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A NEW WEIBULL-LINDLEY DISTRIBUTION IN MODELLING LIFETIME DATA

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Abstract

In literature, Alzaatreh et al. [10] proposed the generalized Weibull-X family of distribution in their study. Based on this study, we introduce a new Weibull-Lindley (NWL) distribution in this study. The pdf (probability density function), distribution, survival functions, hazard rate and cumulative hazard functions are derived and investigated. Besides, many mathematical properties including mode, quantile function, median, Shannon entropy, skewness, kurtosis, and order statistics are also derived. According to the maximum likelihood method, the estimation of parameters is done. In application part, we use real data sets. According to results, our proposed NWL distribution is superior comparison with the Akash, Lindley, New Weibull-F, two-parameter Lindley (TPL), and Weibull-Lindley (WL) distributions.

Keywords: Weibull-Lindley distribution, estimation, maximum likelihood method, entropy, life-time data.

YAŞAM VERİLERİNİN MODELLENMESİ İÇİN YENİ WEIBULL-LINDLEY DAĞILIMI

Özet

Literatürde Alzaatreh ve diğerleri [10] çalışmalarında genelleştirilmiş Weibull-X dağılım ailesini önermişlerdir. Önerilen dağılımdan yararlanarak, bu çalışmada yeni bir Weibull-Lindley (NWL) dağılımı geliştirilmiştir. Olasılık yoğunluk, dağılım, yaşam, hazard ve kantil fonksiyonları, mod, medyan, Shannon entropisi, çarpıklık ve basıklık katsayıları, sıralı istatistikleri gibi birçok matematiksel özellik de elde edilmiştir. Maksimum olabilirlik yöntemine göre parametre tahmini yapılmıştır. Uygulama kısmında gerçek veri setlerini kullanılmış ve önerilen NWL dağılımımız Akash, Lindley, New Weibull-F, iki parametreli Lindley (TPL) ve Weibull-Lindley (WL) dağılımları ile karşılaştırıldığında daha iyi sonuçların elde edildiği görülmüştür.

Anahtar Kelimeler: Weibull-lindley dağılımı, tahmin, maksimum olabilirlik yöntemi, entropi, yaşam verisi.

Cite

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1. Introduction

In recent years, new distributions have been generated by means of fitting lifetime data generated from various disciplines such as geology, ecology, biology, and so on. Statistical distributions can be used to describe and predict real-world events. Even though numerous distributions have been produced, there is always a need for more flexible distributions or distributions that fit specific real-world conditions [1].

In the literature, over 25 novel families of generalized distribution have been examined. Many scholars have pioneered in this field, including Azzalini and Capitania [2], Gupta et al. [3], Azzalini [4], and Marshall and Olkin [5]. The significant work begins after Eugene et al. [6] introduces the beta generator, which is later characterized by Jones [7]. Alexander et al. [8], and Cordeiro and de-Castro [9] proposed competing generators.

Alzaatreh et al. [10] defined a link function that generates T-X families of generalized distributions by means of any pdf as a generator. For a special case, they examined generalized Weibull-X. By fixing T as a Weibull distributed random variable and allowing X to have any form, generalized Weibull-X family has been introduced. Many generalized distributions have been constructed from T-X family of distributions by both fixing distribution of T and picking multiple forms of X.

Data analysis in real life often relies mainly on statistical probability distributions. One of the main objectives for proposing a new distribution is to explain how the lifetime phenomenon arises in fields like insurance, public health, physics, medicine, engineering, computer science, biology, lifetime, and so on. The other reason is the inadequacy of the classical distribution. At this point, the main motivation of this study is to provide a good fit for the real data instead of well-known distributions.

In the next section, the general form for the Weibull-X family of distributions in Alzaatreh et al. [10] is presented and then define the Weibull-Lindley (WL) distribution, based on one-parameter Lindley distribution [11]. The forms of pdf, cdf, survival and hazard functions are obtained and then discussed. In section 4, we will look at the quantile function, median, mode, Shannon entropy, Galton's skewness, order statistics, and Moor's kurtosis, among other statistical aspects of the NWL distribution. We will then use the maximum likelihood method for estimating the parameters. In Section 5, we use the NWL distribution, along with some distributions for fitting two real data sets. Due to Marinho et al. [12], the model fitting is done in R via "Adequacy Model" package.

