Advances in the Theory of Nonlinear Analysis and its Applications 6 (2022) No. 4, 420–432.
https://doi.org/10.31197/atnaa.1127248
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Equivalents of Ordered Fixed Point Theorems of Kirk, Caristi, Nadler, Banach, and others

Sehie Park[a]

[a] *The National Academy of Sciences, Republic of Korea, Seoul 06579; and Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea.*

Abstract

Recently, we improved our long-standing Metatheorem in Fixed Point Theory. In this paper, as its applications, some well-known order theoretic fixed point theorems are equivalently formulated to existence theorems on maximal elements, common fixed points, common stationary points, and others. Such theorems are the ones due to Banach, Nadler, Browder-Göhde-Kirk, Caristi-Kirk, Caristi, Brøndsted, and possibly many others.

Keywords: fixed point theorem; pre-order; metric space; fixed point; stationary point; maximal element.
2010 MSC: 06A75, 47H10, 54E35, 54H25, 58E30, 65K10.

Email address: park35@snu.ac.kr; sehiepark@gmail.com; website:parksehie.com (Sehie Park)

Received February 08, 2022, Accepted June 05, 2022, Online June 07, 2022

1. Introduction

In a short note of Brøndsted [3] in 1976, he observed that certain fixed point theorems may be derived from theorems on the existence of maximal elements in partially ordered sets. Consequently, he obtained very useful three theorems including the Caristi fixed point theorem [5].

In 1982-2000, we had published several articles mainly related to the Caristi fixed point theorem, the Ekeland variational principle for approximate solutions of minimization problems, and their equivalent formulations with some applications; for example, see [21–26]. From the beginning of such studies, we obtained a Metatheorem for some equivalent statements on maximality, fixed points, stationary points, common fixed points, common stationary points, and others.

In the present article, we show that certain order theoretic fixed point theorems are equivalent to theorems on the existence of maximal elements in pre-ordered sets. In fact, those maximal elements are same to fixed points, stationary points, common fixed points, common stationary points, and others for certain types of maps or multimaps with nonempty values. Consequently, we improve and generalize well-known theorems due to Banach, Nadler, Browder-Göhde-Kirk, Caristi-Kirk, Caristi, Brøndsted, and possibly many others following the method in our recent works [27–29].

This article is organized as follows: In Section 2, we introduce our Metatheorem. Sections 3–10 are concerned to apply Metatheorem to theorems due to Browder-Göhde-Kirk [4, 11, 18], Brøndsted [3], Caristi-Kirk [6], Caristi [5], Nadler [8, 20], Banach [2], Tarski-Kantorovitch [9], and Jachymski [13–17], respectively. Finally, Section 11 is for certain remarks as the conclusion.

2. Metatheorem in Fixed Point Theory

We introduce our new version of Metatheorem introduced in [28] in 2022:

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x, y)$ a sentence formula for $x, y \in X$. Then the following eight statements are equivalent:*

(i) *There exists an element $v \in A$ such that $G(v, w)$ for any $w \in X\backslash\{v\}$.*

(ii) *If $T : A \multimap X$ is a multimap such that for any $x \in A\backslash T(x)$ there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x, y)$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If $f : A \to X$ is a map such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x, y)$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(iv) *If $f : A \to X$ is a map such that $\neg G(x, f(x))$ for any $x \in A$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(v) *If $T : A \multimap X$ is a multimap such that $\neg G(x, y)$ holds for any $x \in A$ and any $y \in T(x)\backslash\{x\}$, then T has a stationary element $v \in A$, that is, $\{v\} = T(v)$.*

(vi) *If* $\mathfrak{F}$ *is a family of maps* $f : A \to X$ *satisfying* $\neg G(x, f(x))$ *for all* $x \in A$ *with* $x \neq f(x)$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in A$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$*.*

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : A \multimap X$ *for* $i \in I$ *with an index set* I *such that* $\neg G(x, y)$ *holds for any* $x \in A$ *and any* $y \in T_i(x)\backslash\{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in A$*, that is,* $\{v\} = T_i(v)$ *for all* $i \in I$*.*

(viii) *If* Y *is a subset of* X *such that for each* $x \in A\backslash Y$ *there exists a* $z \in X\backslash\{x\}$ *satisfying* $\neg G(x, z)$*, then there exists a* $v \in A \cap Y$*.*

Here, maps and multimaps have nonempty values and $\neg$ denotes the negation. We give the following proof in [28] for completeness:

Proof. (i)$\Longrightarrow$(ii): Suppose $v \notin T(v)$. Then there exists a $y \in X\backslash\{v\}$ satisfying $\neg G(v, y)$. This is a contradiction.

