INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 15 NO. 2 PAGE 343–358 (2022)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1127959

Hyperbolic Number Forms of the Euler-Savary Equation

Duygu Çağlar and Nurten Gürses*

(Communicated by Kazım İlarslan)

ABSTRACT

This study deals with hyperbolic number forms of the Euler-Savary Equation (ESE) that find one of the four points on a pole ray, provided the other three are known. These hyperbolic number forms are examined under one-parameter planar hyperbolic motions that are examined according to the osculating circles contacting through three infinitesimally close points. The hyperbolic number approach gives more detailed information than the traditional method. Thus, it eliminates sign errors and provides convenience in the application. As a final part, examples are given to show the utility of the practical way in the application.

Keywords: *Euler-Savary Equation, hyperbolic number, one-parameter planar motion, kinematics.*
AMS Subject Classification (2020): *Primary: 53A17 ; Secondary: 11E88.*

1. Introduction

Kinematics deals with the study of relative motion and identifies the possible motion of points, objects, and systems of objects geometrically without consideration of the causes of the motions. In general, the motion of the moving objects relative to the fixed objects is examined according to the time parameter. For mathematics, kinematics is a bridge connecting geometry, physics, and mechanics. When planes are thought of as objects, rotation and translation are used to study planar kinematic geometry [11, 19, 21, 22, 30].

In 1956, one-parameter planar motions were introduced by Blaschke and Müller in the 2-dimensional Euclidean plane $\mathbb{E}^2$ [4]. Also, in [4], the relation between velocities and accelerations is examined, and the Euler-Savary Equation (ESE) is calculated. ESE gives the relation between a point in the moving plane and the center of curvature of the trajectory drawn by this point in the fixed plane [6]. Using a similar approach, one-parameter planar motions are examined in the complex plane $\mathbb{C}$ (see in [4]) and ESE is obtained by presenting the relation between the curvatures of the trajectory curves (see in [1, 23]). In analogy with complex motions, one-parameter motions in the hyperbolic plane $\mathbb{H}$ are defined by [38]. Also, ESE is determined by [13] in $\mathbb{H}$. Additionally, considering generalized complex numbers (see in [20]), one-parameter planar motion and ESE are obtained in generalized complex number plane (see in [17]). In these studies, ESE is calculated based on the radius of the osculating circles and the diameter of the inflection circle. A relative coordinate system is used to talk about ESE.

Additionally, several studies can be found in the literature on combined graphical and analytical methods in kinematic synthesis and analysis [19, 21]. These methods discuss how to observe sign conventions to identify their senses and find magnitudes of point-to-point line segments graphically. These sign conventions often become sources of error. Instead of these well-known methods, it is possible to choose new techniques convenient for digital computation. Just for this purpose, in planar kinematics, the complex number approach with its arithmetic theory is an efficient analytical technique that enables digital computation and automatically takes care of the signs. With this aim, in 1982, Sandor et al. introduced the complex number forms of the ESE using the complex number approach. This technique has the advantage of eliminating the need for sign conventions and is suitable for the application. It enables us to determine the fourth point while three of the

Received : *08-06-2022,* ***Accepted :*** *30-10-2022*
*** Corresponding author***

four points on the pole ray are known in the light of the ESE's four different complex number forms. This approach gives a direct relationship between these four points via osculating circles [11,30–33].

In this paper, we establish a new approach for the ESE by considering the hyperbolic numbers motivated by the studies [31–33]. This study is organized into 4 sections. The first section is devoted to the introduction and the rest is arranged as follows: Section 2 examines the fundamental concepts related to complex and hyperbolic numbers and gives the ESE's complex number forms. In Section 3, we obtain the hyperbolic number forms of the ESE by introducing four different forms that find each of the four special points on the same pole ray: pole point, arbitrary point of moving plane, inflection point, and the center of curvature of the path described by the arbitrary point of moving plane in the fixed plane. The hyperbolic number forms of the ESE are obtained by directly relating to these four points and discussed with the help of the osculating circles. Also, these hyperbolic forms have the advantage of eliminating sign conventions and giving an easy calculation. These advantages are also illustrated with numerical examples in Section 4.

2. Fundamental Notions

This section provides basic information about complex and hyperbolic numbers [2,3,7–9,14–16,18,20,24–29, 34–37]. Also, it includes the complex number forms of the ESE (see in [30–33]).

2.1. Complex Numbers and Hyperbolic Numbers

The set of complex numbers is denoted by $\mathbb{C} := \left\{z = x_1 + ix_2 : x_1, x_2 \in \mathbb{R}, i^2 = -1, i \neq \pm 1\right\}$. Every element of this set with the form $z = x_1 + ix_2$ is called a complex number. Here x_1 and x_2 are called real and imaginary parts of z, respectively. For $z_1 = x_1 + ix_2$, $z_2 = y_1 + iy_2 \in \mathbb{C}$, we have: $z_1 + z_2 = (x_1 + y_1) + i(x_2 + y_2)$, and $z_1 z_2 = (x_1 y_1 - x_2 y_2) + i(x_1 y_2 + x_2 y_1)$, $cz = cx_1 + icx_2$, $c \in \mathbb{R}$. Multiplication is commutative, associative, and distributes over addition. The conjugate of z is $\bar{z} = x_1 - ix_2$. The scalar product of z_1 and z_2 is defined by $\langle z_1, z_2\rangle = x_1 y_1 + x_2 y_2$. The modulus of z, which is the distance from the origin, is calculated as $\|z\| = \sqrt{\langle z, z\rangle} = \sqrt{z\bar{z}} = \sqrt{{x_1}^2 + {x_2}^2}$. The geometric location of points at a fixed distance r from a fixed point z_0 is a circle defined by $d(z_0, z) = \sqrt{\langle z_0 - z, z_0 - z\rangle} = r$. Since the set of complex numbers is isomorphic to the Euclidean plane, $\mathbf{z} = (x_1, x_2)$ can be written for the position vector of z. Moreover, the polar form of z is defined as $z = \|z\|\, e^{i\arg(z)}$, where $\arg(z) = \tan^{-1}\left(\frac{x_2}{x_1}\right)$. Here $\arg(z)$ is the positive angle from the real axis to the vector $\mathbf{z}$. From Euler's formula $e^{i\arg(z)} = \cos(\arg(z)) + i\sin(\arg(z))$ can be written [18,20,36].

The next theorem establishes the projection vector in $\mathbb{C}$.

Theorem 2.1. *In $\mathbb{C}$, if the projection vector of $\overrightarrow{AC}$ onto vector $\mathbf{x}$ is $\overrightarrow{AB}$, then it can be written as:*

$$\overrightarrow{AB} = \cos\left(\sphericalangle\left(\overrightarrow{AB}, \overrightarrow{AC}\right)\right) e^{i(\arg(\overrightarrow{AB}) - \arg(\overrightarrow{AC}))}\overrightarrow{AC},$$

where $\overrightarrow{AC}$ and $\mathbf{x}$ are arbitrary vectors.

Besides, the set of hyperbolic numbers is denoted by $\mathbb{H} := \left\{z = x_1 + jx_2 : x_1, x_2 \in \mathbb{R}, j^2 = 1, j \neq \pm 1, j \notin \mathbb{R}\right\}$. Every element of this set with the form $z = x_1 + jx_2$ is called a hyperbolic number. Here x_1 and x_2 are called real and imaginary parts of z, respectively. For $z_1 = x_1 + jx_2$, $z_2 = y_1 + jy_2 \in \mathbb{H}$, we have: $z_1 + z_2 = (x_1 + y_1) + j(x_2 + y_2)$, $z_1 z_2 = (x_1 y_1 + x_2 y_2) + j(x_1 y_2 + x_2 y_1)$, $cz = cx_1 + jcx_2, c \in \mathbb{R}$. Multiplication is commutative, associative, and distributes over addition. The hyperbolic conjugate of z is $\overline{z} = x_1 - jx_2$. The hyperbolic inner product of z_1 and z_2 is defined by $\langle z_1, z_2\rangle = x_1 y_1 - x_2 y_2$. The hyperbolic modulus of z is given by $\|z\| = \sqrt{|\langle z, z\rangle|} = \sqrt{|z\overline{z}|} = \sqrt{|{x_1}^2 - {x_2}^2|}$. The set of all points in $\mathbb{H}$ that satisfies the equation $d(z_0, z) = \sqrt{|\langle z_0 - z, z_0 - z\rangle|} = r$ is a circle with radius r and center z_0. The asymptotes $y = \pm x$ of the unit circles (dashed lines in Fig. 1) naturally separate $\mathbb{H}$ into four regions labeled branches $\mathbb{H}$-I, $\mathbb{H}$-II, $\mathbb{H}$-III, and $\mathbb{H}$-IV (see in Fig. 1). The hyperbolic numbers, serve as coordinates in the Lorentzian plane in much the same way that the complex numbers serve as coordinates in the Euclidean plane. The relationship between the complex numbers and the Euclidean plane also exists between the hyperbolic numbers and the Lorentzian plane. Thus, the position vector of a hyperbolic number $z = x_1 + jx_2$ is $\mathbf{z} = (x_1, x_2)$. The character of any vector in Lorentzian plane can be classified such as: $\mathbf{z}$ is called spacelike (SL) if $\langle \mathbf{z}, \mathbf{z}\rangle > 0$ or $\mathbf{z} = \mathbf{0}$, timelike (TL) if $\langle \mathbf{z}, \mathbf{z}\rangle < 0$, null if $\langle \mathbf{z}, \mathbf{z}\rangle = 0$ whereby $\mathbf{z} \neq \mathbf{0}$. $\mathbf{z}$ is future pointing timelike (FPTL) if $\langle \mathbf{z}, \mathbf{E}\rangle < 0$, and past pointing timelike (PPTL) if $\langle \mathbf{z}, \mathbf{E}\rangle > 0$ where $\mathbf{E} = (0, 1)$. $\mathbf{z}$ is spacelike-I (SL-I) if $\langle \mathbf{z}, \mathbf{e}\rangle > 0$, and spacelike-III (SL-III) if