2. Weibull-X Family of Distributions

Let X be the random variable with cdf $F(x)$ and the pdf $f(x)$ as well. Besides, the pdf of $T\epsilon[a,b]$ is symbolized as $r(t)$. $W(F(x))$ is the function for the cdf $F(x)$ for any random variable. Using these informations, the cdf for the new distribution is given by

$$G(x) = \int_a^{W(F(x))} r(t)dt, \\ = R\left[W\left(F(x)\right)\right]. \quad (1)$$

In Equation (1), cdf of the random variable T is symbolized as $R(t)$.

Then, the pdf is calculated as

$$g(x) = \frac{d}{dx}\left[W\left(F(x)\right)\right] r\left[W\left(F(x)\right)\right] \quad (2)$$

which is the composite function of $R.W.F(x)$.

By means of the link function $W\left(F(x)\right)$ which acts as a transformer, the pdf $r(t)$ is turned into the new distribution $g(x)$, and thus the T-X family of distribution names is provided [1]. Then, if the random variable X is discrete, the family of distributions will be discrete.

Using different forms for$W\left(F(x)\right)$, the new family of distribution can be proposed. The lifetime distribution has positive domains. For this reason, the various types of lifetime distribution family can be generated using $-\log\left(1-F(x)\right)$, $\frac{F(x)^\alpha}{1-F(x)^\alpha}, \frac{F(x)}{(1-F(x)}$, and $-\log\left(1-F(x)^\alpha\right.$ link functions.

Alzaatreh et al. [10] considered Weibull distribution in case of lifetime distribution using the support of T as $-\log\left(1-F(x)\right)$ in their study and the cdf for the new family of Weibull distribution is obtained by substituting into Eq. (1) as

$$G(x) = R\{-\log[1-F(x)]\} = R\left(H(x)\right) \quad (3)$$

Here, the cdf of T is symbolized as $R(t)$ and $-\log[1-F(x)]$ means the cumulative hazard rate function for X. In this way, the corresponding pdf is obtained below.

$$g(x) = \frac{f(x)}{(1-F(x))} r(\{-\log(1-F(x))\}) = h(x)r\left(H(x)\right). \quad (4)$$

Here, $h(x)$ and $H(x)$ are the hazard rate and cumulative hazard functions for X with$F(x)$, respectively.

The pdf of T is given below in case the random variable T which follows Weibull distribution with parameters β and c as

$$r(t) = \frac{c}{\beta}\left(\frac{t}{\beta}\right)^{c-1} \exp\left(-\left(\frac{t}{\beta}\right)^c\right),$$

where, $t \geq 0, (c, \beta) > 0$. From Eq. (4), we have the pdf of Weibull-X family as

$$g(x) = \frac{cf(x)(-(\log(1-F(x))/\beta)^{c-1}}{\beta(1-F(x))}\left\{\exp - \left[\left(-\frac{\log(1-F(x))}{\beta}\right]^c\right\}. \quad (5)$$

The cdf for the Weibull distribution and Weibull-X are given, respectively, as follows:

$$R(t) = 1 - \exp\left(-\left(\frac{t}{\beta}\right)^c\right),$$

$$G(x) = 1 - \exp\left[-\left\{-\frac{\log(1-F(x))}{\beta}\right\}^c\right]. \quad (6)$$

In this point, let X follows the Lindley distribution with parameter θ. Then, the pdf of the X is given as

$$f(x) = \frac{\theta^2}{(1+\theta)}(1+x)\exp(-\theta x), \quad (7)$$

where $x \geq 0,\ and\ \theta > 0$.

Then, the cdf of the Lindley distribution is obtained as

$$F(x) \ = 1 - \exp(-\theta x)\frac{(\theta+1+\theta x)}{(\theta+1)} \quad (8)$$

The survival function and hazard rate function are defined, respectively, as

$$S(x) = \frac{(\theta+1+\theta x)}{(\theta+1)}\exp(-\theta x), \quad (9)$$

$$h(x) = \frac{f(x)}{S(x)} \\ = \frac{\theta^2(1+x)}{(\theta+1+\theta x)}. \quad (10)$$

The cumulative hazard function $H(x)$ is equal to$\left(-\ln\left(S(x)\right)\right)$. Using this relationship, the cumulative hazard functions is obtained as

$$H(x) = -\ln\left[\frac{(\theta+1+\theta x)}{(\theta+1)} exp(-\theta x)\right]. \quad (11)$$

3. A New Weibull-Lindley (NWL) Distribution

3.1. Density Function of the NWL Distribution

We obtain the pdf for our proposed new distribution as

$$g(x) = \frac{c}{\beta}\left[\frac{(\theta^2(1+x))}{(1+\theta+\theta x)}\right] * \left[\left\{\frac{-log\left(\frac{\exp(-\theta x)(1+\theta+\theta x)}{(\theta+1)}\right)}{\beta}\right\}^{c-1}\right]\left\{\exp\left[-\left\{\frac{-log\left[\frac{(\theta+1+\theta x)\exp(-\theta x)}{(1+\theta)}\right]}{\beta}\right\}^{c}\right]\right\}. \quad (12)$$

In equation (12), β, θ and c>0 and $x \geq 0$.

