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Some Curves on 3-Dimensional Normal almost Contact Pseudo-metric Manifolds

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Abstract. In this study, we characterize Frenet curves in 3-dimensional normal almost contact pseudo-metric manifolds. We give Frenet equations and the Frenet elements of such curves. Also, we obtain the curvatures of non-geodesic Frenet curves on 3-dimensional almost contact pseudo-metric manifolds. Finally we present some corollaries about these curves.

1. Introduction

The differential geometry of curves on manifolds is an attractive topic in differential geometry. Especially the curves in contact and para-contact manifolds drew attention and studied by many authors. Olszak [10], gave the conditions for an a.c.m structure on a manifold to be normal and gave examples for this structure.

Welyczko [14], gave some of the results for Legendre curves to the case of 3-dimensional normal a.c.m. manifolds, especially, quasi-Sasakian manifolds. Acet and Perktaş [1] obtained curvature and torsion of Legendre curves in 3-dimensional (ε, δ) trans-Sasakian manifolds.

Yıldırım [15] obtained the curvatures of non-geodesic Frenet curves on three dimensional normal almost contact manifolds and gave some results for these characterizations. De and Mondal [6] studied ξ-projectively flat and φ-projectively flat 3-dimensional normal almost contact metric manifolds and gave an illustrative example.

Calvaruso and Perrone [3] introduced a systematic study of contact structures with pseudo-Riemannian associated metrics, emphasizing analogies and differences with respect to the Riemannian case. In particular, they classified contact pseudo-metric manifolds of constant sectional curvature, three dimensional locally symmetric contact pseudo-metric manifolds and three-dimensional homogeneous contact Lorentzian manifolds.

Takahashi [11] defined Sasakian manifold with pseudo-Riemannian metric and discussed the classification of Sasakian manifolds. Venkatesha V. [13] examined 3-dimensional normal almost contact pseudo-metric manifold and gave the conditions for these manifolds to be normal. studied the almost contact pseudo-metric manifolds of dimension three which are normal and derived certain necessary and sufficient conditions for an almost contact pseudo-metric manifold to be normal.

This paper is organized as: Section 2 with three subsections, we give basic definitions and propositions of an almost contact pseudo-metric manifold. In the second subsection we give the properties of 3-dimensional

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almost contact pseudo-metric manifolds. We give Frenet equations of a curve in 3-dimensional almost contact pseudo-metric manifolds in the last subsection of this section.

We finally give the Frenet elements of a Frenet curve in such manifolds and give corollaries for the Frenet curves in the third section.

2. Preliminaries

2.1. Normal Almost Contact Pseudo-metric Manifolds

A $(2n+1)$-dimensional smooth connected manifold M is said to be an almost contact manifold if there exists on M a (1,1) tensor field φ, a vector field ξ and a 1-form η such that [2]

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \qquad (1)$$
$$\varphi(\xi) = 0, \quad \eta \circ \varphi = 0.$$

If an almost contact manifold is endowed with a pseudo-Riemannian metric g such that

$$\bar{g}(\varphi X, \varphi Y) = \bar{g}(X, Y) - \varepsilon \eta(X)\eta(Y), \qquad (2)$$

where $\varepsilon = \mp 1$, for all $X, Y \in TM$, then $(\bar{N}, \varphi, \xi, \eta, \bar{g})$ is called an almost contact pseudo-metric manifold[13]. From (2) we have

$$\eta(X) = \varepsilon \bar{g}(X, \xi) \quad \textit{and} \quad \bar{g}(\varphi X, Y) = -\bar{g}(X, \varphi Y). \qquad (3)$$

In particular, for an almost contact pseudo-metric manifold $\bar{g}(\xi, \xi) = \varepsilon$. Thus, the characteristic vector field ξ is a unit vector field, which is either spacelike or timelike, but cannot be ligtlike. The fundamental 2-form of an almost contact pseudo-metric manifold $(\bar{N}, \varphi, \xi, \eta, \bar{g})$ is defined by

$$\Phi(X, Y) = \bar{g}(X, \varphi(Y)), \qquad (4)$$

where $\eta \wedge \Phi^n \neq 0$[13]. An almost contact pseudo-metric manifold is said to be contact pseudo-metric manifold if $d\eta = \Phi$, where

$$d\eta(X, Y) = \frac{1}{2}\left(X\eta(Y) - Y\eta(X) - \eta([X, Y])\right). \qquad (5)$$