(ii)$\Longrightarrow$(iii): Clear.

(iii)$\Longrightarrow$(iv): Clear.

(iv)$\Longrightarrow$(v): Suppose T has no stationary element, that is, $T(x)\backslash\{x\} \neq \emptyset$ for any $x \in A$. Choose a choice function f on $\{T(x)\backslash\{x\} : x \in A\}$. Then f has no fixed element by its definition. However, for any $x \in A$, we have $x \neq f(x)$ satisfying $\neg G(x, f(x))$. Therefore, by (iv), f has a fixed element, a contradiction.

(v)$\Longrightarrow$(vi): Define a multimap $T : A \multimap X$ by $T(x) := \{f(x) : f \in \mathfrak{F}\} \neq \emptyset$ for all $x \in A$. Since $\neg G(x, f(x))$ for any $x \in A$ and any $f \in \mathfrak{F}$, by (v), T has a stationary element $v \in A$, which is a common fixed element of $\mathfrak{F}$.

(vi)$\Longrightarrow$(i): Suppose that for any $x \in A$, there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x, y)$. Choose $f(x)$ to be one of such y. Then $f : A \to X$ has no fixed element by its definition. However, $\neg G(x, f(x))$ for all $x \in A$. Let $\mathfrak{F} = \{f\}$. By (vi), f has a fixed element, a contradiction.

(i)+(v) $\Longrightarrow$ (vii): By (i), there exists a $v \in A$ such that $G(v, w)$ for all $w \in X\backslash\{v\}$. For each $i \in I$, by (v), we have a $v_i \in A$ such that $\{v_i\} = T_i(v_i)$. Suppose $v \neq v_i$. Then $G(v, v_i)$ holds by (i) and $\neg G(v, v_i)$ holds by assumption on (v). This is a contradiction. Therefore $v = v_i$ for all $i \in I$.

(vii) $\Longrightarrow$ (v): Clear.

(i) $\Longrightarrow$ (viii): By (i), there exists a $v \in A$ such that $G(v, w)$ for all $w \neq v$. Then by the hypothesis, we have $v \in Y$. Therefore, $v \in A \cap Y$.

(viii) $\Longrightarrow$ (i): For all $x \in A$, let

$$A(x) := \{y \in X : x \neq y,\ \neg G(x, y)\}.$$

Choose $Y = \{x \in X : A(x) = \emptyset\}$. If $x \notin Y$, then there exists a $z \in A(x)$. Hence the hypothesis of (viii) is satisfied. Therefore, by (viii), there exists a $v \in A \cap Y$. So $A(v) = \emptyset$; that is, $G(v, w)$ for all $w \neq v$. Hence (i) holds.

This completes our proof. □

Note that the element $v \in A$ is the same for all (i)–(viii), and (iv)$\Longrightarrow$(v) adopts the Axiom of Choice. In our previous works [27, 28], we gave many examples of Metatheorem, for which (i)–(viii) are true.

3. Kirk Fixed Point Theorem

In 1965, Kirk [16] obtained the following:

Theorem 3.1. (Kirk) *Let X be a Banach space and suppose that C is a nonempty weakly compact convex subset of X which has the normal structure property. Then, any nonexpansive map $f : C \to C$ (that is, $||f(x) - f(y)|| \leq ||x - y||$ for all $x, y \in C$) has a fixed point.*

Kirk's original proof was based on Zorn's Lemma. Later Fuchssteiner [10] used Zermelo's theorem in the context of nonexpansive map.

In the same year 1965, Browder [4] and Göhde [11] obtained the following:

Theorem 3.2. *Let X be a uniformly convex Banach space and suppose that C is a nonempty closed bounded convex subset of X. Then, any nonexpansive map $f : C \to C$ has a fixed point.*

This is known as the Browder-Göhde-Kirk fixed point theorem for nonexpansive maps. Early in 1981, Granas [12] gave a proof of Theorem 3.2 for Hilbert spaces by applying the Hartman-Stampacchia theorem.

In 2002, using Zermelo's theorem, Jachymski [16] established a common fixed point theorem for two progressive maps on a partially ordered set. This result yields the Kirk fixed point theorem for nonexpansive maps independent of the Axiom of Choice.