$\langle \mathbf{z}, \mathbf{e} \rangle < 0$ where $\mathbf{e} = (1, 0)$. Moreover, the polar forms of z are given as follows:

$$\begin{cases} z = \;\; \|z\| \, e^{j \arg(z)} & \text{if } z \text{ is on } \mathbb{H}\text{-I}, \\ z = -\|z\| \, e^{j \arg(z)} & \text{if } z \text{ is on } \mathbb{H}\text{-III}, \end{cases} \tag{2.1}$$

where $\arg(z) = \tanh^{-1}\left(\frac{x_2}{x_1}\right)$. Here $\arg(z)$ represents the angle from the real axis to the vector $\mathbf{z}$. And

$$\begin{cases} z = \;\; j \, \|z\| \, e^{j \arg(z)} & \text{if } z \text{ is on } \mathbb{H}\text{-II}, \\ z = -j \, \|z\| \, e^{j \arg(z)} & \text{if } z \text{ is on } \mathbb{H}\text{-IV}, \end{cases} \tag{2.2}$$

where $\arg(z) = \coth^{-1}\left(\frac{x_2}{x_1}\right) = \tanh^{-1}\left(\frac{x_1}{x_2}\right)$. Here $\arg(z)$ represents the angle from the imaginary axis to the vector $\mathbf{z}$ [2,3,7–9,14–16,20,24–29,34,35,37]. Using the hyperbolic form of the Euler's formula, $e^{j \arg(z)}$ in equation (2.1) and equation (2.2) can be written as $e^{j \arg(z)} = \cosh(\arg(z)) + j \sinh(\arg(z))$.

For every region in $\mathbb{H}$, the hyperbolic numbers approach the asymptotes relating to its argument value, giving rise to positive and negative directed arguments (see in Fig. 1). Hyperbolic rotations through hyperbolic angles occur along the hyperbola. The positive and negative directed hyperbolic rotation can be seen in Fig. 2 [20,27].

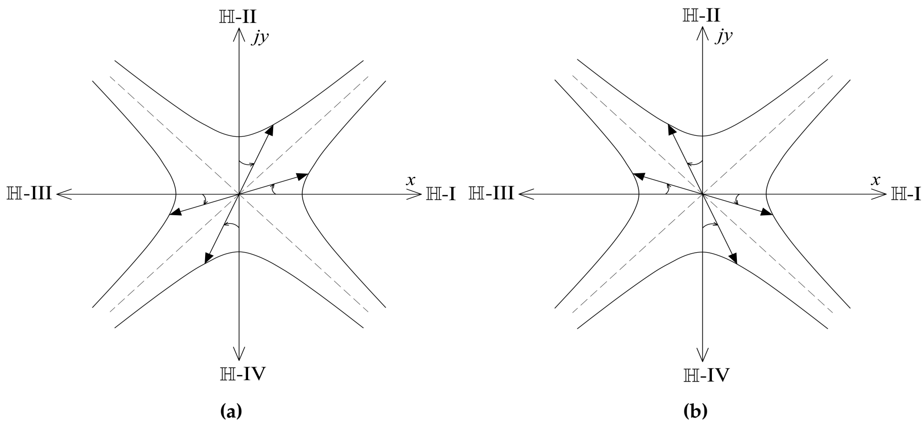


Figure 1. (a) Positive directed argument (b) Negative directed argument

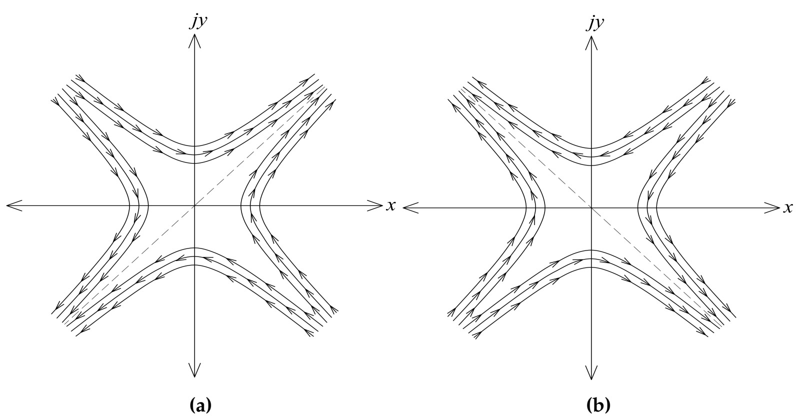


Figure 2. (a) Positive directed rotation, (b) Negative directed rotation

Theorem 2.2. *The oriented hyperbolic angle function $\sphericalangle(.,.)$ has the following properties [25]:*

i) $\sphericalangle(\mathbf{z},\mathbf{z}) = \sphericalangle(\mathbf{z},-\mathbf{z}) = 0$
ii) $\sphericalangle(\mathbf{z_1},\mathbf{z_2}) = -\sphericalangle(\mathbf{z_2},\mathbf{z_1})$
iii) $\sphericalangle(\mathbf{z_1},\mathbf{z_2}) = \sphericalangle(-\mathbf{z_1},\mathbf{z_2}) = \sphericalangle(\mathbf{z_1},-\mathbf{z_2}) = \sphericalangle(-\mathbf{z_1},-\mathbf{z_2})$
iv) $\sphericalangle(\mathbf{z_1},\mathbf{z_2}) + \sphericalangle(\mathbf{z_2},\mathbf{z_3}) = \sphericalangle(\mathbf{z_1},\mathbf{z_3})$

The following theorem related to the projection vector in $\mathbb{H}$ is given without proof since it can be easily verified using study [24].

Theorem 2.3. *In $\mathbb{H}$, if the projection vector of $\overrightarrow{AC}$ onto vector $\mathbf{x}$ is $\overrightarrow{AB}$, then it can be written as follows:*

$$\overrightarrow{AB} = \cosh\left(\sphericalangle\left(\overrightarrow{AB},\overrightarrow{AC}\right)\right) e^{j(\arg(\overrightarrow{AB})-\arg(\overrightarrow{AC}))}\overrightarrow{AC},$$

where $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are two noncollinear SL vectors and $\overrightarrow{BC}$ is TL vector. Similarly

$$\overrightarrow{AB} = j\sinh\left(\sphericalangle\left(\overrightarrow{AB},\overrightarrow{AC}\right)\right) e^{j(\arg(\overrightarrow{AB})-\arg(\overrightarrow{AC}))}\overrightarrow{AC},$$

where $\overrightarrow{BC}$ and $\overrightarrow{AC}$ are two noncollinear SL vectors and $\overrightarrow{AB}$ is TL vector.

It is worthy of note that, the equations in the above theorem also hold if the words "SL" and "TL" are reversed.

2.2. Complex Number Forms of the ESE

In this section, four different complex forms of the ESE* are discussed [31].

Consider one-parameter planar motion of the moving plane Σ with respect to the fixed plane of reference Σ'. At each time t, the fixed and moving pole curves, p and π, are tangent to each other at pole point I. The osculating circles of p and π have radius (or radius of curvature) ρ_p, ρ_π and center O_p, O_π, respectively. As the motion progresses, π rolls on p without slipping. Therefore, the osculating circle of radius ρ_π (is called the moving osculating circle) rolls without slipping on the osculating circle of radius ρ_p (is called the fixed osculating circle) through three infinitesimally close positions, i.e. they contact through three infinitesimally close points. Let $\{O_p; x, iy\}$ be the fixed coordinate system linked to Σ and let A be a point on Σ. Then the position vector of the first position of A according to $\{O_p; x, iy\}$ is: $\overrightarrow{O_pA} = \mathbf{w} + \mathbf{z}$, where $\mathbf{w} = \boldsymbol{\rho}_p - \boldsymbol{\rho}_\pi$, $\mathbf{z} = \boldsymbol{\rho}_\pi + \mathbf{a}$ and $\mathbf{a} = \overrightarrow{IA}$ is pole ray emanating from I in the direction θ with respect to the iy-axis (the common normal of the pole curves at I). Also, $\boldsymbol{\rho}_p = \overrightarrow{O_pI}$ and $\boldsymbol{\rho}_\pi = \overrightarrow{O_\pi I}$. After π has rolled on p infinitesimally small angle ϕ; A displaces to A_k and position vector of A_k is given by:

$$\overrightarrow{O_pA_k} = \mathbf{w}e^{i\psi} + \mathbf{z}e^{i\phi}, \tag{2.3}$$

where the angles ψ and ϕ are infinitesimally small rotations of $\mathbf{w}$ and $\mathbf{z}$ respectively. Meanwhile, the points A and O_π displace infinitesimally close points which enables to compute the velocity vectors [31]: The velocity vector of O_π is given in two ways: $\mathbf{V}_{O_\pi} = i\dot{\phi}(-\boldsymbol{\rho}_\pi)$ or $\mathbf{V}_{O_\pi} = i\dot{\psi}(\boldsymbol{\rho}_p - \boldsymbol{\rho}_\pi)$. Thus, it is clear that: $\dot{\psi} = \boldsymbol{\rho}\dot{\phi}$ where $\boldsymbol{\rho} = \frac{-\boldsymbol{\rho}_\pi}{\boldsymbol{\rho}_p - \boldsymbol{\rho}_\pi}$. The velocity vector of A which is moved from the first position to A_k, is also found as $\mathbf{V}_A = i\,\dot{\psi}\,\mathbf{w} + i\,\dot{\phi}$ or $\mathbf{V}_A = i\,\dot{\phi}\,\mathbf{a}$. By taking the second derivative of equation (2.3) and applying necessary calculations give the following acceleration vector:

$$\boldsymbol{\mathcal{A}}_A = i\ddot{\phi}\mathbf{a} - \dot{\phi}^2\mathbf{a} - i\dot{\phi}\mathbf{u}.$$

Here, the first term is the tangential acceleration component, the second term is the centripetal component and the third term is the invariant acceleration component where $\mathbf{u}$ is the transfer velocity vector. Transfer velocity is defined as the time rate of change in position along p of the instant center I as it π rolls on p.

