal{P}_n)^{GL_n(\mathbb{F}_2)}$.

So, $\widetilde{\varphi_n}$ induces the homomorphism

$$\varphi_n : \mathrm{Tor}_n^{\mathcal{A}}(\mathbb{F}_2, \sum{}^{-1}\mathbb{F}_2) \longrightarrow (\mathcal{A}\mathcal{P}_n)^{GL_n(\mathbb{F}_2)}.$$

Let $(\mathcal{A}\mathcal{P}_n)_m^{GL_n(\mathbb{F}_2)}$ be the subspace of $(\mathcal{A}\mathcal{P}_n)_m$ consisting of all the $GL_n(\mathbb{F}_2)$-invariant classes of degree m, and let the space $\mathbb{F}_2 \otimes_{GL_n(\mathbb{F}_2)} PH_m((\mathbb{R}\mathcal{P}^{\infty})^n)$ be the dual to $(\mathcal{A}\mathcal{P}_n)_m^{GL_n(\mathbb{F}_2)}$. Then, the dual of φ_n :

$$\psi_n := (\varphi_n)^* : \mathbb{F}_2 \otimes_{GL_n(\mathbb{F}_2)} PH_*((\mathbb{R}\mathcal{P}^{\infty})^n) \longrightarrow \mathrm{Ext}_{\mathcal{A}}^{n,n+*}(\mathbb{F}_2, \mathbb{F}_2).$$

is also called the n-th Singer algebraic transfer.

Singer illustrated the importance of the algebraic transfer by proving that ψ_n is an isomorphism with $n = 1, 2$ and at other degrees with $n = 3, 4$, but he refuted this for ψ_5 at degree 9, and then proposed the following conjecture.

Conjecture 3.8 (**Singer** [19]). *For any $n \geqslant 0$, the algebraic transfer ψ_n is a monomorphism.*

The hit problem's importance and relevance may be found throughout Singer's work. Moreover, using the modular representation theory of linear groups, Boardman confirmed in [1] that ψ_3 is also an isomorphism. Many authors have studied the Singer algebraic transfer for $n \geqslant 4$ (see Minami [10], Bruner-Ha-Hung [2], Phuc [15], Sum-Tin [23], Tin [29] and others). However, Singer's conjecture remains open for $n \geqslant 4$.

Based on the results for the hit problem, combining the computation of the cohomology groups $\mathrm{Ext}_{\mathcal{A}}^{5,5+*}(\mathbb{F}_2,\mathbb{F}_2)$ by Lin [8] and Chen [3], we studied and verified the Singer's conjecture for the algebraic transfer in degrees $5(2^s-1)+2^s m$, with s an arbitrary positive integer, and $m \in \{1,2,3\}$. We have the following.

Theorem 3.9 (see [23] and [29]). *Let s be an arbitrary positive integer. Singer's conjecture is true for $n=5$ and the generic degrees $d_s = 5(2^s-1)+2^s m$, where $m \in \{1,2,3\}$.*

In the current article, using the result in Tin-Sum [28] (see Theorem 1.3), we also see that

$$(\mathcal{A}\mathcal{P}_6)^{GL_6(\mathbb{F}_2)}_{6(2^s-1)+2\cdot 2^s} \cong (\mathcal{A}\mathcal{P}_6)^{GL_6(\mathbb{F}_2)}_{6(2^2-1)+2\cdot 2^2}, \quad \text{for all } s>1.$$

By passing to the dual, we obtain the following theorem.

Theorem 3.10. *We have an isomorphism of $\mathbb{F}_2$-vector spaces:*

$$\mathbb{F}_2 \otimes_{GL_6(\mathbb{F}_2)} PH_{6(2^s-1)+2\cdot 2^s}((\mathbb{R}\mathcal{P}^\infty)^6) \cong \mathbb{F}_2 \otimes_{GL_6(\mathbb{F}_2)} PH_{6(2^2-1)+2\cdot 2^2}((\mathbb{R}\mathcal{P}^\infty)^6),$$

for any positive integers $s \geqslant 2$.

Remarkability, from the result of the above theorem, we only need to study the vector space $\mathbb{F}_2 \otimes_{GL_6(\mathbb{F}_2)} PH_{6(2^s-1)+2\cdot 2^s}((\mathbb{R}\mathcal{P}^\infty)^6)$ for $s \leqslant 2$.

4. Conclusions and Future work

In this article, we study the hit problem for the polynomial algebra as a module over the Steenrod algebra in some generic degrees and its application to the sixth algebraic transfer of Singer. In the future, we will verify the Singer's conjecture for the sixth algebraic transfer in degree $6(2^s-1)+2\cdot 2^s$, with s an arbitrary positive integer, by combining the computations of the cohomology groups of the Steenrod algebra in these cases.