Motivated by the Browder-Göhde-Kirk theorem and our Metatheorem, we have the following equivalency:

Theorem 3.3. *Let C be a closed convex subset in a uniformly convex Banach space and $\phi : C \to C$ be a map.*

Then the following statements are equivalent:

(i) *There exists an element $v \in C$ such that $||\phi(v) - \phi(w)|| > ||v - w||$ for any $w \in C \backslash \{v\}$.*

(ii) *If $T : C \multimap C$ is a multimap such that for any $x \in C \backslash T(x)$ there exists a $y \in C \backslash \{x\}$ satisfying $||\phi(x) - \phi(y)|| \leq ||x - y||$, then T has a fixed element $v \in C$, that is, $v \in T(v)$.*

(iii) *If $f : C \to C$ is a map such that for any $x \in C$ with $x \neq f(x)$, there exists a $y \in C \backslash \{x\}$ satisfying $||\phi(x) - \phi(y)|| \leq ||x - y||$, then f has a fixed element $v \in C$, that is, $v = f(v)$.*

(iv) *If $f : C \to C$ is a map such that $||\phi(x) - \phi(f(x))|| \leq ||x - f(x)||$ for all $x \in C$, then f has a fixed element $v \in C$, that is, $v = f(v)$.*

(v) *If* $T : C \multimap C$ *is a multimap such that* $||\phi(x) - \phi(y)|| \leq ||x - y||$ *holds for any* $x \in C$ *and any* $y \in T(x)\backslash\{x\}$*, then* T *has a stationary element* $v \in C$*, that is,* $\{v\} = T(v)$*.*

(vi) *If* $\mathfrak{F}$ *is a family of maps* $f : C \to C$ *satisfying* $||\phi(x)-\phi(f(x))|| \leq ||x-f(x)||$ *for all* $x \in C$ *with* $x \neq f(x)$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in C$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$*.*

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : C \multimap C$ *for* $i \in I$ *with an index set* I *such that* $||\phi(x) - \phi(y)|| \leq ||x - y||$ *holds for any* $x \in C$ *and any* $y \in T_i(x)\backslash\{x\}$ *for all* $i \in I$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in C$*, that is,* $\{v\} = T_i(v)$ *for all* $i \in I$*.*

(viii) *If* Y *is a subset of* C *such that for each* $x \in C\backslash Y$ *there exists a* $z \in C\backslash\{x\}$ *satisfying* $||\phi(x) - \phi(z)|| \leq ||x - z||$*, then there exists a* $v \in C \cap Y = Y$*.*

Proof. In Metatheorem, let $G(v, w)$ be the statement $||\phi(v) - \phi(w)|| > ||v - w||$ and let $A = X = C$. Then each of (i)–(viii) follows from the corresponding ones in Metatheorem. This completes our proof. □

Note that, for the particular case when $\phi = f$ is nonexpansive, the case (iii) extends the Browder-Göhde-Kirk fixed point theorem.

4. Brønsted Principle

In a short note in 1976, Brøndsted [3] observed that certain fixed point theorems may be derived from the following statement on the existence of maximal elements in partially ordered sets.

(A) *Let* $(E, \preccurlyeq)$ *be a partially ordered set which admits at least one maximal element. Let* $f : E \to E$ *be a progressive map (that is,* $x \preccurlyeq f(x)$ *for all* $x \in E$*). Then* f *admits at least one fixed point.*

Recall that the main point in our previous work [29] is to show how certain fixed point theorem can be deduced from the simple observation (A).

In [29], we adopted the following instead of (A), where a pre-order is required to be only reflexivity and transitivity.

Brønsted Principle. *Let* $(E, \preccurlyeq)$ *be a pre-ordered set and* $f : E \to E$ *be a progressive map. Then* E *admits a maximal element* $v \in E$ *if and only if* v *is a fixed element of* f*.*

In most applications of this principle, a maximal element in a partially ordered set is achieved by the upper bound of a chain. Moreover, this is better than (A) and the necessity is trivial.

In [28], we applied this principle to the following consequence of Metatheorem:

Theorem 4.1. *Let* (X, d) *be a complete metric space, and* $\varphi : X \to \mathbb{R}$ *lower semicontinuous and bounded below. Define a partial order* $\preccurlyeq$ *on* X *by*

$$x \preccurlyeq y \quad \textit{iff} \quad d(x,y) \leq \varphi(x) - \varphi(y).$$

Then the following equivalent statements hold:

(i) *There exists a maximal element* $v \in X$*; that is,* $v \not\preccurlyeq w$ *or* $d(v,w) > \varphi(v) - \varphi(w)$ *for any* $w \in X \backslash \{v\}$*.*

(ii) *If* $T : X \multimap X$ *is a multimap such that for any* $x \in X \backslash T(x)$ *there exists a* $y \in X \backslash \{x\}$ *satisfying* $d(x,y) \leq \varphi(x) - \varphi(y)$*, then* T *has a fixed element* $v \in X$*, that is,* $v \in T(v)$*.*