*In the classical approach, during one-parameter planar motions in $\mathbb{C}$, ESE is given by $\left(\frac{1}{\overline{IA}} - \frac{1}{\overline{IO_A}}\right)\mathrm{Im}(e^{i\alpha}) = \frac{1}{\overline{IO_\pi}} - \frac{1}{\overline{IO_p}} = -\frac{d\nu}{ds}$, where I is the pole point, O_π and O_p are the centers of curvatures of polodes at their point of contact, O_A is the center of path curvature of A of the moving plane described in the fixed plane, α is the argument of the pole ray, $d\nu$ is the infinitesimal small rotation angle and ds is the scalar arc element of the pole curves. Here overlines are used to indicate that the particular quantity is directed. For detailed information, see the studies [1,4,6,17,22,23]

This vector is calculated as $\mathbf{u} = i\dot{\psi}\boldsymbol{\rho}_p = i\dot{\phi}\rho_p\boldsymbol{\rho}$. Also, considering the inflection point[†] and denoting the normal component of $\boldsymbol{\mathcal{A}}_A$ as $\boldsymbol{\mathcal{A}}_A^n$, we have: $\boldsymbol{\mathcal{A}}_A^n = -\dot{\phi}^2\mathbf{a} + (\cos\theta)e^{i\theta}(-i\dot{\phi}\mathbf{u}) = 0$. Simplifying and letting $-i\mathbf{u}/\dot{\phi} = \boldsymbol{\delta}$, we obtain: $\mathbf{J}_A = \overrightarrow{IJ_A} = \cos\theta e^{i\theta}\boldsymbol{\delta}$ where J_A is the inflection point.

Consequently, considering the four points O_A (the center of path curvature of A), J_A, A and I on the pole ray and correlating the vectors which are all collinear with the ray as $\mathbf{O}_A = \overrightarrow{IO_A}$, $\mathbf{J}_A = \overrightarrow{IJ_A}$, $\mathbf{a} = \overrightarrow{IA}$, and $\boldsymbol{\rho}_A = \overrightarrow{O_AA}$, the following four complex ESE forms provide a way to find any one of the four points: O_A, J_A, A and I, if the other three are known, respectively [31–33]:

$$\begin{cases} \boldsymbol{\rho}_A = \dfrac{\|\mathbf{a}\|^2}{\|\mathbf{a}-\mathbf{J}_A\|}e^{i\arg(\mathbf{a}-\mathbf{J}_A)}, & \text{(ESE-1)} \\ \mathbf{J}_A = \mathbf{a} - \dfrac{\|\mathbf{a}\|^2}{\|\boldsymbol{\rho}_A\|^2}\boldsymbol{\rho}_A, & \text{(ESE-2)} \\ \mathbf{a} = \dfrac{\mathbf{J}_A\mathbf{O}_A}{\mathbf{O}_A+\mathbf{J}_A}, & \text{(ESE-3)} \\ \mathbf{a} = \left|\left(\left\|\overrightarrow{J_AA}\right\|\|\boldsymbol{\rho}_A\|\right)^{1/2}\right|\left(\pm e^{i\arg\boldsymbol{\rho}_A}\right)\cdot & \text{(ESE-4)} \end{cases}$$

3. Construction of Hyperbolic Number Forms of the ESE

Motivated by studies [31–33], the main goal of this section is to examine ESE with the hyperbolic number approach[‡]. While calculating hyperbolic number forms of the ESE, during the one-parameter planar hyperbolic motion, types of pole curves (SL, TL) and types of pole rays (SL-I, SL-III, FPTL, PPTL) must be taken into consideration. Throughout this section, four different hyperbolic number forms of the ESE will be examined for SL pole curves considering different types of pole rays. For TL pole curves, the calculations will be conducted with a similar approach, and the results will be compared in Corollary 3.2 for the sake of brevity.

Let us consider the one-parameter planar hyperbolic motion of moving plane Σ with respect to the fixed plane of reference Σ'. At each time t, the fixed and moving SL pole curves, p and π, are tangent to each other at pole point I. The osculating circles of p and π have radius ρ_p, ρ_π and center O_p, O_π, respectively. Here $\overrightarrow{O_pI} = \boldsymbol{\rho}_p$ and $\overrightarrow{O_\pi I} = \boldsymbol{\rho}_\pi$. As the hyperbolic motion progresses, the osculating circle of radius ρ_π rolls without slipping on the osculating circle of radius ρ_p through three infinitesimally close positions. Let $\{O_p; x, jy\}$ be the fixed coordinate system linked to Σ and let A be a point on Σ.

Remark 3.1. Considering SL pole curves, the following probabilities can be given for the radius vectors $\boldsymbol{\rho}_p$ and $\boldsymbol{\rho}_\pi$ of the osculating circles: 1st probability: FPTL & PPTL (see in Fig. 3), 2nd probability: PPTL & FPTL, 3rd probability: PPTL & PPTL, 4th probability: FPTL & FPTL.

Throughout this section 1st probability is taken into consideration, namely $\boldsymbol{\rho}_p$ is FPTL and $\boldsymbol{\rho}_\pi$ is PPTL.

The position vector of the first position of A according to the fixed coordinate system is $\overrightarrow{O_pA} = \mathbf{w} + \mathbf{z}$, where

$$\mathbf{w} \quad = \overrightarrow{O_pO_\pi} = \overrightarrow{O_pI} + \overrightarrow{IO_\pi} = \boldsymbol{\rho}_p - \boldsymbol{\rho}_\pi$$

and

$$\mathbf{z} \quad = \overrightarrow{O_\pi A} = \overrightarrow{O_\pi I} + \overrightarrow{IA} = \boldsymbol{\rho}_\pi + \mathbf{a}.$$

Also $\mathbf{a} = \overrightarrow{IA}$ is pole ray emanating from I in the direction[§] θ with respect to the jy-axis. Here jy-axis[¶] is the common normal of pole curves at I and θ is the angle between pole ray with the common normal of pole curves (From Theorem 2.2-iii, $\sphericalangle(jy, \mathbf{a}) = \sphericalangle(-jy, \mathbf{a}) = \theta$).

[†] Points with the center of the path curvature at infinity is called the inflection point [4] and the normal component of their acceleration vector is zero [31].

[‡] In the classical approach, during one-parameter hyperbolic planar motions, ESE for SL-I (FPTL) pole ray on SL (TL) pole curves is given by $\left(\dfrac{1}{\overline{IA}} - \dfrac{1}{\overline{IO_A}}\right)\mathrm{Im}(e^{j\alpha}) = \dfrac{1}{\overline{IO_\pi}} - \dfrac{1}{\overline{IO_p}} = -\dfrac{d\nu}{ds}$, where ds represents the scalar arc element and $d\nu$ represents the infinitesimal hyperbolic angle of the motion of the pole curves. Here overlines are used to indicate that the particular quantity is directed. Here, if the pole curves are SL (TL) and pole ray is FPTL (SL-I), $\mathrm{Im}(e^{j\alpha})$ is changed as $\mathrm{Im}(je^{j\alpha})$. For detailed information, see the studies [13,17,38].

[§] Throughout the study, the positive direction is examined.

[¶] For TL pole curves, x-axis is the common normal of the pole curves at I.

While π has rolled on p with infinitesimally small hyperbolic angle ϕ; A displaces from the first position to A_k and the position vector of A_k is given by (see the visualization in Fig. 4 for the SL-I pole ray):

$$\overrightarrow{O_pA_k} = \mathbf{w}e^{j\psi} + \mathbf{z}e^{j\phi}. \tag{3.1}$$

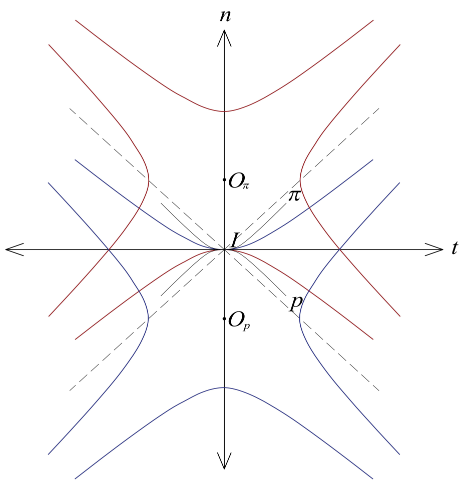
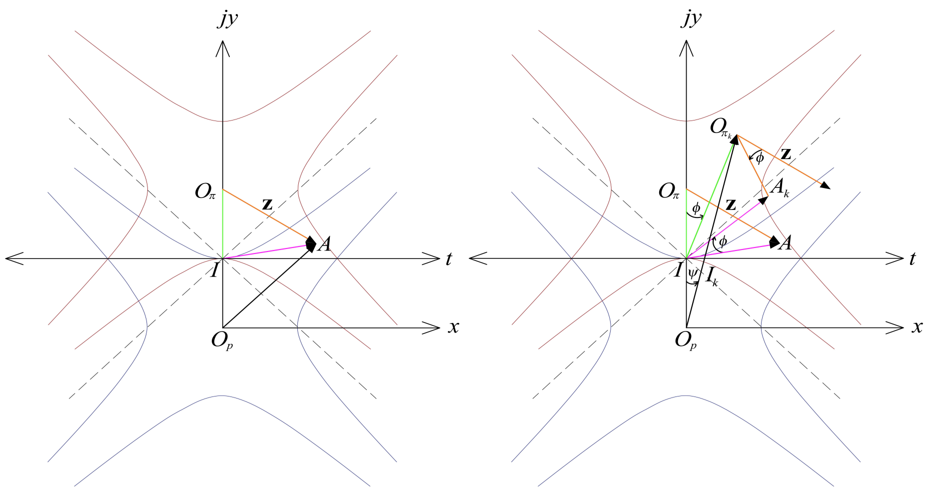


Figure 3. The position of SL pole curves with osculating circles||



(a) **(b)**

Figure 4. (a) The first position, (b) The position after rotation with infinitesimally small hyperbolic angle ϕ for SL-I pole ray