Figure 1 shows different shapes of the density function in case of considering various parameter sets.

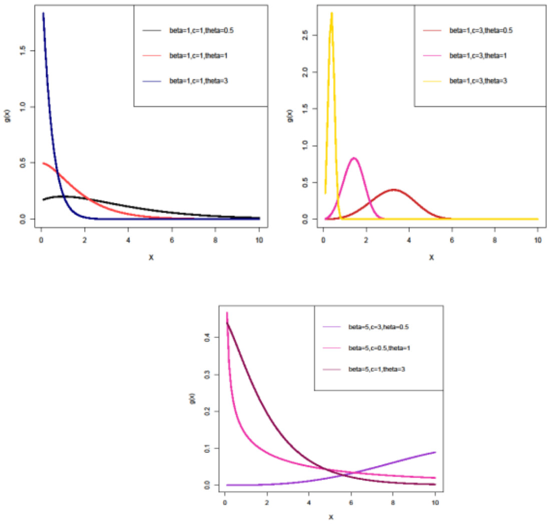

Figure 1. Plots of density function for the NWL distribution for various parameter values.

The pdf with various parameter values are showed in Figure 1. According to the results, the pdf of NWL distribution has various, increasing or decreasing, shapes. If all parameters are greater or equal to 1, we can see an exponential shape except in case of (β=1,c=3,and θ=3). In this case, curve rises then decline sharply for lower values of xs. When parameters equal to (β=1,c=3,and θ=0.5), similar to normal distribution, the pdf has a bell-shape.

3.2. Cumulative Density Function of the NWL Distribution

By substituting equation (8) into (6), cdf of NWL is obtained as

$$G(x) = 1 - \exp\left[-\left\{-\frac{log\left(1-\left(1-\frac{(\theta+1+\theta x)}{(\theta+1)}exp(-\theta x)\right)\right)}{\beta}\right\}^{c}\right], \quad (13)$$

Here we have $(\theta, \beta, c) > 0$ and$x \geq 0$. Some shapes of NWL distribution with several sets of parameters are described in Figure 2.

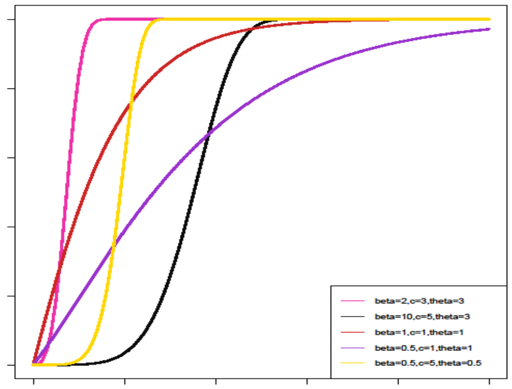

Figure 2. Distribution function curves for NWL distribution for various parameter values.

As seen from the Figure 2, cdf of NWL increases to one under different rates in case of considering several sets of parameters.

3.3. Survival Function of the NWL Distribution

Survival function of the NWL distribution is calculated as follows:

$$S(x) = 1 - G(x)$$

and then the survival function for the NWL distribution is obtained as

$$S(x) = \exp\left[-\left\{-\frac{log\left(1-\left(1-\frac{(\theta+1+\theta x)}{(\theta+1)}exp(-\theta x)\right)\right)}{\beta}\right\}^{c}\right]. \quad (14)$$

Using various sets of parameters, we obtain the survival plots for the NWL distribution in Figure 3 as below:

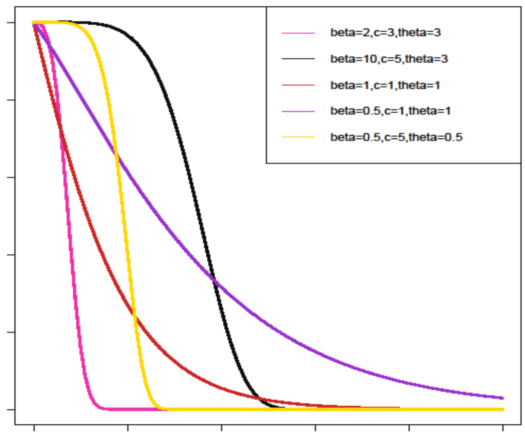

Figure 3. Survival curves for NWL distribution with various parameter values.