(iii) *If* $f : X \to X$ *is a map such that for any* $x \in X$ *with* $x \neq f(x)$*, there exists a* $y \in X \backslash \{x\}$ *satisfying* $d(x,y) \leq \varphi(x) - \varphi(y)$*, then* f *has a fixed element* $v \in X$*, that is,* $v = f(v)$*.*

(iv) *If* $f : X \to X$ *is a map such that* $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ *for all* $x \in X$*, then* f *has a fixed element* $v \in X$*, that is,* $v = f(v)$*.*

(v) *If* $T : X \multimap X$ *is a multimap such that* $d(x,y) \leq \varphi(x) - \varphi(y)$ *holds for any* $x \in X$ *and any* $y \in T(x) \backslash \{x\}$*, then* T *has a stationary element* $v \in X$*, that is,* $\{v\} = T(v)$*.*

(vi) *If* $\mathfrak{F}$ *is a family of maps* $f : X \to X$ *satisfying* $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ *for all* $x \in X$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in X$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$*.*

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : X \multimap X$ *for* $i \in I$ *with an index set* I *such that* $d(x,y) \leq \varphi(x) - \varphi(y)$ *holds for any* $x \in A$ *and any* $y \in T_i(x) \backslash \{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in X$*, that is,* $\{v\} = T_i(v)$ *for all* $i \in I$*.*

(viii) *If* Y *is a subset of* X *such that for each* $x \in X \backslash Y$ *there exists a* $z \in X \backslash \{x\}$ *such that* $d(x,y) \leq \varphi(x) - \varphi(y)$*, then there exists an element* $v \in Y$*.*

Note that (iv) is given in [3] as (B) and the celebrated Caristi fixed point theorem, and others except (i) and (viii) are its generalizations. Therefore (i)–(viii) hold.

5. Caristi-Kirk Fixed Point Theorem

From Theorem 4.1(ii), we immediately have the following in [6]:

Theorem 5.1. (Caristi-Kirk) *Let* (X,d) *be a complete metric space and* $T : X \multimap X$ *be a multimap such that for each* $x \in X$*, there exists* $y \in T(x)$ *satisfying*

$$d(x,y) + \varphi(y) \leq \varphi(x)$$

where $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ *is a proper, lower semicontinuous and bounded below functional. Then,* T *has a fixed point, that is, there exists* $v \in X$ *such that* $v \in T(v)$*.*

Note that Theorem 4.1(v) is a variant of the Caristi-Kirk theorem and (vii) is its collective version.

6. Caristi Fixed Point Theorem

Theorem 4.1(iii) is a single-valued case of (ii) and (iv) is the following in [5]:

Theorem 6.1. (Caristi) *Let M be a complete metric space, suppose $\varphi : M \to \mathbb{R}$ is lower semi-continuous and bounded below, and suppose $g : M \to M$ satisfies:*

$$d(x, g(x)) \le \varphi(x) - \varphi(g(x)), \quad x \in M.$$

Then g has a fixed point.

In Theorem 4.1, note that (iv) is given in [3] as (B), which is the celebrated Caristi fixed point theorem, and others except (i) and (viii) are its generalizations. Note that (ii) and (v) are multimap versions of (iii) and that (v) and (vii) are for families of multimaps. Therefore all of (ii)–(vii) can be considered generalizations of the Caristi fixed point theorem.

For the history of proofs of the Caristi theorem, see Jachymski [14] in 1998.

7. Nadler Fixed Point Theorem

From Jachymski [15] in 2001: Let (X, d) be a metric space, $\mathrm{CB}(X)$ denotes the family of nonempty closed bounded subsets of X, and $\mathrm{Cl}(X)$ denotes the family of all nonempty closed subsets of X (not necessarily bounded). For $A, B \in \mathrm{Cl}(X)$, set

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\},\ \sup\{d(b, A) : b \in B\}\},$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$. Then H is called a generalized Hausdorff metric since it may have infinite values.

Definition 7.1. Let (X, d) be a metric space and let $T : X \multimap X$. T is called a contraction [20] if for a fixed constant $k < 1$ and for each $x, y \in X$, $H(T(x), T(y)) \le kd(x, y)$. Such a map T is also known as the Nadler contraction.

Using the concept of Hausdorff metric, Nadler [20] proved the following theorem on the existence of fixed points for multimaps, known as the Nadler contraction principle.

Theorem 7.2. (Nadler) *Let (X, d) be a complete metric space. Then, each contraction $T : X \to CB(X)$ has a fixed point.*

As Jachymski [15] noted: In contrast to its single-valued counterpart, fixed points in Theorem 7.2 need not be unique. Indeed, if X is bounded, then the map $T(x) = X$ for all $x \in X$ satisfies the conditions of Theorem 7.2.