3.1. Examining Velocities and Acceleration

Since π has rolled on p by the infinitesimally small hyperbolic angle ϕ, the points A and O_π displace infinitesimally close points. Thus, the velocity vectors of the points are found by taking the time dependent derivative of the position vectors. With relation $\overrightarrow{O_pO_{\pi_k}} = \overrightarrow{O_pI} + \overrightarrow{IO_{\pi_k}}$ (see in Fig. 4), the velocity vector of point O_π can be found in two ways. The first derivative of this equation is $\dot{\overrightarrow{O_pO_{\pi_k}}} = \dot{\overrightarrow{IO_{\pi_k}}}$. Since infinitesimally close displacement occurs at the same time t, $\boldsymbol{\rho}_p$ does not depend on t. During this displacement O_π, moves to O_{π_k},

|| t and n are the common tangent and common normal of SL pole curves, respectively.

and its position vector becomes:

$$\overrightarrow{O_pO_{\pi_k}} = \mathbf{w}e^{j\psi} \text{ or } \overrightarrow{IO_{\pi_k}} = -\boldsymbol{\rho}_\pi e^{j\phi}.$$

By taking derivatives, we have:

$$\dot{\overrightarrow{O_pO_{\pi_k}}} = j\dot{\psi}(\boldsymbol{\rho}_p - \boldsymbol{\rho}_\pi)e^{j\psi} \text{ or } \dot{\overrightarrow{IO_{\pi_k}}} = j\dot{\phi}(-\boldsymbol{\rho}_\pi)e^{j\phi}.$$

The above equations are valid throughout the two infinitesimally small displacements, including the very beginning when $\phi = \psi = 0$. Thus, we express:

$$\mathbf{V}_{O_\pi} = j\dot{\phi}(-\boldsymbol{\rho}_\pi) = j\dot{\psi}(\boldsymbol{\rho}_p - \boldsymbol{\rho}_\pi) \tag{3.2}$$

or

$$\mathbf{V}_{O_\pi} = j\dot{\phi}(-\boldsymbol{\rho}_\pi) = j\dot{\psi}\mathbf{w}.$$

Then, by equation (3.2), it is clear that: $\dot{\phi}(-\boldsymbol{\rho}_\pi) = \dot{\psi}(\boldsymbol{\rho}_p - \boldsymbol{\rho}_\pi)$. Thus, we obtain:

$$\dot{\psi} = \boldsymbol{\rho}\dot{\phi}, \text{ where } \boldsymbol{\rho} = \frac{-\boldsymbol{\rho}_\pi}{\boldsymbol{\rho}_p - \boldsymbol{\rho}_\pi}. \tag{3.3}$$

Here $\boldsymbol{\rho}$ is either a positive or a negative real, its algebraic sign depends on the position of the osculating circles and the relative magnitudes of ρ_π and ρ_p. It is expressed in vector form for assurance of its accurate algebraic sign in digital computation.

Similarly, A moves to infinitesimally point A_k during hyperbolic motion. Via the first derivative of $\overrightarrow{O_pA_k} = \overrightarrow{O_pI} + \overrightarrow{IA_k}$ (see in Fig. 4), the velocity vector of A is also found in two ways. Firstly, the position vector of A at position A_k can be written as equation (3.1) or $\overrightarrow{IA_k} = \mathbf{a}e^{j\phi}$. Taking the derivative of these equations give:

$$\dot{\overrightarrow{O_pA_k}} = j\dot{\psi}\mathbf{w}e^{j\psi} + j\dot{\phi}\mathbf{z}e^{j\phi} \text{ or } \dot{\overrightarrow{IA_k}} = j\dot{\phi}\mathbf{a}e^{j\phi}.$$

Above equations are valid for $\phi = \psi = 0$. Thus, we express the velocity vector of A at the first position as:

$$\mathbf{V}_A = j\dot{\psi}\mathbf{w} + j\dot{\phi}\mathbf{z} = j\dot{\phi}\mathbf{a}.$$

By taking the second derivative of equation (3.1) and letting $\phi = \psi = 0$, we obtain the acceleration vector of point A at the first position as:

$$\boldsymbol{\mathcal{A}}_A = j\ddot{\phi}\mathbf{a} + \dot{\phi}^2\mathbf{a} + \dot{\phi}^2\boldsymbol{\rho}_\pi\left(1 - \boldsymbol{\rho}\right). \tag{3.4}$$

Since moving pole curve π rolls on fixed pole curve p, the rate of change in position of pole point I along pole curve p over time is called the transfer rate of the pole point. So using equation (3.3), the transfer velocity $\mathbf{u} = j\dot{\psi}\boldsymbol{\rho}_p$ can be rewritten as:

$$\mathbf{u} = j\dot{\phi}\boldsymbol{\rho}_p\boldsymbol{\rho}. \tag{3.5}$$

Therefore, equation (3.4) is rewritten as follows:

$$\boldsymbol{\mathcal{A}}_A = j\ddot{\phi}\mathbf{a} + \dot{\phi}^2\mathbf{a} - j\dot{\phi}\mathbf{u}. \tag{3.6}$$

Here the first term is the tangential acceleration component, the second term is the centripetal component. The third term is the invariant acceleration component since it is independent of the choice of A and I, (see in Fig. 5 (a)).

Corollary 3.1. *Algebraic expressions of velocity vectors are the same for all probabilities, however, vector type varies.* $\mathbf{V}_{O_\pi}$ *is SL-I in 1st (Fig. 3) and 3rd probabilities, SL-III in 2nd and 4th probabilities for each pole ray. Additionally, for each probability,* $\mathbf{V}_A$ *is FPTL, PPTL, SL-I and SL-III for SL-I, SL-III, FPTL and PPTL pole rays, respectively.*

3.2. Obtaining Hyperbolic Number Forms of the ESE

In this section, consider the following four points on the pole ray: the pole point I, an arbitrary point A of the moving plane, the center of curvature O_A of the path of A, and the inflection point J_A on the pole ray. Let us find the fourth one when three of them are known.

Considering mentioned points, the following four vectors which are all collinear with the ray are written as $\overrightarrow{IA} = \mathbf{a}$, $\overrightarrow{IJ_A} = \mathbf{J}_A$, $\overrightarrow{IO_A} = \mathbf{O}_A$ and $\overrightarrow{O_A A} = \boldsymbol{\rho}_A$. Here $\boldsymbol{\rho}_A$ is the vector radius of path curvature.

The center of path curvature, O_A, is fixed and then the angular velocity (denote with $\dot{\gamma}$) of the vector radius of curvature ρ_A pivoted at O_A is given by: $\dot{\gamma} = \dot{\phi}\dfrac{\mathbf{a}}{\boldsymbol{\rho}_A}\cdot$ The quantity $\dfrac{\mathbf{a}}{\boldsymbol{\rho}_A}$ is a positive or negative real, according as $\mathbf{a}$ and $\boldsymbol{\rho}_A$ have the same or opposite sense. Thus, we obtain:

$$\boldsymbol{\mathcal{A}}_A^n = (\dot{\gamma})^2 \boldsymbol{\rho}_A = (\dot{\phi})^2 \frac{\|\mathbf{a}\|^2}{\|\boldsymbol{\rho}_A\|^2} \boldsymbol{\rho}_A \cdot \tag{3.7}$$

In this step, we consider the following four cases:

Case 1: Hyperbolic ESE Forms for SL-I Pole Ray:

Let us first find the normal component of equation (3.6) using the projection vector (see Theorem 2.3) to calculate the inflection point.

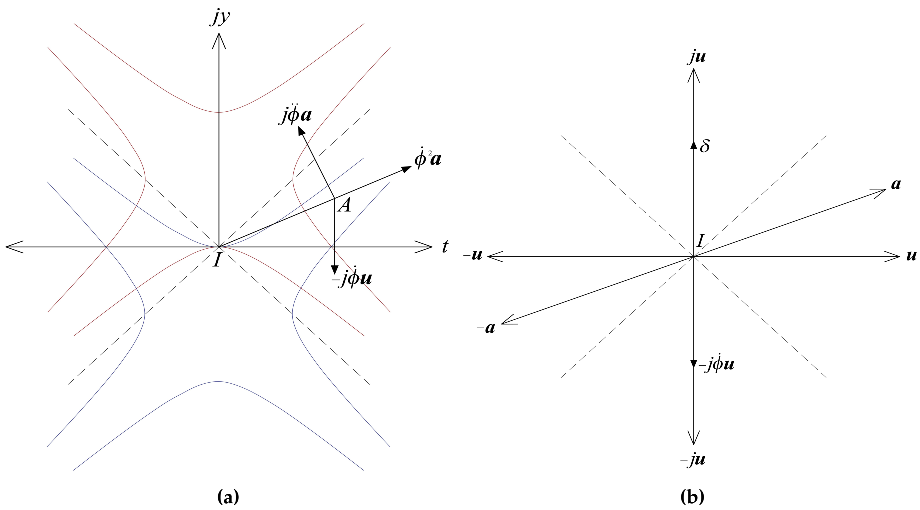


Figure 5. (a) The acceleration, (b) The projection vector of SL-I pole ray

For the SL-I pole ray, the normal component of $-j\dot{\phi}\mathbf{u}$ (PPTL) is obtained by its projection onto $-\mathbf{a}$ (SL-III) such that (see Fig. 5 (b)):

$$(-j\dot{\phi}\mathbf{u})^n = j \sinh\theta e^{j(\arg(-\mathbf{a}) - \arg(-j\mathbf{u}))}(-j\dot{\phi}\mathbf{u}).$$

Substituting the above equation into equation (3.6), we have:

$$\boldsymbol{\mathcal{A}}_A^n = \dot{\phi}^2\mathbf{a} + j \sinh\theta e^{j(\arg(-\mathbf{a}) - \arg(-j\mathbf{u}))}(-j\dot{\phi}\mathbf{u}). \tag{3.8}$$