[3]In an almost contact pseudo-metric manifold $(\bar{N}, \varphi, \xi, \eta, \bar{g})$ there always exists a special kind of local pseudo-orthonormal basis $\{e_i, \varphi e_i, \xi\}_{i=1}^{n}$, called a local φ-basis.

Let $\bar{N}$ be a (2n+1)-dimensional almost contact pseudo-metric manifold with structure (φ, ξ, η) and consider the manifold $\bar{N} \times R$. We denote a vector field on $\bar{N} \times R$ by $X, f\frac{d}{dt}$, where $X \in T\bar{N}$, t is the coordinate on $\mathfrak{R}$ and f is a C^∞ function on $\bar{N} \times R$. Then the structure J on $\bar{N} \times R$ defined by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}, \qquad (6)$$

is an almost complex structure. If the almost complex structure J is integrable, then we say that the almost contact pseudo-metric structure (φ, ξ, η) is normal. Necessary and sufficient condition for integrability of J is

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0, \qquad (7)$$

where $[\varphi, \varphi]$ is the Nijenhius torsion of φ.[3]

Proposition 2.1. *[12] An almost contact pseudo-metric manifold is normal if and only if*

$$(\nabla_{\varphi X}\varphi)Y - \varphi(\nabla_X \varphi)Y + (\nabla_X \eta)(Y)\xi = 0, \qquad (8)$$

where ∇ is the Levi-Civita connection.

2.2. Three dimensional normal almost contact pseudo-metric(n.a.c.p-m) manifold

Lemma 2.2. *[13] A three dimensional n.a.c.p-m manifold $\bar{N}$ is normal if and only if*

$$\nabla_{\varphi X}\xi = \varphi\nabla_X\xi. \tag{9}$$

Theorem 2.3. *[13] For a three dimensional n.a.c.p-m manifold $\bar{N}$, the following three conditions are mutually equivalent:*
(1) *$\bar{N}$ is normal*
(2) *there exist smooth functions α, β on $\bar{N}$ such that*

$$\nabla_X\xi = \alpha\{X - \eta(X)\xi\} - \beta\varphi X, \tag{10}$$

(3) *there exist smooth functions α, β on $\bar{N}$ such that*

$$(\nabla_{X\varphi})Y = \alpha\{\varepsilon\bar{g}(\varphi X, Y)\xi - \eta(Y)\varphi X\} + \beta\{\varepsilon\bar{g}(X, Y)\xi - \eta(Y)X\}. \tag{11}$$

In particular, the functions appearing above are given by

$$2\alpha = div(\xi), \quad 2\beta = tr(\varphi\nabla_X). \tag{12}$$

Corollary 2.4. *[13] For a three dimensional n.a.c.p-m manifold, the vector field ξ is geodesic, i.e., $\nabla_\xi\xi = 0$ and $d\eta = \varepsilon\beta\Phi$.*

From (11) we can give the following definition.

Definition 2.5. *[13] A three dimensional n.a.c.p-m manifold is called*
(i) *cosymplectic if $\alpha = \beta = 0$,*
(ii) *quasi-Sasakian if $\alpha = 0$ and $\beta \neq 0$, and β-Sasakian pseudo-metric manifold if $\alpha = 0$ and β is non-zero constant. If $\beta = \varepsilon$ it is the Sasakian pseudo-metric manifold,*
(iii) *an almost α-Kenmotsu pseudo-metric manifold if $\beta = 0$ and $\alpha \neq 0$, and α-Kenmotsu pseudo-metric manifold if $\beta = 0$ and α is a non-zero constant. If $\alpha = 1$ it is the Kenmotsu pseudo-metric manifold.*