Jachymski recalled a more general form of Nadler's theorem established by Covitz and Nadler [8] as follows:

Theorem 7.3. (Nadler Fixed Point Theorem) *Let (X, d) be a complete metric space and $T : X \to \mathrm{Cl}(X)$. Assume there is an $h \in [0, 1)$ such that*

$$H(T(x), T(y)) \le hd(x, y) \quad for\ all\ x, y \in X.$$

Then T has a fixed point.

In 1998, Jachymski [14] showed that Nadler's contraction $\mathcal{T}$ on a complete metric space (X, d) admits a selection $T : X \to X$, which is a Caristi map on (X, d) generated by a Lipschitzian function f. Hence, Caristi's fixed point theorem yields Nadler's theorem.

Motivated by Theorem 7.3 and Metatheorem, we have the following extended form of [27, Theorem 3.2]:

Theorem 7.4. *Let X be a complete metric space, $F : X \to \mathrm{Cl}(X)$ be a multimap, and $0 < h < 1$. Then the following equivalent statements hold:*

(i) *There exists an element $v \in X$ such that $H(F(v), F(w)) > hd(v, w)$ for any $w \in X\backslash\{v\}$.*

(ii) *If $T : X \multimap X$ is a multimap such that, for any $x \in X\backslash T(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $H(F(x), F(y)) \leq hd(x, y)$, then T has a fixed element $v \in X$, that is, $v \in T(v)$.*

(iii) *If $f : X \to X$ is a map such that, for any $x \in X$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $d(F(x), F(y)) \leq hd(x, y)$, then f has a fixed element $v \in X$, that is, $v = f(v)$.*

(iv) *If $f : X \to X$ is a map such that $d(F(x), F(f(x))) \leq hd(x, f(x))$ for any $x \in X$, then f has a fixed element $v \in X$, that is, $v = f(v)$.*

(v) *If $T : X \multimap X$ is a multimap such that $H(F(x), F(y)) \leq hd(x, y)$ holds for any $x \in X$ and any $y \in T(x)\backslash\{x\}$, then T has a stationary element $v \in X$, that is, $\{v\} = T(v)$.*

(vi) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ such that, for all $x \in X$ with $x \neq f(x)$ there exists a $y \in X\backslash\{x\}$ satisfying $d(F(x), F(y)) \leq hd(x, y)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(vii) *If $\mathfrak{F}$ is a family of multimaps $T_i : X \to X$, $i \in I$, satisfying $H(F(x), F(y)) \leq hd(x, y)$ for all $x \in X$ and any $y \in T_i(x)\backslash\{x\}$, $i \in I$, then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = T_i(v)$ for all $i \in I$.*

(viii) *If Y is a subset of X such that for each $x \in X\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying $H(F(x), F(z)) \leq hd(x, z)$, then there exists a $v \in X \cap Y = Y$.*

Proof. Note that, in Metatheorem, put $A = X$ and let $G(v, w)$ be the statement $H(F(v), F(w)) > hd(v, w)$. Then each of (i)–(viii) follows from the corresponding ones in Metatheorem. Note that, when $T = F$ is a single-valued map f, (iv) holds by the Banach contraction principle. Then we have a maximal element in (i) by Brøndsted principle. This completes our proof. □

Note that Theorem 7.4(ii) or (v) extend Nadler's theorem and (iii) implies the Banach contraction principle. Therefore, in some sense, these two theorems are equivalent in view of Theorem 7.4.

8. Extensions of the Banach contraction principle

Recall the following well-known concept:

Definition 8.1. Let (X, d) be a metric space and let $f : X \to X$. f is called a *contraction* if for a given constant $k < 1$ and for each $x, y \in X$, $d(f(x), f(y)) \leq kd(x, y)$. Such a mapping f is also known as *Banach contraction.*

In this section, we will discuss the most basic fixed point theorem in analysis, known as the Banach Contraction Principle. According to Jachymski [15]: "It is due to Banach and appeared in his Ph.D. thesis (1920, published in 1922 [2]). The principle was first stated and proved by Banach for the contraction maps in the setting of complete normed linear spaces. At about the same time the concept of an abstract metric space was introduced by Hausdorff, which then provided the general framework for the principle for contractions in a complete metric space."

Theorem 8.2. (Banach Contraction Principle) *Let (X, d) be a complete metric space, then each contraction $f : X \to X$ has a unique fixed point.*

It is well-known that this follows from the Caristi fixed point theorem which was extended by Theorem 4.1(iii). This also extends the Banach principle as follows:

Theorem 8.3. *Let (X, d) be a complete metric space and $k \in [0, 1)$. If $f : X \to X$ is a continuous map such that for any $x \in X$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $d(f(x), f(y)) \leq kd(x, y)$, then f has a fixed element $v \in X$, that is, $v = f(v)$.*

Proof. From $d(f(x), f(y)) \leq kd(x, y)$, we can deduce

$$d(x, y) \leq \varphi(x) - \varphi(y) \text{ with } \varphi(x) = \frac{1}{1-k} d(x, f(x)),$$

where $\varphi : X \to \mathbb{R}^+ = [0, \infty)$.