Here $\boldsymbol{\mathcal{A}}_A^n = 0$ gives $\mathbf{J}_A$ as SL-I:

$$\mathbf{J}_A = j \sinh\theta e^{j(\arg(-\mathbf{a}) - \arg(-j\mathbf{u}))}\left(\frac{j\mathbf{u}}{\dot{\phi}}\right)\cdot \tag{3.9}$$

Using Theorem 2.2, this equation is also rewritten as:

$$\mathbf{J}_A = j \sinh\theta e^{j(\arg(\mathbf{a}) - \arg(j\mathbf{u}))}\left(\frac{j\mathbf{u}}{\dot{\phi}}\right)\cdot \tag{3.10}$$

Simplifying and letting

$$\frac{j\mathbf{u}}{\dot{\phi}} = \boldsymbol{\delta}, \tag{3.11}$$

it is said that *the projection vector of* $\boldsymbol{\delta}$ *onto the pole ray is* $\mathbf{J}_A$. The geometrical locus of inflection points is a circle with diameter δ for all values of θ. This circle is called inflection circle. Using equations (3.7) and (3.8), we have:

$$\boldsymbol{\mathcal{A}}_A^n = (\dot{\phi})^2 \frac{\|\mathbf{a}\|^2}{\|\boldsymbol{\rho}_A\|^2}\boldsymbol{\rho}_A = \dot{\phi}^2\mathbf{a} + j\sinh\theta e^{j(\arg(-\mathbf{a})-\arg(-j\mathbf{u}))}(-j\dot{\phi}\mathbf{u}).$$

Substituting equation (3.9) into the last equation, we obtain:

$$\frac{\boldsymbol{\rho}_A}{\|\boldsymbol{\rho}_A\|^2} = \frac{\mathbf{a} - \mathbf{J}_A}{\|\mathbf{a}\|^2}. \tag{3.12}$$

It is worth to note that, for SL pole rays, the types of $\boldsymbol{\rho}_A$ and $\overrightarrow{J_A A}$ are same since $\arg\boldsymbol{\rho}_A = \arg(\mathbf{a} - \mathbf{J}_A)$. In other words, equation (3.12) reveals the fact that O_A and J_A are always on the same side of A. Thus $\mathbf{a}$ and $\mathbf{J}_A$ are SL-I, and $\mathbf{O}_A$ is SL-I or SL-III (see Fig. 6). $\boldsymbol{\rho}_A$ can be solved from equation (3.12) such that:

$$\|\boldsymbol{\rho}_A\| = \frac{\|\mathbf{a}\|^2}{\|\mathbf{a} - \mathbf{J}_A\|}.$$

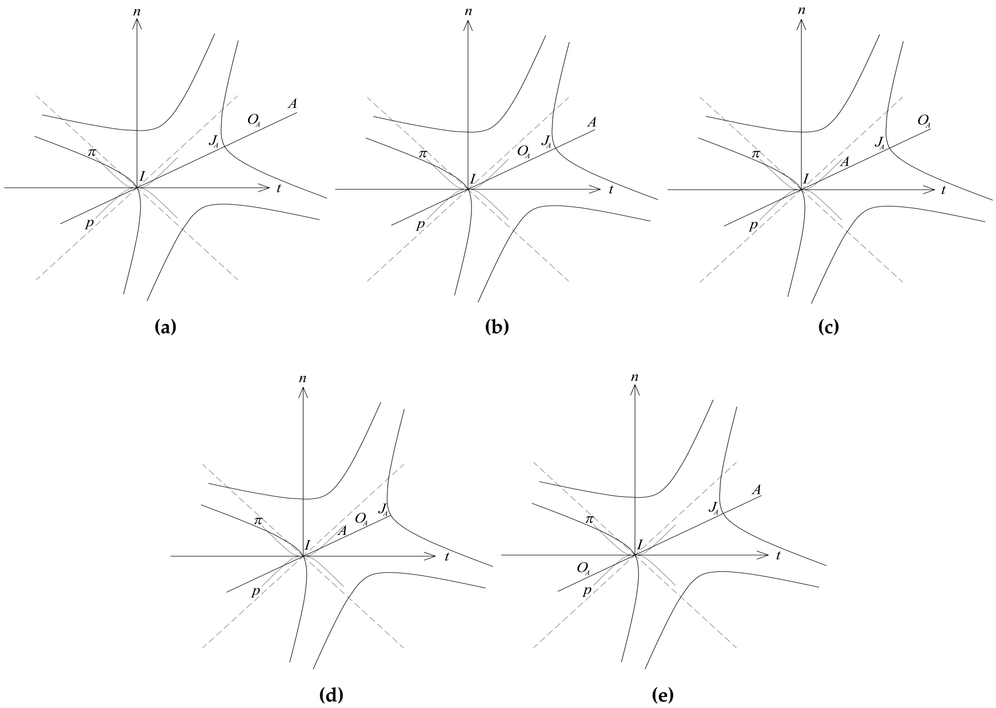


Figure 6. Different positions of four points for the ESE for SL-I pole ray (The circles in figures are inflection circles)

It should be noted that the equation $\frac{\|\boldsymbol{\rho}_A\| e^{j\arg(\boldsymbol{\rho}_A)}}{\|\boldsymbol{\rho}_A\|^2} = \frac{\|\mathbf{a}-\mathbf{J}_A\| e^{j\arg(\mathbf{a}-\mathbf{J}_A)}}{\|\mathbf{a}\|^2}$ is used where $\boldsymbol{\rho}_A$, $\overrightarrow{J_A A}$ are SL-I and $\boldsymbol{\rho}_A$ or $\overrightarrow{J_A A}$ are SL-III. Also, equation (2.1) is used for $\boldsymbol{\rho}_A$ (SL-I or SL-III) in the last equation. Hence, the first hyperbolic form of the ESE for SL-I pole ray is obtained as:

$$\boldsymbol{\rho}_A = \pm\frac{\|\mathbf{a}\|^2}{\|\mathbf{a} - \mathbf{J}_A\|} e^{j\arg(\mathbf{a}-\mathbf{J}_A)}. \qquad \text{(ESE-1)} \tag{3.13}$$

This equation is in sign-proof hyperbolic-number notation. ESE-1 is applicable to find O_A when I, A and J_A are known. Unlike the traditional expression of the ESE, the argument part of ESE-1 indicates that the path of A is always concave towards J_A. The choice of the sign is based on the fact that O_A and J_A are always on the same side of A similar to equation (3.12).

The second hyperbolic form of the ESE is produced by solving equation (3.12) for $\mathbf{J}_A$:

$$\mathbf{J}_A = \mathbf{a} - \frac{\|\mathbf{a}\|^2}{\|\boldsymbol{\rho}_A\|^2}\boldsymbol{\rho}_A, \qquad \text{(ESE-2)} \tag{3.14}$$

for the case when points I, A and O_A are known, and J_A is sought.

The third hyperbolic form of the ESE can be obtained by rearranging equation (3.14) and solving it for $\mathbf{a}$:

$$\frac{1}{\mathbf{J}_A} = \frac{1}{\mathbf{a} - \dfrac{\|\mathbf{a}\|^2}{\|\boldsymbol{\rho}_A\|^2}\boldsymbol{\rho}_A} = \frac{1}{\mathbf{a} - \dfrac{\|\mathbf{a}\|^2 e^{j2\arg\mathbf{a}}}{\|\boldsymbol{\rho}_A\|^2 e^{j2\arg\mathbf{a}}}\left(\pm\|\boldsymbol{\rho}_A\|\, e^{j\arg\boldsymbol{\rho}_A}\right)}.$$

Let's examine the expression $\dfrac{e^{j\arg\boldsymbol{\rho}_A}}{e^{j2\arg\mathbf{a}}}$ for vector $\boldsymbol{\rho}_A$ (SL-I or SL-III). Using relationship $e^{j\arg\boldsymbol{\rho}_A} = \begin{cases} e^{j\arg\mathbf{a}} \\ -e^{j\arg\mathbf{a}} \end{cases}$, we have:

$$\frac{1}{\mathbf{J}_A} = \frac{1}{\mathbf{a} - \dfrac{\|\mathbf{a}\|^2 e^{j2\arg\mathbf{a}}}{\pm\|\boldsymbol{\rho}_A\|}\dfrac{1}{e^{j\arg\boldsymbol{\rho}_A}}} = \frac{1}{\mathbf{a} - \dfrac{\mathbf{a}^2}{\boldsymbol{\rho}_A}} = \frac{\boldsymbol{\rho}_A}{\mathbf{a}(\boldsymbol{\rho}_A - \mathbf{a})}.$$

Then, with the necessary regulations, we obtain:

$$\frac{1}{\mathbf{J}_A} = \frac{1}{\mathbf{a}} + \frac{1}{\boldsymbol{\rho}_A - \mathbf{a}} = \frac{1}{\mathbf{a}} - \frac{1}{\mathbf{O}_A}.$$

Solving it for $\mathbf{a}$ gives $\dfrac{1}{\mathbf{a}} = \dfrac{1}{\mathbf{J}_A} + \dfrac{1}{\mathbf{O}_A}$, and so

$$\mathbf{a} = \frac{\mathbf{J}_A\mathbf{O}_A}{\mathbf{O}_A + \mathbf{J}_A} \qquad \text{(ESE-3)} \tag{3.15}$$

is obtained. This form finds the location of A when the locations of points I, J_A and O_A are known on the pole ray. Equation (3.15) is in vector form and therefore is suitable for automatic computation since it contains information about the sense of the vectors without the need for traditional sign conventions.