From Figure 3, survival curves decline sharply for various parameters at lower values of xs. When parameters equal to $(\beta = 2, c = 3, and\ \theta = 3)$, this curve stays constant. We can see also exponential decline when parameters equal to$(\beta = 1, c = 1, and\ \theta = 1), and\ (\beta = 0.5, c = 1, and\ \theta = 1)$.

3.4. Hazard Function of the NWL Distribution

From the density and survival functions in Eq. (12) and Eq. (14), respectively, the hazard function of the NWL distribution is obtained by

$$h(x)=\frac{g(x)}{S(x)}=\frac{c}{\beta}\left[\frac{(\theta^2(1+x))}{(1+\theta+\theta x)}\right]\left[\left\{\frac{-\log\left(\frac{\exp(-\theta x)(1+\theta+\theta x)}{(\theta+1)}\right)}{\beta}\right\}^{c-1}\right]. \quad (15)$$

In Figure 4, the different shapes of hazard function for NWL distribution are displayed using various sets of parameters.

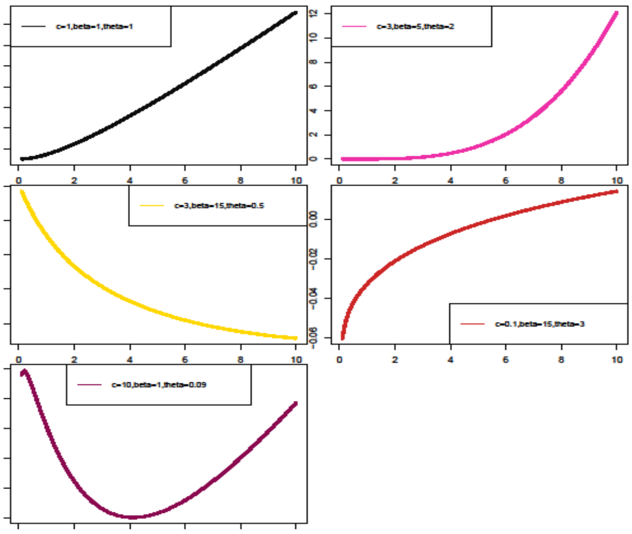

Figure 4. Hazard curves for NWL distribution for various parameter values.

According to the Figure 4, we can conclude that the hazard curves for NWL distribution is exponentially increasing when parameters equal to$(c=3,\beta=5, and\ \theta=2)$, exponentially decreasing if parameters equal to$(c=3,\beta=15, and\ \theta=0.5)$. Besides, hazard curves is constantly increasing when parameters equal to$(c=1,\beta=1, and\ \theta=1)$, and shape of "Bathtube" when parameters equal to$(c=10,\beta=1, and\ \theta=0.09)$.

3.5. Cumulative Hazard Function of the NWL Distribution

Cumulative hazard function, $H(x)$, is obtained by

$$H(x)=-\ln(S(x))$$

$$=-\ln\left(\exp\left[-\left\{-\frac{\log\left(1-\left(1-\frac{(\theta+1+\theta x)}{(\theta+1)}\exp(-\theta x)\right)\right)}{\beta}\right\}^{c}\right]\right). \quad (16)$$

4. Some Mathematical Properties of the NWL Distribution

Now, the quantile function, median, Moor's kurtosis, Galton's skewness, mode, Shannon entropy, and order statistics are derived and discussed as some of the mathematical properties of the NWL distribution.

4.1. Quantile Function of the NWL Distribution

Quantile function is obtained by inverting cdf equation of the NWL distribution as follows:

$$y=G(x)=1-\exp\left[-\left\{-\frac{\log\left(1-\left(1-\frac{(\theta+1+\theta x)}{(\theta+1)}\exp(-\theta x)\right)\right)}{\beta}\right\}^{c}\right],$$

$$=1-\exp\left[-\left\{-\frac{\log(1-F(x))}{\beta}\right\}^{c}\right],$$

and quantile function of the NWL distribution is obtained by

$$Q_p=1-\frac{1}{\theta}-\frac{1}{\theta}W_{-1}[(1+\theta)$$

$$*\left\{\exp(-1-\theta)-\exp\left\{\beta\ln(1-x)^{\frac{1}{c}}\right\}\right]. \quad (17)$$

In this equation, $\beta,\theta, and\ c>0$ and W_{-1} is describe as the negative branch for the Lambert W function.