If $f : X \to X$ is a map such that for any $x \in X$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $d(f(x), f(y)) \leq kd(x, y)$, then f has a fixed element $v \in X$ by Theorem 4.1(iii); that is, $v = f(v)$. □

A. Latif [19] gave an example of the situation in which $f : X \to X$ is not necessarily a contraction, but f^n is a contraction for some n.

Example 8.4. Let $f : [0, 2] \to [0, 2]$ be defined by

$$f(x) = 0 \text{ for } x \in [0, 1]; \; f(x) = 1 \text{ for } x \in (1, 2].$$

Then, $f^2(x) = 0$ for all $x \in [0, 2]$, and so, f^2 is a contraction on [0, 2].

Note that f is not continuous and thus not a contraction. Moreover, this example does not satisfy Theorem 4.1(iii) in case $f = \varphi$.

We give an example of Theorem 8.3 based on [1, 7]:

Definition 8.5. Let (X, d) be a metric space. For any $x, y \in X$, the segment between x and y is defined by

$$[x, y] = \{z \in X : d(x, z) + d(z, y) = d(x, y)\}.$$

Definition 8.6. Let (X, d) be a metric space. A continuous map $f : X \to X$ is called a *directional contraction* if there exists $\alpha \in (0, 1)$ such that for any $x \in X$ with $f(x) \neq x$, there exists $z \in [x, f(x)]\backslash\{x\}$ such that $d(f(x), f(z)) \leq \alpha d(x, z)$.

Corollary 8.7. (Clarke's Fixed Point Theorem) *Let (X, d) be a complete metric space and $f : X \to X$ be a directional contraction map. Then, f has a fixed point.*

9. Tarski-Kantorovitch Theorem

A selfmap f of a partially ordered set $(P, \preccurlyeq)$ is said to be $\preccurlyeq$-*continuous* if for every countable chain C having a supremum, the image $f(C)$ has a supremum and $\sup f(C) = f(\sup C)$. It is easily seen that a $\preccurlyeq$-continuous map is isotone. $(P, \preccurlyeq)$ is said to be $\preccurlyeq$-*complete* if every countable chain has a supremum.

Following Dugundji and Granas [9, p.15], Jachymski [15] introduced:

Theorem 9.1. (Tarski-Kantorovitch) *Let $(P, \preccurlyeq)$ be a $\preccurlyeq$-complete partially ordered set and a map $f : P \to P$ be $\preccurlyeq$-continuous. If there exists $p_0 \in P$ such that $p_0 \preccurlyeq f(p_0)$, then f has a fixed point; moreover, $p_* := \sup\{f^n(p_0) : n \in \mathbb{N}\}$ is fixed under f.*

By applying Metatheorem, this suggests the following extension of the Brøndsted principle:

Theorem 9.2. *Let $(X, \preccurlyeq)$ be a pre-ordered set and $G(x, y)$ is $x \not\preccurlyeq y$ for $x, y \in X$. Then the following eight statements are equivalent:*

(i) *There exists an element $v \in X$ such that $v \not\preccurlyeq w$ for any $w \in X\backslash\{v\}$.*

(ii) *If $T : X \multimap X$ is a multimap such that for any $x \in X\backslash T(x)$ there exists a $y \in X\backslash\{x\}$ satisfying $x \preccurlyeq y$, then T has a fixed element $v \in X$, that is, $v \in T(v)$.*

(iii) *If $f : X \to X$ is a map such that for any $x \in X$ with $x \preccurlyeq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $x \preccurlyeq y$, then f has a fixed element $v \in X$, that is, $v = f(v)$.*

(iv) *If $f : X \to X$ is a map such that $x \preccurlyeq f(x)$ for any $x \in X$, then f has a fixed element $v \in X$, that is, $v = f(v)$.*

(v) *If $T : X \multimap X$ is a multimap such that $x \preccurlyeq y$ holds for any $x \in X$ and any $y \in T(x)\backslash\{x\}$, then T has a stationary element $v \in X$, that is, $\{v\} = T(v)$.*

(vi) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $x \preccurlyeq f(x)$ for all $x \in X$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(vii) *If $\mathfrak{F}$ is a family of multimaps $T_i : X \multimap X$ for $i \in I$ with an index set I such that $x \preccurlyeq y$ holds for any $x \in X$ and any $y \in T_i(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = T_i(v)$ for all $i \in I$.*

(viii) *If* Y *is a subset of* X *such that for each* $x \in X\backslash Y$ *there exists a* $z \in X\backslash\{x\}$ *satisfying* $x \preccurlyeq z$*, then there exists a* $v \in X \cap Y = Y$*.*

Consequently, in a pre-ordered set, the existence of maximal elements is equivalent to the corresponding ones for fixed elements, stationary elements, common fixed elements, common stationary elements, and locations of such elements.