The last form of the ESE obtains the possible locations of I for the given locations of A, O_A and J_A: This can be done by starting with equation (3.14). We can write

$$\mathbf{J}_A - \mathbf{a} = \overrightarrow{AJ_A} = -\frac{\|\mathbf{a}\|^2}{\|\boldsymbol{\rho}_A\|^2}\boldsymbol{\rho}_A.$$

With the necessary regulations

$$\|\mathbf{a}\|^2 = -\overrightarrow{AJ_A}\frac{\|\boldsymbol{\rho}_A\|^2}{\boldsymbol{\rho}_A} = \overrightarrow{J_AA}\frac{\|\boldsymbol{\rho}_A\|^2}{\boldsymbol{\rho}_A}$$

is obtained. The right side of the last equation must be examined to write $\|\mathbf{a}\|$, that is, to take the square root of the equation. Thus, pass on to polar form in the last equation

$$\|\mathbf{a}\|^2 = \left\|\overrightarrow{J_AA}\right\|\|\boldsymbol{\rho}_A\|$$

is obtained as $\boldsymbol{\rho}_A$ and $\overrightarrow{J_AA}$ are SL-I or SL-III. Since $\left\|\overrightarrow{J_AA}\right\|\|\boldsymbol{\rho}_A\|$ has positive sign, we can write $\|\mathbf{a}\| = \sqrt{\left\|\overrightarrow{J_AA}\right\|\|\boldsymbol{\rho}_A\|}$. Thus, the last hyperbolic number form

$$\mathbf{a} = \left|\left(\left\|\overrightarrow{J_AA}\right\|\|\boldsymbol{\rho}_A\|\right)^{1/2}\right|\left(\pm e^{j\arg\boldsymbol{\rho}_A}\right), \qquad \text{(ESE-4)}$$

where $e^{j\arg\mathbf{a}} = \pm e^{j\arg\boldsymbol{\rho}_A}$ is written since $\arg\mathbf{a} = \pm\arg\boldsymbol{\rho}_A$ (To find the point I, ESE-4 must not contain I.).

Case 2: Hyperbolic ESE Forms for SL-III Pole Ray:

For SL-III pole ray, the normal component of $-j\dot{\phi}\mathbf{u}$ (PPTL) is obtained by its projection onto $\mathbf{a}$ (SL-III) (Theorem 2.3) and calculated as:

$$(-j\dot{\phi}\mathbf{u})^n = j\sinh\theta e^{j(\arg(\mathbf{a})-\arg(-j\mathbf{u}))}(-j\dot{\phi}\mathbf{u}).$$

Thus we have:

$$\boldsymbol{\mathcal{A}}_A^n = \dot{\phi}^2\mathbf{a} + j\sinh\theta e^{j(\arg(\mathbf{a})-\arg(-j\mathbf{u}))}(-j\dot{\phi}\mathbf{u}). \tag{3.16}$$

Via Theorem 2.2, we obtain $\mathbf{J}_A$ as SL-I by letting $\boldsymbol{\mathcal{A}}_A^n = 0$:

$$\mathbf{J}_A = j\sinh\theta e^{j(\arg(\mathbf{a})-\arg(-j\mathbf{u}))}\left(\frac{j\mathbf{u}}{\dot{\phi}}\right)$$

or

$$\mathbf{J}_A = j\sinh\theta e^{j(\arg(-\mathbf{a})-\arg(j\mathbf{u}))}\left(\frac{j\mathbf{u}}{\dot{\phi}}\right). \tag{3.17}$$

$\boldsymbol{\mathcal{A}}_A^n$ is easily written by using equations (3.6) and (3.16), so equation (3.12) is reobtained. Then, four hyperbolic number forms of the ESE are given as follows:

$$\begin{cases} \boldsymbol{\rho}_A = & \pm\dfrac{\|\mathbf{a}\|^2}{\|\mathbf{a}-\mathbf{J}_A\|}e^{j\arg(\mathbf{a}-\mathbf{J}_A)}, & \text{(ESE-1)} \\ \mathbf{J}_A = & \mathbf{a} - \dfrac{\|\mathbf{a}\|^2}{\|\boldsymbol{\rho}_A\|^2}\boldsymbol{\rho}_A, & \text{(ESE-2)} \\ \mathbf{a} = & \dfrac{\mathbf{J}_A\mathbf{O}_A}{\mathbf{O}_A+\mathbf{J}_A}, & \text{(ESE-3)} \\ \mathbf{a} = & -\left|\left(\left\|\overrightarrow{J_AA}\right\|\|\boldsymbol{\rho}_A\|\right)^{1/2}\right|\left(\pm e^{j\arg\boldsymbol{\rho}_A}\right). & \text{(ESE-4)} \end{cases} \tag{3.18}$$

*Case 3: Hyperbolic ESE Forms for FPTL Pole Ray**:*

For FPTL pole ray, the normal component of $-j\dot{\phi}\mathbf{u}$ (PPTL) is obtained by its projection onto $-\mathbf{a}$ (PPTL) (see Theorem 2.3) and calculated as:

$$\boldsymbol{\mathcal{A}}_A^n = \dot{\phi}^2\mathbf{a} + \cosh\theta e^{j(\arg(-\mathbf{a})-\arg(-j\mathbf{u}))}(-j\dot{\phi}\mathbf{u}). \tag{3.19}$$

Thus, we have $\mathbf{J}_A$ as FPTL (via Theorem 2.2):

$$\mathbf{J}_A = \cosh\theta e^{j(\arg(-\mathbf{a})-\arg(-j\mathbf{u}))}\left(\frac{j\mathbf{u}}{\dot{\phi}}\right)$$

or

$$\mathbf{J}_A = \cosh\theta e^{j(\arg(\mathbf{a})-\arg(j\mathbf{u}))}\left(\frac{j\mathbf{u}}{\dot{\phi}}\right). \tag{3.20}$$

Then using equations (3.7) and (3.19), we can write:

$$\boldsymbol{\mathcal{A}}_A^n = (\dot{\phi})^2\frac{\|\mathbf{a}\|^2}{\|\boldsymbol{\rho}_A\|^2}\boldsymbol{\rho}_A = \dot{\phi}^2\mathbf{a} + \cosh\theta e^{j(\arg(-\mathbf{a})-\arg(-j\mathbf{u}))}(-j\dot{\phi}\mathbf{u}).$$

**This case was presented as an oral presentation at the International Conference on Mathematics and Its Applications in Science and Engineering (ICMASE 2020) which was held between 9-10 July 2020. The abstract was published in the abstract book of the congress with the title "Hyperbolic-Number Forms of the Euler Savary Equation: The Consideration of Future Pointing Timelike Pole Rays for Spacelike Pole Curves" [10].

It is clear that equation (3.12) is reobtained. The hyperbolic number forms of the ESE for FPTL pole ray are given by:

$$\begin{cases} \boldsymbol{\rho}_A = & \pm j \dfrac{\|\mathbf{a}\|^2}{\|\mathbf{a}-\mathbf{J}_A\|} e^{j\arg(\mathbf{a}-\mathbf{J}_A)}, & \text{(ESE-1)} \\ \mathbf{J}_A = & \mathbf{a} - \dfrac{\|\mathbf{a}\|^2}{\|\boldsymbol{\rho}_A\|^2}\boldsymbol{\rho}_A, & \text{(ESE-2)} \\ \mathbf{a} = & \dfrac{\mathbf{J}_A\mathbf{O}_A}{\mathbf{O}_A+\mathbf{J}_A}, & \text{(ESE-3)} \\ \mathbf{a} = & j\left|\left(\left\|\overrightarrow{J_A A}\right\|\|\boldsymbol{\rho}_A\|\right)^{1/2}\right|\left(\pm e^{j\arg\boldsymbol{\rho}_A}\right). & \text{(ESE-4)} \end{cases} \tag{3.21}$$

Case 4: Hyperbolic ESE Forms for PPTL Pole Ray:

For PPTL pole ray, the normal component of $-j\dot{\phi}\mathbf{u}$ (PPTL) is obtained by its projection onto $\mathbf{a}$ (PPTL) (see Theorem 2.3) and calculated as:

$$\boldsymbol{\mathcal{A}}_A^n = \dot{\phi}^2\mathbf{a} + \cosh\theta e^{j(\arg(\mathbf{a})-\arg(-j\mathbf{u}))}(-j\dot{\phi}\mathbf{u}). \tag{3.22}$$

Thus, we get $\mathbf{J}_A$ as FPTL (via Theorem 2.2):

$$\mathbf{J}_A = \cosh\theta e^{j(\arg(\mathbf{a})-\arg(-j\mathbf{u}))}\left(\frac{j\mathbf{u}}{\dot{\phi}}\right)$$

or

$$\mathbf{J}_A = \cosh\theta e^{j(\arg(-\mathbf{a})-\arg(j\mathbf{u}))}\left(\frac{j\mathbf{u}}{\dot{\phi}}\right).$$

$\boldsymbol{\mathcal{A}}_A^n$ is easily written using equations (3.7) and (3.22), so equation (3.12) is reobtained. Then, four hyperbolic number forms of the ESE are given as follows:

$$\begin{cases} \boldsymbol{\rho}_A = & \pm j \dfrac{\|\mathbf{a}\|^2}{\|\mathbf{a}-\mathbf{J}_A\|} e^{j\arg(\mathbf{a}-\mathbf{J}_A)}, & \text{(ESE-1)} \\ \mathbf{J}_A = & \mathbf{a} - \dfrac{\|\mathbf{a}\|^2}{\|\boldsymbol{\rho}_A\|^2}\boldsymbol{\rho}_A, & \text{(ESE-2)} \\ \mathbf{a} = & \dfrac{\mathbf{J}_A\mathbf{O}_A}{\mathbf{O}_A+\mathbf{J}_A}, & \text{(ESE-3)} \\ \mathbf{a} = & -j\left|\left(\left\|\overrightarrow{J_A A}\right\|\|\boldsymbol{\rho}_A\|\right)^{1/2}\right|\left(\pm e^{j\arg\boldsymbol{\rho}_A}\right). & \text{(ESE-4)} \end{cases} \tag{3.23}$$

Remark 3.2. For the 2nd-4th probabilities mentioned in Remark 3.1, the hyperbolic number forms of the ESE are obtained in the same algebraic form. However, they have varying vector types.

Remark 3.3. Hyperbolic ESE forms considering TL pole curves considering different types of pole rays can be deduced with a similar approach immediately (see Example 4.5).

Let us end this section by comparing the results in the following corollary considering hyperbolic ESE forms for SL and TL pole curves.