Lemma 2.6. *[13] For a three dimensional n.a.c.p-m manifold $\xi(\beta) + 2\alpha\beta = 0$ holds.*

2.3. Frenet Curves

Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold with Levi-Civita connection ∇ and $\vartheta : I \to \bar{N}$ be a unit speed curve parametrized by arc length s in $\bar{N}$ where I is an open interval. A unit speed curve ϑ is called timelike or spacelike if its casual character is -1 or 1, respectively. Also, ϑ is called a Frenet curve if $\bar{g}(\vartheta', \vartheta') \neq 0$. A Frenet curve ϑ admits an orthonormal frame field $\{t = \vartheta', n, b\}$ along ϑ. Then the following Frenet equations holds:

$$\begin{aligned}
\nabla_{\vartheta'} t &= \kappa n,\\
\nabla_{\vartheta'} n &= -\kappa t + \varepsilon\tau b,\\
\nabla_{\vartheta'} b &= -\varepsilon\tau n,
\end{aligned}$$

where $\kappa = |\nabla_{\vartheta'}\vartheta'|$ is the geodesic curvature of ϑ and τ is geodesic torsion. The vector fields t, n and b are called the tangent vector field, the principal normal vector field and the binormal vector field of ϑ, respectively.
A Frenet curve ϑ is a geodesic if and only if $\kappa = 0$. A Frenet curve ϑ with constant geodesic curvature and zero geodesic torsion is called a pseudo-circle. A pseudo-helix is a Frenet curve ϑ whose geodesic curvature and torsion are constant.

A curve in a 3-dimensional n.a.c.p-m manifold is said to be slant if its tangent vector field has constant angle with the Reeb vector field,i.e.$\eta(\vartheta') = \varepsilon\bar{g}(\vartheta', \xi) = cos\theta = constant$. If the condition $\eta(\vartheta') = \varepsilon\bar{g}(\vartheta', \xi) = 0$ holds then ϑ is a Legendre curve[14].

3. Main Results

Let us consider a 3-dimensional normal almost contact pseudo-metric manifold $\bar{N}$. Let $\vartheta : I \to \bar{N}$ be a non-geodesic ($\kappa \neq 0$) Frenet curve given with the arc-parameter s and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{N}$. From the basis $(\vartheta^{'}, \varphi\vartheta^{'}, \xi)$ we obtain an orthonormal basis $\{z_1, z_2, z_3\}$ defined by

$$\begin{aligned} z_1 &= \vartheta^{'}, \\ z_2 &= \frac{\varphi\vartheta^{'}}{\sqrt{1-\varepsilon\varrho^2}}, \\ z_3 &= \frac{\xi - \varepsilon\varrho\vartheta^{'}}{\sqrt{1-\varepsilon\varrho^2}}, \end{aligned} \tag{13}$$

where

$$\eta(\vartheta^{'}) = \varepsilon\bar{g}(\vartheta^{'}, \xi) = \varepsilon\varrho. \tag{14}$$

Moreover we have

$$\bar{\nabla}_{\vartheta'} z_1 = \nu z_2 + \mu z_3 \tag{15}$$

such that

$$\nu = \bar{g}(\bar{\nabla}_{\vartheta'} z_1, z_2) \tag{16}$$

is a function. Then we obtain μ by

$$\mu = \bar{g}(\bar{\nabla}_{\vartheta'} z_1, z_3) = \frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \alpha\sqrt{1-\varepsilon\varrho^2}. \tag{17}$$

So, we have

$$\bar{\nabla}_{\vartheta'} z_2 = -\nu z_1 + \left(\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right) z_3 \tag{18}$$

and

$$\bar{\nabla}_{\vartheta'} z_3 = -\mu z_1 - \left(\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right) z_2. \tag{19}$$