10. Jachymski Stationary Point Theorem

In 2011, Jachymski [17] gave a stationary point theorem for some multimap on a metric space. The existence of fixed points of such maps characterizes the metric completeness and yields the order-theoretic Cantor theorem and the Ekeland variational principle.

The following (i) is known as the order theoretic Cantor theorem due to Granas-Horvath; see [17, Theorem 3]:

Theorem 10.1. *Let* (X, d) *be a complete metric space endowed with a partial order* $\preccurlyeq$*. Assume that for any* $x \in X$*, the set* $\{y \in X : x \preccurlyeq y\}$ *is closed and given* $\varepsilon > 0$*, there is* $y \in X$ *such that* $x \preccurlyeq y$ *and* $\mathrm{diam}\{z \in X : y \preccurlyeq z\} < \varepsilon$*.*

Then the following equivalent statements hold:

(i) *There exists a maximal element* $v \in X$*, that is,* $v \not\preccurlyeq w$ *for any* $w \in X\backslash\{v\}$*.*

(ii) *If* $T : A \multimap X$ *is a multimap such that for any* $x \in X\backslash T(x)$ *there exists a* $y \in X\backslash\{x\}$ *satisfying* $x \preccurlyeq y$*, then* T *has a fixed element* $v \in X$*, that is,* $v \in T(v)$*.*

(iii) *If* $f : X \to X$ *is a map such that for any* $x \in X$ *with* $x \neq f(x)$*, there exists a* $y \in X\backslash\{x\}$ *satisfying* $x \preccurlyeq y$*, then* f *has a fixed element* $v \in$*, that is,* $v = f(v)$*.*

(iv) *If* $f : X \to X$ *is a map such that* $x \preccurlyeq f(x)$ *for any* $x \in X$*, then* f *has a fixed element* $v \in$*, that is,* $v = f(v)$*.*

(v) *If* $T : X \multimap X$ *is a multimap such that* $x \preccurlyeq y$ *holds for any* $x \in X$ *and any* $y \in T(x)\backslash\{x\}$*, then* T *has a stationary element* $v \in X$*, that is,* $\{v\} = T(v)$*.*

(vi) *If* $\mathfrak{F}$ *is a family of maps* $f : X \to X$ *satisfying* $x \preccurlyeq f(x)$ *for all* $x \in X$ *with* $x \neq f(x)$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in X$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$*.*

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : X \multimap X$ *for* $i \in I$ *with an index set* I *such that* $x \preccurlyeq y$ *holds for any* $x \in X$ *and any* $y \in T_i(x)\backslash\{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in X$*, that is,* $\{v\} = T_i(v)$ *for all* $i \in I$*.*

(viii) *If* Y *is a subset of* X *such that for each* $x \in X\backslash Y$ *there exists a* $z \in X\backslash\{x\}$ *satisfying* $x \preccurlyeq z$*, then there exists a* $v \in X \cap Y = Y$*.*

Proof. Recall that (i) is true by [17, Theorem 3]. Now Metatheorem with $X = A$ works for $x \not\preccurlyeq y$ instead of $G(x, y)$. □

11. Conclusion

Our original Metatheorem first appeared in [21] in 1985. The one in [27] is a particular form without (iv) and indicated that it holds for pre-ordered (quasi-ordered or pseudo-ordered) sets. The present Metatheorem and Theorem 3.2 were appeared in our previous article entitled *Applications of various maximum principles* [28] with several applications.

In this article, we introduce our Metatheorem in [28] and show that it can be applied to equivalent formulations of a number of known theorem as we did in our previous work [27]. In such equivalent formulations, certain maximal points are actually same to fixed points, stationary points, collectively fixed points, collectively stationary points, and we have some information on the location of such points. No one recognized this fact yet. Therefore, if we have a theorem on any of such points, then we can deduce at least seven equivalent theorems on other types of points.

In many fields of mathematical sciences, there are plentiful number of theorems concerning maximal points or fixed points that can be applicable our Metatheorem. Some of such theorems can be seen in our previous works in [21]–[29]. Therefore, our Metatheorem is a machine to find new equivalent theorems with trivial proofs. This is like an industrial revolution of making new equivalent statements.