4.2. Median of the NWL Distribution

The median is calculated by solving$Q_p=0.5$. For this reason, the equation is obtained as

$$1-\frac{1}{\theta}-\frac{1}{\theta}W_{-1}[(1+\theta)\left\{-\exp\left\{\beta\ ln(1-x)^{\frac{1}{c}}\right\}\exp(-1-\theta)\right]=0.5.$$

We can get median of this distribution for specified values for parameter, since the computations will be laborious due to the complex Lambert function.

4.3. Kurtosis and Skewness

Moments of NWL are difficult to obtain like many other distributions due to involving an expansion. For this reason, the kurtosis and the skewness are recommend in this case. The measure of skewness and kurtosis are defined by Galton [13] and Moors [14], respectively, are given as follows:

$$\text{Moors kurtosis } K=\frac{Q_{\frac{7}{8}}-2Q_{\frac{5}{8}}+Q_{\frac{3}{8}}-Q_{\frac{1}{8}}}{Q_{\frac{6}{8}}+Q_{\frac{2}{8}}},$$

$$\text{Galton skewness } S=\frac{Q_{\frac{6}{8}}-2Q_{\frac{4}{8}}+Q_{\frac{2}{8}}}{Q_{\frac{6}{8}}+Q_{\frac{2}{8}}}.$$

Using different sets of parameter values, Moors Kurtosis and Galton Skewness are computed and shown Table 1 and Figure 5.

Table 1. Kurtosis and Skewness values for NWL distribution

Parameters	Moors Kurtosis	Galton Skewness
β=1,C=1,θ=0.5	0.187042864	0.187042864
β=2,C=2,θ=0.5	0.044174674	2.851095
β=3,C=3,θ=0.5	-0.00619929	4.221864
β=4,C=4,θ=0.5	-0.03236153	3.330798
β=5,C=5,θ=0.5	-0.04848698	7.000002

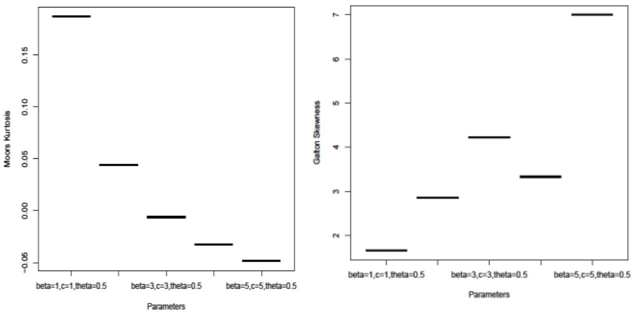

Figure 5. Kurtosis and skewness values for NWL distribution for several values of parameters.

Table 1 and Figure 5 shows that, the NWL distribution has normal approximation with lower values of parameters and positively skewed. Besides, as the value of parameter decreases higher values of Moors kurtosis are observed.

4.4. The Mode of the NWL Distribution

Using the Equation (12), the mode of NWL distribution is obtained with the solution of the equation $\frac{d}{dx}\left(g(x)\right) = 0$. As a result, by taking the derivative, we get the mode of the NWL distribution as follows:

$$\left\{\frac{(c-1)}{x} - \frac{\theta}{(\theta+1+x)} + \frac{1}{(1+x)} - \frac{(c+1)}{\log\left[\left(\frac{(\theta+1+\theta x)}{(\theta+1)}\exp(-\theta x)\right]\right.} * \left\{\frac{-\theta}{(\theta+1+\theta x)} + \theta\right\} - \left(\frac{c}{\beta}\right)\left(\frac{\theta}{(1+\theta+\theta x)} - \theta\right)\left\{\frac{-1}{\beta^c}\left[-\frac{(\theta+1+\theta x)}{(\theta+1)}\exp(-\theta x)\right]^c\right\}^{1-\frac{1}{c}}\right\} * g(x) = 0 \quad (18)$$

4.5. Shannon Entropy of the NWL Distribution

General formula for Shannon entropy of X was proposed in Alzaatreh et al. [10] in the case of the Weibull-X family of distribution. The pdf for the measure of the variation of uncertainty is given by

$\eta_X = -\mu_T + \eta_T - E[log f\{F^{-1}(1-e^{-T})\}]$.