The fundamental forms of the tangent vector ϑ' on the basis of the equation (13) is

$$[\omega_{ij}(\vartheta')] = \begin{bmatrix} 0 & \nu & \mu \\ -\nu & 0 & -\beta + \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} \\ -\mu & \beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} & 0 \end{bmatrix} \tag{20}$$

and the Darboux vector connected to the vector ϑ' is

$$\omega(\vartheta') = \left(-\beta + \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right) z_1 - \mu z_2 + \nu z_3. \tag{21}$$

Then, we have

$$\bar{\nabla}_{\vartheta'} z_i = \omega(\vartheta') \wedge \varepsilon z_i \quad (1 \leq i \leq 3). \tag{22}$$

Furthermore, for any vector field $Z = \sum_{i=1}^{3} \theta^i z_i \in \chi(\bar{N})$ is strictly dependent on the curve ϑ on $\bar{N}$, there exists the following equation

$$\bar{\nabla}_{\vartheta'} Z = \omega(\vartheta') \wedge Z + \varepsilon\sum_{i=1}^{3} z_i[\theta^i] z_i. \tag{23}$$

3.1. Frenet Elements of the curve ϑ

Let $\vartheta : I \to \bar{N}$ be a non-geodesic ($\kappa \neq 0$) Frenet curve given with the arc parameter s and the elements $\{t, n, b, \kappa, \tau\}$.

From (15) we have

$$\kappa n = \bar{\nabla}_{\vartheta'} z_1 = \nu z_2 + \mu z_3. \tag{24}$$

From the equations (17) and (23) we find

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \alpha\sqrt{1-\varepsilon\varrho^2}\right)^2}. \tag{25}$$

On the other hand

$$\begin{aligned}\bar{\nabla}_{\vartheta'} n =& \left(\frac{\nu}{\varepsilon\kappa}\right)' z_2 + \frac{\nu}{\varepsilon\kappa}\nabla_{\vartheta'} z_2 + \left(\frac{\mu}{\varepsilon\kappa}\right)' z_3 + \frac{\mu}{\varepsilon\kappa}\nabla_{\vartheta'} z_3 \\ =& -\kappa t + \varepsilon\tau B.\end{aligned} \tag{26}$$

By using the equations (18) and (19) we find

$$\begin{aligned}\tau b =& \left[\left(\frac{\nu}{\varepsilon\kappa}\right)' + \frac{\mu}{\varepsilon\kappa}\left(\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right)\right] z_2 \\ &+ \left[\left(\frac{\mu}{\varepsilon\kappa}\right)' + \frac{\nu}{\varepsilon\kappa}\left(\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right)\right] z_3.\end{aligned} \tag{27}$$

By a direct computation we find following equation

$$\left[\left(\frac{\nu}{\varepsilon\kappa}\right)'\right]^2 + \left[\left(\frac{\mu}{\varepsilon\kappa}\right)'\right]^2 = \left[-\left(\frac{\nu}{\varepsilon\kappa}\right)'\frac{\mu}{\varepsilon\kappa} + \frac{\nu}{\varepsilon\kappa}\left(\frac{\mu}{\varepsilon\kappa}\right)'\right]^2. \tag{28}$$

If we take the norm of the this equation and use the equations (17) and (28) in (27) we get

$$\tau = \left|\left(\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right) - \sqrt{\left[\left(\frac{\nu}{\varepsilon\kappa}\right)'\right]^2 + \left[\left(\frac{\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \alpha\sqrt{1-\varepsilon\varrho^2}}{\varepsilon\kappa}\right)'\right]^2}\right|. \tag{29}$$

Theorem 3.1. *Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold and ϑ be a Frenet curve on $\bar{N}$. Then t, n and b can be given as*

$$\begin{aligned}t =& \vartheta' = z_1, \\ n =& \frac{\nu}{\varepsilon\kappa} z_2 + \frac{\mu}{\varepsilon\kappa} z_3, \\ b =& \frac{1}{\varepsilon\tau}\left[\left(\frac{\nu}{\kappa}\right)' - \frac{\mu}{\kappa}\left(\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right)\right] z_2 \\ &+ \frac{1}{\varepsilon\tau}\left[\left(\frac{\mu}{\kappa}\right)' + \frac{\nu}{\kappa}\left(\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right)\right] z_3.\end{aligned} \tag{30}$$