References

[1] Q.H. Ansari, Ekeland's variational principle and its extensions with applications, Chapter 3 in: Topics in Fixed Point Theory (S. Almezel, Q.H. Ansari, M.A. Khamsi, eds.), Springer, 2014.

[2] S. Banach, Theorie des Opérations Linéaires, Chelsea Publ. Co., New York, Reprinted from the first edition, Hafner, Lwow, Ukraine (current), 1932.

[3] A. Brøndsted, Fixed point and partial orders, Shorter Notes, Proc. Amer. Math. Soc. 60 (1976), 365–366.

[4] F.E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. NAS. USA 54 (1965), 1041–1045.

[5] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc. 215 (1976), 241-251.

[6] J. Caristi and W. A. Kirk, Geometric fixed point theory and inwardness conditions, The Geometry of Metric and Linear Spaces (Conf. Michigan State Univ., 1974), Lecture Notes in Math., vol. 490, Springer-Verlag, New York, (1975), 74-83.

[7] F. Clarke, Pointwise contraction criteria for the existence of fixed points. MRC Technical Report 1658, July 1976, University of Wisconsin, Madison, Wisconsin, 1976.

[8] H. Covitz and S.B. Nadler, Jr, Multi-valued contraction mappings in generalized metric spaces, Israel J. Math. 8 (1970), 5-11.

[9] J. Dugundji and A. Granas, Fixed Point Theory I, Polish Scientific Publ., Warszawa, 1982.

[10] B. Fuchssteiner, Iterations and fixpoints, Pacific J. Math. 68 (1977), 73-80.

[11] D. Göhde, Zum Prinzip der kontraktiven Abbildung, Math. Nach. 30 (1965), 251–258.

[12] A. Granas, KKM-maps and their appliations to nonlinear problems, in: The Scottish Book – Mathematics from the Scottish Café (R.D. Mauldin, ed.), Birkhäuser, Boston-Basel-Stuttgart (1981), 45–61.

[13] J. Jachymski, Fixed point theorems in metric and uniform spaces via the Knaster-Tarski Principle, Nonlinear Anal. 32 (1998), 225–233.

[14] J.R. Jachymski, Caristi's fixed point theorem and selections of set-valued contractions, J. Math. Anal. Appl. 227 (1998), 55-67.

[15] J. Jachymski, Order theoretic aspects of metric fixed point theory, Chapter 18 in: Handbook of Metric Fixed Point Theory (W.A. Kirk and B. Sims, eds.), Kluwer Acad. Publ. (2001), 613–641.
[16] J. Jachymski, Another proof of the Browder-Göhde-Kirk theorem via ordering argument, Bull. Austral. Math. Soc. 65 (2002), 105–107.
[17] J. Jachymski, A stationary point theorem characterizing metric completeness, Appl. Math. Letters 23 (2011), 169–171.
[18] W.A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72 (1965), 1004–1006.
[19] A. Latif, Banach contraction principle and its generalizations, Chapter 2 in: Topics in Fixed Point Theory (S. Almezel, Q.H. Ansari, M.A. Khamsi, eds.), Springer Inter. Publ. Switzerland (2014), 33–64.
[20] S. B. Nadler, Jr., Multi-valued contraction mappings, Pacific J. Math. 30 (1969), 475-488.
[21] S. Park, Some applications of Ekeland's variational principle to fixed point theory, Approximation Theory and Applications (S.P. Singh, ed.), Pitman Res. Notes Math. 133 (1985), 159–172.
[22] S. Park, Countable compactness, l.s.c. functions, and fixed points, J. Korean Math. Soc. 23 (1986), 61–66.
[23] S. Park, Equivalent formulations of Ekeland's variational principle for approximate solutions of minimization problems and their applications, Operator Equations and Fixed Point Theorems (S.P. Singh et al., eds.), MSRI-Korea Publ. 1 (1986), 55–68.
[24] S. Park, Equivalent formulations of Zorn's lemma and other maximum principles, J. Korean Soc. Math. Edu. 25 (1987), 19–24.
[25] S. Park, Partial orders and metric completeness, Proc. Coll. Natur. Sci. SNU 12 (1987), 11–17.
[26] S. Park, On generalizations of the Ekeland-type variational principles, Nonlinear Anal. 39 (2000), 881–889.
[27] S. Park, Equivalents of various maximum principles, Results in Nonlinear Analysis 5(2) (2022), 169–174.
[28] S. Park, Applications of various maximum principles, J. Fixed Point Theory (2022) 2022:3, 1–23.
[29] S. Park, Equivalents of generalized Brøndsted principle, J. Informatics Math. Sci., to appear.