5. Appendix

In the appendix, we provide an algorithm in MAGMA [35] to verify the dimension result of the vector space $(\mathcal{A}\mathcal{P}_6)_{6(2^2-1)+2\cdot 2^2}$. And then, we obtain the dimension result of the vector space $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;26)} \cap (\mathcal{A}\mathcal{P}_6^+)_{6(2^2-1)+2\cdot 2^2}$.

```
N := 26;
R<x1, x2, x3, x4, x5, x6> := PolynomialRing(GF(2), 6);
P<t> := PolynomialRing (R);
Sqhom := hom < R -> P | [x + t * x^2 : x in [x1, x2, x3, x4, x5, x6]] >;
function Sq(j, x)
    c := Coefficients (Sqhom(x));
    if j+1 gt ♯c then
        return R ! 0;
    else
        return c[j+1];
    end if;
end function;
M := MonomialsOfDegree(R, N);
V := VectorSpace(GL(2), ♯ M);
```

```
function MtoV(m)
If IsZero(m) then
return V ! 0;
else
cv := [Index(M, mm) : mm in  Terms(m)];
return  CharacteristicVector(V, cv);
end if;
end function;
M1 :=   MonomialsOfDegree(R, N-1);
M2 :=   MonomialsOfDegree(R, N-2);
M4 :=   MonomialsOfDegree(R, N-4);
M8 :=   MonomialsOfDegree(R, N-8);
M16 :=   MonomialsOfDegree(R, N-16);
print  [ ♯ m  :   m  in  [M1, M2, M4, M8, M16] ];
S1  :=  [Sq(1, x)  :  x  in  M1];
print "S1";
H1  :=  sub <V  |  [M to V(x)  :  x  in  S1]>;
print  Dimension(H1);
S2  :=  [Sq(2, x)  :  x  in  M2];
print "S2";
H2  :=  sub <V  |  [M to V(x)  :  x  in  S2]>;
print Dimension(H2);
H  := H1  +  H2;
print Dimension(H);
S4  :=  [Sq(4, x)  :  x  in  M4];
print "S4";
H4  :=  sub <V  |  [M to V(x)  :  x  in  S4]>;
print Dimension(H4);
H  := H  +  H4;
print Dimension(H);
S8  :=  [Sq(8, x)  :  x  in  M8];
print  "S8";
H8  :=  sub <V  |  [M to V(x)  :  x  in  S8]>;
print Dimension(H8);
H  := H  +  H8;
print Dimension(H);
S16  :=  [Sq(16, x)  :  x  in  M16];
print "S16";
H16  :=  sub <V  |  [M to V(x)  :  x  in  S16]>;
print Dimension(H16);
H  := H  +  H16;
print Dimension(H);
print  "\n", Dimension(V);
print Dimension(H);
print Dimension(V)  -  Dimension(H);
```

According to the above calculations, we get

$$\dim\,(\mathcal{A}\mathcal{P}_6)_{6(2^2-1)+2\cdot 2^2} = Dimension(V) - Dimension(H) = 9765.$$

Hence, we obtain

$$\dim\,(\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;26)} \cap (\mathcal{A}\mathcal{P}_6^+)_{26}) = 9765 - \dim(\mathcal{A}\mathcal{P}_6^0)_{26} - \dim \mathrm{Im}(\widetilde{Sq}^0_*)_{(6;26)} = 3636.$$

Note that the result on the dimension of $\mathrm{Ker}(\widetilde{Sq}^0_*)_{(6;26)} \cap (\mathcal{AP}_6^+)_{6(2^2-1)+2\cdot 2^2}$ can also be verified by using a computer calculation program in SAGE (Software for Algebra and Geometry Experimentation), see Viet [32], and [36].

Acknowledgment. I would like to express my warmest thanks to my advisor, Prof. Nguyen Sum (Sai Gon University, Vietnam) for helpful conversations. This research is supported by Ho Chi Minh City University of Technology and Education (HCMUTE), Vietnam. Finally, I would like to express my warmest thanks to the referees for many helpful suggestions and detailed comments, which have led to the improvement of the paper's exposition.

References

[1] J. M. Boardman, *Modular representations on the homology of power of real projective space*, in: M. C. Tangora (Ed.), Algebraic Topology, Oaxtepec, 1991, in: Contemp. Math. **146**, 49-70, 1993.

[2] R.R. Bruner, L.M. Ha, , N.H.V. Hung, *On behavior of the algebraic transfer*, Trans. Amer. Math. Soc. **357**, 473-487, 2005.

[3] T.W. Chen, *Determination of* $\mathrm{Ext}^{5,*}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$, Topology Appl. **158**, 660-689, 2011.

[4] M.D. Crossley, N.D. Turgay, *Conjugation invariants in the mod 2 dual Leibniz-Hopf algebra*, Comm. Algebra **41** (9), 3261-3266, 2013.

[5] M.D. Crossley, *The Steenrod algebra and other copolynomial Hopf algebras*, Bull. London Math. Soc. **32** (5), 609-614, 2000.

[6] A.S. Janfada, R.M.W. Wood, *Generating* $H^*(\mathrm{BO}(3), \mathbb{F}_2)$ *as a module over the Steenrod algebra*, Math. Proc. Camb. Phil. Soc. **134**, 239-258, 2003.

[7] M. Kameko, *Products of projective spaces as Steenrod modules*, Ph.D. Thesis, The Johns Hopkins University, ProQuest LLC, Ann Arbor, MI, 1990. 29 pp.