Moreover we can write

$$\xi = \varepsilon\varrho t + \frac{\mu\sqrt{1-\varepsilon\varrho^2}}{\kappa} n - \varepsilon\frac{\sqrt{1-\varepsilon\varrho^2}}{\tau}\left[\left(\frac{\mu}{\kappa}\right)' + \frac{\nu}{\kappa}\left(\beta - \varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right)\right] b. \tag{31}$$

Theorem 3.2. *Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold and ϑ be a Frenet curve on $\bar{N}$. ϑ is a slant curve on $\bar{N}$ if and only if the Frenet elements $\{t, n, b, \kappa, \tau\}$ of this curve ϑ are as follows*

$$\begin{aligned}
t =& z_1 = \vartheta',\\
n =& z_2 = \frac{\varphi\vartheta'}{\sqrt{1-\varepsilon cos^2\theta}},\\
b =& z_3 = \frac{\xi - \varepsilon cos\theta\vartheta'}{\sqrt{1-\varepsilon cos^2\theta}},\\
\kappa =& \sqrt{\alpha^2(1-\varepsilon cos^2\theta)+\nu^2},\\
\tau =& \left|\left(\beta - \varepsilon\frac{cos\theta\nu}{\sqrt{1-\varepsilon\cos^2\theta}}\right) - \sqrt{\left[\left(\frac{\nu}{\varepsilon\kappa}\right)'\right]^2 + \left[\left(\frac{\alpha\sqrt{1-\varepsilon\cos^2\theta}}{\varepsilon\kappa}\right)'\right]^2}\right|.
\end{aligned} \tag{32}$$

Proof. Let the curve ϑ be a slant curve on $\bar{N}$. By considering the condition $\varrho = \eta(\vartheta') = cos\theta = constant$ in the equations (13), (25) and (29) we arrive at (32). If (32) holds, it is obvious from the definition of slant curves, ϑ is slant. □

From Theorem 3.2, we easily give the above corollaries.

Corollary 3.3. *Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold and ϑ be a slant curve on $\bar{N}$. If κ is a non-zero constant, then $\tau = \left|\left(\beta - \varepsilon\frac{cos\theta\nu}{\sqrt{1-\varepsilon\cos^2\theta}}\right)\right|$ and ϑ is a pseudo-helix on $\bar{N}$.*

Corollary 3.4. *Let $\bar{N}$ be a three dimensional n.a.c.p-m and ϑ be a slant curve on this manifold $\bar{N}$. If κ is not constant and $\tau = 0$ then ϑ is a plane curve on $\bar{N}$ and the following equation satisfies*

$$\bar{g}(\nabla_{\vartheta'}z_2, z_3) = \frac{\nu^2(\frac{\alpha}{\nu})'\sqrt{1-\varepsilon\cos^2\theta}}{\nu^2+\alpha^2(1-\varepsilon\cos^2\theta)}. \tag{33}$$

Theorem 3.5. *Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold and ϑ be a Frenet curve on $\bar{N}$. ϑ is a Legendre curve($\varrho = \eta(\vartheta') = 0$) on this manifold if and only if the Frenet elements $\{t, n, b, \kappa, \tau\}$ of this curve ϑ are as follows*

$$\begin{aligned}
t =& z_1 = \vartheta',\\
n =& z_2 = \varphi\vartheta',\\
b =& z_3 = \xi,\\
\kappa =& \sqrt{\nu^2+\alpha^2},\\
\tau =& \left|\beta - \sqrt{\left[\left(\frac{\nu}{\varepsilon\kappa}\right)'\right]^2 + \left[\left(\frac{\alpha}{\varepsilon\kappa}\right)'\right]^2}\right|.
\end{aligned} \tag{34}$$

Proof. Let the curve ϑ be a Legendre curve on $\bar{N}$. By considering $\varrho = \eta(\vartheta') = 0$ in the equations (13), (25) and (29) we arrive at(34). If the equations in (34) hold, from the definition of Legendre curves it is obvious that the curve ϑ is a Legendre curve on $\bar{N}$. □

Corollary 3.6. *Let the curve ϑ is a Legendre curve in three dimensional n.a.c.p-m manifold $\bar{N}$. If κ is non-zero constant and $\tau = 0$ then ϑ is a plane curve on $\bar{N}$ and $\beta = 0$.*

Moreover we can give the following corollaries.