Corollary 3.2. *For hyperbolic ESE forms considering SL and TL pole curves, the following statements are given:*

- *Algebraic expressions of the position, velocity and acceleration vectors are same; however, they have varying types.*
- *Algebraic expressions of $\mathbf{J}_A$ are different; however, vector types of $\mathbf{J}_A$ are the same.*
- *The algebraic expression of four different hyperbolic ESE forms is the same for every type of pole ray.*

Thus, it is concluded that the type of pole curves is not a distinctive property while calculating hyperbolic ESE forms, but the type of pole ray gives distinctive properties.

4. Implementation

Computing ESE considering the hyperbolic number approach gives an easy calculation method. It is based on vector calculations and provides a detailed examination of the locations of the points in $\mathbb{H}$.

Example 4.1. During one-parameter planar hyperbolic motion, let us determine the location of the point O_A considering the points $I = 0$, $A = 2 + j$, $O_\pi = 2j$, $O_p = -j$ and $\theta = 0.1$. It is seen that pole curves are SL and pole ray is SL-I. Thus, the argument of pole ray does not equal the angle between pole curves and their normal. Then, from equation (2.1), we can write :

$$A = \sqrt{3}e^{j\tanh^{-1}\left(\frac{1}{2}\right)}.$$

Using equations (3.10), we have:

$$J_A = \frac{2}{3}\sinh(0.1)\left(\cosh\left(\tanh^{-1}\left(\frac{1}{2}\right)\right) + j\sinh\left(\tanh^{-1}\left(\frac{1}{2}\right)\right)\right) = \frac{32}{415} + \frac{16}{415}j,$$

where $\tanh^{-1}\left(\frac{1}{2}\right) \simeq 0.549$ and $\boldsymbol{\delta} = \boldsymbol{\rho}\boldsymbol{\rho}_p = \left(0, \frac{2}{3}\right)$ (see equations (3.5) and (3.11)). Here, it is clear that $\mathbf{J}_A$ is obtained as SL-I.

We know I, A and J_A, so we will find O_A. By applying ESE-1 (see equation (3.13)) $\boldsymbol{\rho}_A = \pm\left(\frac{830}{399}, \frac{415}{399}\right)$ is written where $e^{j\arg(\mathbf{a}-\mathbf{J}_A)} = e^{j\arg(\mathbf{a})}$. Since O_A and J_A are always on the same side of A, $\boldsymbol{\rho}_A$ is SL-I and $O_A = -\frac{32}{399} - \frac{16}{399}j$, where $\mathbf{O}_A = \mathbf{a} - \boldsymbol{\rho}_A$. Thus, this example is example for Fig. 6-(e) in 1st probability (see in Fig. 3).

In Example 4.2-Example 4.4, the application of the hyperbolic number forms of the ESE to the Cardan mechanism is examined specifically. The Cardan (Cardanic) motion generated by a circle rolling within another circle with twice its radius [5, 12]. 3rd probability (in Example 4.2 and Example 4.3) and 4th probability (in Example 4.4) are discussed (see Remark 3.1). The solutions are calculated practically with the hyperbolic number forms of the ESE.

Example 4.2. During one-parameter planar hyperbolic motion, let us determine the location of the point O_A considering the points $I = 0$, $A = 1 + 2j$, $O_\pi = j$ and $O_p = 2j$. It is seen that pole curves are SL and pole ray is FPTL. From equation (2.2), we can write:

$$A = j\sqrt{3}e^{j\coth^{-1}(2)}.$$

Considering equation (3.20) via (3.5) and (3.11), we have:

$$J_A = 2j\cosh\left(\coth^{-1}(2)\right)e^{j\coth^{-1}(2)} = \frac{4}{3} + \frac{8}{3}j,$$

where $\coth^{-1}(2) \simeq 0.549 = \theta$ and $\boldsymbol{\delta} = \boldsymbol{\rho}\boldsymbol{\rho}_p = (0, 2)$. Here, it is clear that $\mathbf{J}_A$ is obtained as FPTL.

We know I, A and J_A, so we will find O_A. By applying ESE-1 (see equation (3.21)) $\boldsymbol{\rho}_A = \pm(3, 6)$ is written where $e^{j\arg(\mathbf{a}-\mathbf{J}_A)} = e^{j\arg(\mathbf{a})}$. Since O_A and J_A are always on the same side of A, $\boldsymbol{\rho}_A$ is PPTL and $O_A = 4 + 8j$, where $\mathbf{O}_A = \mathbf{a} - \boldsymbol{\rho}_A$.

Additionally, it is possible to find A by using the calculated values of $O_A = j4\sqrt{3}e^{j\coth^{-1}(2)}$ and $J_A = j\frac{4}{\sqrt{3}}e^{j\coth^{-1}(2)}$ considering ESE-3 (see equation (3.21)) as:

$$A = \frac{\frac{4}{\sqrt{3}}4\sqrt{3}e^{j\coth^{-1}(2)}e^{j\coth^{-1}(2)}}{j\left(\frac{4}{\sqrt{3}} + 4\sqrt{3}\right)e^{j\coth^{-1}(2)}} = 1 + 2j.$$

Example 4.3. During one-parameter planar hyperbolic motion, let us determine the location of the point O_A considering the points $I = 0$, $A = -4 - 3j$, $O_\pi = 2j$ and $O_p = 4j$ and $\theta = 0.2$. It is seen that pole curves are SL and pole ray is SL-III. Thus, the argument of pole ray does not equal the angle between pole curves and their normal. Then, from equation (2.1), we can write:

$$A = -\sqrt{7}e^{j\tanh^{-1}\left(\frac{3}{4}\right)}.$$

Using equations (3.17), (3.5) and (3.11), we have:

$$J_A = -4\sinh(0.2)e^{j\tanh^{-1}\left(\frac{3}{4}\right)} = -\frac{610}{501} - \frac{305}{334}j,$$

where $\tanh^{-1}(3/4) \simeq 0.973$ and $\boldsymbol{\delta} = \boldsymbol{\rho}\boldsymbol{\rho}_p = (0,-4)$. Here, it is clear that $\mathbf{J}_A$ is SL-III.

We know I, A and J_A, so we will find O_A. By applying ESE-1 (see equation (3.18)) we obtain $\boldsymbol{\rho}_A = \pm\left(\frac{4008}{697}, \frac{3006}{697}\right)$ where $e^{j\arg(\mathbf{a}-\mathbf{J}_A)} = e^{j\arg(\mathbf{a})}$. Since O_A and J_A are always on the same side of A, $\boldsymbol{\rho}_A$ is SL-III and $O_A = \frac{1220}{697} + \frac{915}{697}j$, where $\mathbf{O}_A = \mathbf{a} - \boldsymbol{\rho}_A$.

Additionally, let us take consider $O_A = \frac{305\sqrt{7}}{697}e^{j\tanh^{-1}\left(\frac{3}{4}\right)}$ and $J_A = -\frac{305\sqrt{7}}{1002}e^{j\tanh^{-1}\left(\frac{3}{4}\right)}$ and use ESE-3 (see equation (3.18)) to obtain A as:

$$A = \frac{-\frac{305\sqrt{7}}{697}\frac{305\sqrt{7}}{1002}e^{j\tanh^{-1}\left(\frac{3}{4}\right)}e^{j\tanh^{-1}\left(\frac{3}{4}\right)}}{\left(\frac{305\sqrt{7}}{697} - \frac{305\sqrt{7}}{1002}\right)e^{j\tanh^{-1}\left(\frac{3}{4}\right)}} = -4 - 3j$$

which was assumed in this example.

Example 4.4. During one-parameter planar hyperbolic motion, let us determine the location of the point O_A considering the points $I = 0$, $A = -2-3j$, $O_\pi = -j$ and $O_p = -2j$. It is seen that pole curves are SL and pole ray is PPTL. In this case, the argument of pole ray equals the angle between pole curves and their normal. Similar to the above example, from equation (2.2), we can write:

$$A = -j\sqrt{5}e^{j\coth^{-1}\left(\frac{3}{2}\right)}.$$

Using equation (3.20), we have:

$$J_A = -2j\cosh\left(\coth^{-1}\left(\frac{3}{2}\right)\right)e^{j\coth^{-1}\left(\frac{3}{2}\right)} = -\frac{12}{5} - \frac{18}{5}j,$$

where $\coth^{-1}\left(\frac{3}{2}\right) \simeq 0.804 = \theta$ and $\boldsymbol{\delta} = \boldsymbol{\rho}\boldsymbol{\rho}_p = (0,-2)$ (see equations (3.5) and (3.11)). Here, it is clear that $\mathbf{J}_A$ is PPTL.

We know I, A and J_A, so we will find O_A. By applying ESE-1 (see equation (3.23)) we obtain $\boldsymbol{\rho}_A = \pm(10, 15)$ where $e^{j\arg(\mathbf{a}-\mathbf{J}_A)} = e^{j\arg(\mathbf{a})}$. Since O_A and J_A are always on the same side of A, $\boldsymbol{\rho}_A$ is FPTL and $O_A = -12-18j$, where $\mathbf{O}_A = \mathbf{a} - \boldsymbol{\rho}_A$.

Additionally, let us take consider $O_A = -j6\sqrt{5}e^{j\coth^{-1}\left(\frac{3}{2}\right)}$ and $J_A = -j\frac{6}{\sqrt{5}}e^{j\coth^{-1}\left(\frac{3}{2}\right)}$ and use ESE-3 (see equation (3.23)) to obtain A:

$$A = \frac{6\sqrt{5}\frac{6}{\sqrt{5}}e^{j\coth^{-1}\left(\frac{3}{2}\right)}e^{j\coth^{-1}\left(\frac{3}{2}\right)}}{-j\left(6\sqrt{5} + \frac{6}{\sqrt{5}}\right)e^{j\coth^{-1}\left(\frac{3}{2}\right)}} = -2 - 3j$$

which was assumed in this example.