[8] W.H. Lin, $\mathrm{Ext}^{4,*}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$ *and* $\mathrm{Ext}^{5,*}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$, Topology Appl. **155**, 459-496, 2008.

[9] J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. **67**, 150-171, 1958.

[10] N. Minami, *The iterated transfer analogue of the new doomsday conjecture*, Trans. Amer. Math. Soc. **351**, 2325-2351, 1999.

[11] L.X. Mong, N. Sum, *The hit problem for the polynomial algebra of five variables in a type of generic degrees*, Preprint, 22 pages, 2022.

[12] M.F. Mothebe, P. Kaelo, O. Ramatebele, *Dimension formulae for the polynomial algebra as a module over the Steenrod algebra in degrees less than or equal to 12*, J. Math. Research, **8**, 92-100, 2016.

[13] T.N. Nam, *$\mathcal{A}$-générateurs génériques pour l'algébre polynomiale*, Adv. Math. **186**, 334-362, 2004.

[14] F.P. Peterson, *Generators of* $H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$ *as a module over the Steenrod algebra*, Abstracts Amer. Math. Soc. **833**, 55-89, 1987.

[15] D.V. Phuc, *On Peterson's open problem and representations of the general linear groups*, J. Korean Math. Soc. **58**(3), 643-702, 2021.

[16] S. Priddy, *On characterizing summands in the classifying space of a group*, I. Amer. Jour. Math. **112**, 737-748, 1990.

[17] J. Repka, P. Selick, *On the subalgebra of* $H_*((\mathbb{R}P^\infty)^n; \mathbb{F}_2)$ *annihilated by Steenrod operations*, J. Pure Appl. Algebra **127**, 273-288, 1998.

[18] J.P. Serre, *Cohomologie modulo 2 des complexes d'Eilenberg-Maclane*, Comment. Math. Helv. **27**, 198-232, 1953.

[19] W.M. Singer, *The transfer in homological algebra*, Math. Zeit. **202**, 493-523, 1989.

[20] J.H. Silverman, *Hit polynomials and the canonical antiautomorphism of the Steenrod algebra*, Proc. Amer. Math. Soc. **123**, 627-637, 1995.

[21] N.E. Steenrod, D.B.A. Epstein, *Cohomology operations*, Annals of Mathematics Studies 50, Princeton University Press, Princeton N.J, 1962.

[22] N. Sum, *The negative answer to Kameko's conjecture on the hit problem*, Adv. Math. **225**, 2365-2390, 2010.

[23] N. Sum, N.K. Tin, *Some results on the fifth Singer transfer* East-West J. Math. **17**(1), 70-84, 2015.

[24] N.Sum, *On the Peterson hit problem*, Adv. Math. **274**, 432-489, 2015.

[25] N. Sum, N.K. Tin, *The hit problem for the polynomial algebra in some weight vectors*, Topology Appl. **290**, 107579, 2021.

[26] N.D. Turgay, *An alternative approach to the Adem relations in the mod 2 Steenrod algebra*, Turkish J. Math. **38**(5), 924-934, 2014.

[27] N.D. Turgay, S. Kaji, *The* mod 2 *dual Steenrod algebra as a subalgebra of the* mod 2 *dual Leibniz-Hopf algebra*, J. Homotopy Relat. Struct. **12** (3), 727-739, 2017.

[28] N.K. Tin, N. Sum, *Kameko's homomorphism and the algebraic transfer*, C. R. Acad. Sci. Paris, Ser. I, **354**, 940-943, 2016.

[29] N.K. Tin, *Hit problem for the polynomial algebra as a module over Steenrod algebra in some degrees*, Asian-European J. Math. **15** (1), 2250007, 2022.

[30] N.K. Tin, *A note on the $\mathcal{A}$-generators of the polynomial algebra of six variables and applications*, Turkish J. Math. 46(5), 1911-1926, 2022.

[31] N.K. Tin, *On the hit problem for the Steenrod algebra in the generic degree and its applications*, Rev. R. Acad. de Cienc. Exactas Fís. Nat. Ser. A Mat. (RACSAM) **116** (2), 2022.

[32] V.H. Viet, *Application of SAGE Computer Algebra System for the hit problem*, Master thesis, Quynhon University, 2015.

[33] G. Walker, R.M.W. Wood, *Young tableaux and the Steenrod algebra*, Proceedings of the International School and Conference in Algebraic Topology Hanoi 2004, Geom. Topol. Monogr., Geom. Topol. Publ., Coventry **11**, 379-397, 2007.

[34] R.M.W. Wood, *Steenrod squares of polynomials and the Peterson conjecture*, Math. Proc. Cambridge Phil. Soc. **105**, 307-309, 1989.

[35] MAGMA Computational Algebra System (version 2. 25-8), The Computational Algebra Group at the University of Sydney, 2020, http://magma.maths.usyd.edu.au/magma.

[36] SAGE Mathematics Software (version 5.4.1. Stein, William et al.), The Sage Development, 2012, http://www.sagemath.org.