Corollary 3.7. *Let $\bar{N}$ be a three dimensional n.a.c.p-m manifold and ϑ be a Frenet curve on this manifold. If $\bar{N}$ is cosymplectic, then from the equations* (25) *and* (29) *the curvatures of ϑ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon \varrho'}{\sqrt{1-\varepsilon\varrho^2}} \right)^2} \tag{35}$$

and

$$\tau = \left| \varepsilon \frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} + \sqrt{\left[\left(\frac{\nu}{\kappa} \right)' \right]^2 + \left[\left(\frac{\varrho'}{\kappa\sqrt{1-\varepsilon\varrho^2}} \right)' \right]^2} \right|. \tag{36}$$

i) *If ϑ is a slant, then we get*

$$\kappa = \nu \quad and \quad \tau = \left| \varepsilon \frac{cos\theta}{\sqrt{1-\varepsilon\cos^2\theta}} \right| \kappa. \tag{37}$$

ii) *If ϑ is a Legendere curve, then we get*

$$\kappa = \nu \quad and \quad \tau = 0. \tag{38}$$

Corollary 3.8. *Let ϑ be a curve on three dimensional quasi Sasakian pseudo-metric manifold $\bar{N}$. Then, the curvatures of ϑ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon \varrho'}{\sqrt{1-\varepsilon\varrho^2}} \right)^2} \tag{39}$$

and

$$\tau = \left| \beta - \varepsilon \frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} + \sqrt{\left[\left(\frac{\nu}{\kappa} \right)' \right]^2 + \left[\left(\frac{\varrho'}{\kappa\sqrt{1-\varepsilon\varrho^2}} \right)' \right]^2} \right|. \tag{40}$$

If the curve ϑ is a slant curve on $\bar{N}$, then we get

$$\kappa = \nu \quad and \quad \tau = \left| \beta - \varepsilon \frac{cos\theta}{\sqrt{1-\varepsilon\cos^2\theta}} \right| \kappa. \tag{41}$$

If the curve ϑ is a Legendre curve on $\bar{N}$, then we obtain

$$\kappa = \nu \quad and \quad \tau = |\beta|. \tag{42}$$

Corollary 3.9. *Let ϑ be a curve on three dimensional β-Sasakian pseudo-metric manifold $\bar{N}$. Then, the curvatures of ϑ are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon \varrho'}{\sqrt{1-\varepsilon\varrho^2}} \right)^2} \tag{43}$$

and

$$\tau = \left| \beta - \varepsilon \frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} + \sqrt{\left[\left(\frac{\nu}{\kappa} \right)' \right]^2 + \left[\left(\frac{\varrho'}{\kappa\sqrt{1-\varepsilon\varrho^2}} \right)' \right]^2} \right|. \tag{44}$$

The curvatures of ϑ are

$$\kappa = \nu \quad and \quad \tau = \left| \beta - \varepsilon \frac{cos\theta}{\sqrt{1-\varepsilon\cos^2\theta}} \right| \kappa \tag{45}$$

where ϑ is a slant curve in three dimensional β-Sasakian pseudo-metric manifold $\bar{N}$ and

$$\kappa = \nu \quad and \quad \tau = |\beta| \tag{46}$$

where ϑ is a Legendre curve in three dimensional β-Sasakian pseudo-metric manifold $\bar{N}$.