Example 4.5. During one-parameter planar hyperbolic motion, let us determine the location of the point A considering the points $I = 0$, $O_A = 4 + j$, $O_\pi = 1$ and $O_p = -1$. It is seen that pole curves are TL and pole ray is SL. Thus, the argument of pole ray equals the angle between pole curves and their normal. Hence $\mathbf{J}_A = \cosh\theta e^{j(\theta-0)}\boldsymbol{\delta}$ gives

$$J_A = \frac{1}{2}\cosh\left(\tanh^{-1}\left(\frac{1}{4}\right)\right)e^{j\tanh^{-1}\left(\frac{1}{4}\right)} = \frac{8}{15} + \frac{2}{15}j,$$

where $\tanh^{-1}\left(\frac{1}{4}\right) \simeq 0,255$, $\boldsymbol{\rho} = \left(\frac{1}{2}, 0\right)$ and $\boldsymbol{\delta} = \left(\frac{1}{2}, 0\right)$. It is seen $\mathbf{J}_A$ is SL-I. We know I, O_A and J_A, so we can find A by applying ESE-3 (equations (3.15) or (3.18)). Then the following vector is obtained as SL-I:

$$\mathbf{a} = \frac{\mathbf{J}_A\mathbf{O}_A}{\mathbf{O}_A + \mathbf{J}_A} = \left(\frac{8}{17}, \frac{2}{17}\right).$$

5. Conclusion

The ESE is a well-known formulation obtained in various forms and has great importance in kinematics. The classical ESE is generally used for deriving the radius of curvature and the center of curvature of the path traced by a point in a planar rolling-contact mechanism. Using combined graphical and algebraic techniques to determine the ESE often causes sign errors. To eliminate the need for the traditional sign conventions and make them suitable for digital computation, the complex number forms of the ESE are obtained by Sandor [31]. This complex number method is useful for applying path-curvature theory to higher-pair rolling contact mechanisms, like cams, gears, linkages, etc.

The remarkable relationship between the complex numbers-Euclidean plane and the hyperbolic numbers-Lorentzian plane gives researchers many new aspects. This viewpoint concentrates on investigating studies dealing with hyperbolic numbers based on the studies on complex numbers. Features of the hyperbolic plane, like the idea of regions, the shapes of the vectors, and the fact that the circle is an Euclidean hyperbola, make these studies interesting.

The main motivation for this paper is whether the hyperbolic number approach could be used for the ESE, inspired by the advantages of the complex number approach. Based on the affirmative answer, hyperbolic ESE forms are obtained by using the basic properties of the hyperbolic plane and vector calculations. Hyperbolic forms of the ESE gives directly the relationship between the four points I (the pole point), A (an arbitrary point), J_A (the inflection point), and O_A (the center of curvature of the path of A) on the same pole ray in the hyperbolic plane. While doing this observation, types of pole curves (SL, TL) and types of pole rays (SL-I, SL-III, FPTL, PPTL) are considered to make a detailed computation. When three of these four points are known in the hyperbolic plane, the fourth one can be found easily. Also, applying the hyperbolic vector approach provides a detailed examination of the locations of the points in the hyperbolic plane. Thus, it enables the advantage of eliminating sign errors and convenience in practice.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Akbıyık, M., Yüce, S.: *Euler Savary's formula on complex plane* $\mathbb{C}^*$. Appl. Math. E-Notes **16**, 65-71 (2016).
[2] Birman, G. S., Nomizu, K.: *Trigonometry in Lorentzian geometry*. Am Math Mon **91** (9), 543-549 (1984).
[3] Birman, G. S.: *On* L^2 *and* L^3. Elem. Math. **43** (2), 46-50 (1988).
[4] Blaschke, W., Müller, H. R.: Ebene kinematik. Verlag Von R. Oldenbourg. München (1956).
[5] Bottema, O.: *Cardan motion in elliptic geometry* . Canadian Journal of Mathematics, **27**(1), 37-43 (1975).
[6] Buckley, R., Whitfield, E. V.: *The Euler-Savary formula*. Math. Gaz. **33** (306), 297-299 (1949).
[7] Catoni, F., Zampetti, P.: *Two-dimensional space-time symmetry in hyperbolic functions*. Nuovo Cimento Soc. Ital. Fis. B **115** (12), 1433-1440 (2000).
[8] Catoni, F., Cannata, R., Catoni, V., Zampetti, P.: *Hyperbolic trigonometry in two-dimensional space-time geometry*. Nuovo Cimento Soc. Ital. Fis. B **118** (5), 475-492 (2003).
[9] Cockle, J.: *On a new imaginary in algebra*. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **34** (226), 37-47 (1849).
[10] Çağlar, D., Gürses, N.: *New hyperbolic-number forms of the Euler Savary equation: The consideration of future pointing timelike pole rays for spacelike pole curves*. In: International Conference on Mathematics and Its Applications in Science and Engineering (ICMASE), July 09-10 2020, Ankara, TURKIYE. Abstract Book 39-41 (2020).
[11] Erdman, A. G., Sandor, G. N.: Mechanism design analysis and synthesis. Prentice-Hall Inc. (1997).
[12] Eren, K., Ersoy, S.: *Cardan positions in the Lorentzian plane*. Honam Mathematical Journal, **40**(1), 187-198 (2018).

[13] Ersoy, S., Akyiğit, M.: *One-parameter homothetic motion in the hyperbolic plane and Euler-Savary formula*. Adv. Appl. Clifford Algebr. **21** (2), 297-313 (2011).
[14] Fjelstad, P.: *Extending special relativity via the perplex numbers*. Amer. J. Phys. **54** (5), 416-422 (1986).
[15] Fjelstad, P., Gal, S. G.: *n-dimensional hyperbolic complex numbers*. Adv. Appl. Clifford Algebr. **8** (1), 47-68 (1998).
[16] Fjelstad, P., Gal, S. G.: *Two-dimensional geometries, topologies, trigonometries and physics generated by complex-type numbers*. Adv. Appl. Clifford Algebr. **11** (1), 81-107 (2001).
[17] Gürses, N., Akbıyık, M., Yüce, S.: *One-parameter homothetic motions and Euler-Savary formula in generalized complex number plane*. Adv. Appl. Clifford Algebr. **26** (1), 115-136 (2016).
[18] Hahn, L. S.: Complex numbers and geometry. American Mathematical Soc. (2019).
[19] Hall, A. S.: Kinematics and linkage design. Waveland Press Inc. (1986).
[20] Harkin, A. A., Harkin, J. B.: *Geometry of generalized complex numbers*. Math. Mag. **77** (2), 118-129 (2004).
[21] Hartenberg, R., Danavit, J.: Kinematic synthesis of linkages. New York: McGraw-Hill. (1964).
[22] Hirschhorn, J.: Kinematics and dynamics of plane mechanisms. McGraw-Hill. (1962).
[23] Masal, M., Tosun, M., Pirdal, A. Z.: *Euler Savary formula for the one parameter motions in the complex plane* $\mathbb{C}$. Int. J. Phys. Sci. **5** (1), 6-10 (2010).
[24] Nešović, E., Petrović-Torgašev, M.: *Some trigonometric relations in the Lorentzian plane*. Kragujevac J. Math. **33** (25), 219-225 (2003).
[25] Nešović, E.: *Hyperbolic angle function in the Lorentzian plane*. Kragujevac J. Math. **28** (28), 139-144 (2005).
[26] Nešović, E., Petrović-Torgašev, M., Verstraelen L.: *Curves in Lorentzian spaces*. Bolletino dell Unione Mat. Ital. **8** (3), 685-696 (2005).
[27] Nitta, T., Kuroe, Y.: *Hyperbolic gradient operator and hyperbolic back-propagation learning algorithms*. IEEE Trans. Neural Netw. Learn. Syst. **29** (5), 1689-1702 (2017).
[28] O'Neill, B.: Semi-Riemannian geometry with applications to relativity. Academic Press. (1983).
[29] Rochon, D., Shapiro, M.: *On algebraic properties of bicomplex and hyperbolic numbers*. Anal. Univ. Oradea, Fasc. Math **11** (71), 110 (2004).
[30] Rothbart, H., Brown, T. H.: Mechanical design handbook, measurement, analysis, and control of dynamic systems. McGraw-Hill Education. (2006).
[31] Sandor, G. N., Erdman, A. G., Hunt, L., Raghavacharyulu, E.: *New complex-number forms of the Euler-Savary equation in a computer-oriented treatment of planar path-curvature theory for higher-pair rolling contact*. J. Mech. Des. **104**, 227-232 (1982).
[32] Sandor, G. N., Erdman, A. G., Raghavacharyulu, E.: *Double-valued solutions of the Euler-Savary equation and its counterpart in Bobillier's construction*. Mech. Mach. Theory **20** (2), 145-148 (1985).
[33] Sandor, G. N., Xu, Y., Weng, T.: *A graphical method for solving the Euler-Savary equation*. Mech. Mach. Theory **25** (2), 141-147 (1990).
[34] Sobczyk, G.: *The hyperbolic number plane*. College Math. J. **26** (4), 268-280 (1995).
[35] Sobczyk, G.: New foundations in mathematics: The geometric concept of number. Birkhauser. (2013).
[36] Yaglom, I. M.: Complex numbers in geometry. Academic Press. (1968).
[37] Yaglom, I. M.: A simple non-Euclidean geometry and its physical basis. Springer-Verlag. (1979).
[38] Yüce, S., Kuruoğlu, N.: *One-parameter plane hyperbolic motions*. Adv. Appl. Clifford Algebr. **18** (2), 279-285 (2008).

Affiliations

DUYGU ÇAĞLAR
ADDRESS: Yildiz Technical University, Faculty of Arts and Sciences, Department of Mathematics, 34220, İstanbul-Türkiye.
E-MAIL: duygu.caglar@std.yildiz.edu.tr
ORCID ID: 0000-0003-2036-9684

NURTEN GÜRSES
ADDRESS: Yildiz Technical University, Faculty of Arts and Sciences, Department of Mathematics, 34220, İstanbul-Türkiye.
E-MAIL: nbayrak@yildiz.edu.tr
ORCID ID: 0000-0001-8407-854X