Corollary 3.10. *From* (25) *and* (29) *the curvatures of* ϑ *on tree dimensional Sasakian pseudo-metric manifold* $\bar{N}$ *are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}}\right)^2} \tag{47}$$

and

$$\tau = \left|\varepsilon\left(1-\varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}}\right) - \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\varrho'}{\kappa\sqrt{1-\varepsilon\varrho^2}}\right)'\right]^2}\right|. \tag{48}$$

i) *If* ϑ *is a slant curve, then we have*

$$\kappa = \nu \quad and \quad \tau = \left|\varepsilon\left(1-\varepsilon\frac{cos\theta}{\sqrt{1-\varepsilon\cos^2\theta}}\right)\right|\kappa. \tag{49}$$

ii) *If* ϑ *is a Legendere curve, then we get*

$$\kappa = \nu \quad and \quad \tau = 1. \tag{50}$$

Corollary 3.11. *Let* ϑ *be a curve on three dimensional* α*-Kenmotsu pseudo-metric manifold* $\bar{N}$*. Then the curvatures of* ϑ *are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \alpha\sqrt{1-\varepsilon\varrho^2}\right)^2} \tag{51}$$

and

$$\tau = \left|\varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} + \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \alpha\sqrt{1-\varepsilon\varrho^2}}{\varepsilon_2\kappa}\right)'\right]^2}\right|. \tag{52}$$

If ϑ *is a slant curve on* $\bar{N}$*, then we obtain*

$$\kappa = \sqrt{\nu^2 + \alpha^2(1-\varepsilon\cos^2\theta)}, \tag{53}$$

$$\tau = \left|\varepsilon\frac{\nu cos\theta}{\sqrt{1-\varepsilon\cos^2\theta}} + \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\alpha\sqrt{1-\varepsilon\cos^2\theta}}{\kappa}\right)'\right]^2}\right|. \tag{54}$$

If ϑ *is a Legendre curve on* $\bar{N}$*, then we get*

$$\kappa = \sqrt{\nu^2+\alpha^2} \quad and \quad \tau = \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\alpha}{\kappa}\right)'\right]^2}. \tag{55}$$

Corollary 3.12. *Let* ϑ *be a curve on three dimensional Kenmotsu pseudo-metric manifold* $\bar{N}$*. Then, the curvatures of* ϑ *are*

$$\kappa = \sqrt{\nu^2 + \left(\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \sqrt{1-\varepsilon\varrho^2}\right)^2} \tag{56}$$

and

$$\tau = \left|\varepsilon\frac{\varrho\nu}{\sqrt{1-\varepsilon\varrho^2}} + \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\frac{\varepsilon\varrho'}{\sqrt{1-\varepsilon\varrho^2}} - \sqrt{1-\varepsilon\varrho^2}}{\varepsilon_2\kappa}\right)'\right]^2}\right|. \tag{57}$$

The curvatures of ϑ *are*

$$\kappa = \sqrt{\nu^2 + (1 - \varepsilon \cos^2 \theta)}, \tag{58}$$

$$\tau = \left| \varepsilon \frac{\nu cos\theta}{\sqrt{1 - \varepsilon \cos^2 \theta}} + \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{\sqrt{1 - \varepsilon \cos^2 \theta}}{\kappa}\right)'\right]^2} \right|. \tag{59}$$

where ϑ *is a slant curve in three dimensional Kenmotsu pseudo-metric manifold* $\bar{N}$ *and*

$$\kappa = \sqrt{\nu^2 + 1} \quad and \quad \tau = \sqrt{\left[\left(\frac{\nu}{\kappa}\right)'\right]^2 + \left[\left(\frac{1}{\kappa}\right)'\right]^2} \tag{60}$$

where ϑ *is a Legendre curve in three dimensional Kenmotsu pseudo-metric manifold* $\bar{N}$.

4. Conclusion

In this paper we constructed the Frenet apparatus of a non-geodesic Frenet curve on three dimensional normal almost contact pseudo-metric manifold. We gave some theorems about these curves and find their Frenet elements $\{t, n, b, \kappa, \tau\}$. Moreover we gave corollaries for these curves to be slant curve and Legendre curve. So, we characterized some curves on three dimensional normal almost contact pseudo-metric manifolds by using their Frenet elements.

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