Here, μ_T and η_T are the mean and Shannon entropy of T with $r(t)$. By substituting the Shannon entropy and the mean of the Weibull distribution, Shannon entropy is obtained as follows:

$$\eta_X = -\beta\Gamma\left(1+\frac{1}{c}\right) + \gamma\left(1-\frac{1}{c}\right) - \log\left(\frac{c}{\beta}\right) + 1 - E[log f\{F^{-1}(1-e^{-T})\}].$$

Here, c and β are the parameters for the Weibull distribution. Besides, for the Lindley distribution, $F(.)$ is cdf and $f(.)$ is pdf for our case, respectively.

Using the cdf and pdf of Lindley distribution, the $F^{-1}(x)$ and $F^{-1}(1-e^{-T})$ are obtained and then the expression of the $E[f\{F^{-1}(1-e^{-T})\}]$ can also be written as follows:

$$E[f\{F^{-1}(1-e^{-T})\}] = E\{2\log(\theta) - \log(1+\theta) + \log\left\{\left(1+\left[-1-\frac{1}{\theta}-\frac{1}{\theta}W_{-1}\{\exp(-1-\theta)(1+\theta)(-e^{-T})\}\right]\right) + \left(-\theta\left[-1-\frac{1}{\theta}-\frac{1}{\theta}W_{-1}\{\exp(-1-\theta)(1+\theta)(-e^{-T})\}\right]\right)\right\}.$$

As a result, Shannon entropy for NWL distribution is obtained by

$$\eta_X = -\beta\Gamma\left(1+\frac{1}{c}\right) + \gamma\left(1-\frac{1}{c}\right) - \log\left(\frac{c}{\beta}\right) + -E\{2\log(\theta) - \log(1+\theta) + \log\left\{\left(1+\left[-1-\frac{1}{\theta}-\frac{1}{\theta}W_{-1}\{\exp(-1-\theta)(1+\theta)(-e^{-T})\}\right]\right) + \left(-\theta\left[-1-\frac{1}{\theta}-\frac{1}{\theta}W_{-1}\{\exp(-1-\theta)(1+\theta)(-e^{-T})\}\right]\right)\right\}, \quad (19)$$

where $(\beta, \theta, and, c) > 0$.

4.6. Order Statistics of the NWL Distribution

The random variable's order statistic has a variety of uses, including modeling extreme data sets, estimating various statistical quantities, and more. For the NWL distribution, we can obtain distribution of order statistic. Using Equations (12) – (13), n-order statistics represent as the$X_{(1)}, X_{(2)},\ldots X_{(n)}$ and pdf of the r^{th} order is given as follows:

$$f(x_r) = \frac{n!}{(r-1)!(n-r)!}\sum_{i=1}^{n-r}(-1)^i\binom{n-r}{i} g(x)\left(G(x)\right)^{(i+r-1)}, \quad (20)$$

In Equation (20), $G(x)$ is cdf and $g(x)$ is pdf of any continuous distribution in our case being NWL distribution.

The pdf of the r^{th}order statistic for NWL, distribution can be written as follows:

$$f(x_r) = \frac{n!}{(r-1)!(n-r)!}\sum_{i=1}^{n-r}(-1)^i\binom{n-r}{i}\frac{c}{\beta}\left[\frac{\left(\theta^2(1+x)\right)}{(1+\theta+\theta x)}\right]\left[\left\{\frac{-log\left(\frac{\exp(-\theta x)(1+\theta+\theta x)}{(\theta+1)}\right)}{\beta}\right\}^{c-1}\right] \left\{\exp\left[-\left\{\frac{-log\left[\frac{(\theta+1+\theta x)\exp(-\theta x)}{(1+\theta)}\right]}{\beta}\right\}^{c}\right]\right\} * \left\{1-\exp\left[-\left\{-\frac{\log\left(1-\left(1-\frac{(\theta+1+\theta x)}{(\theta+1)}\exp(-\theta x)\right)\right)}{\beta}\right\}^{c}\right]\right\}^{(i+r-1)} \quad (21)$$

In Equation (21), the smallest's (1^{st} order statistic) and the largest's ((n^{th} order statistic)) pdf are obtained by substituting 1 and n instead of r.

4.7. Estimation of Parameters

The first step of the maximum likelihood estimation is writing of the mathematical expression, which is known as the likelihood function of the sample data. The likelihood of the data set, in broad terms, is the probability of obtaining that particular set of data given the probability distribution model chosen [15].

Using Equation (12), the likelihood function is

$$L(x,\psi) = \prod_{i=1}^{n}\left\{\frac{c}{\beta}\left[\frac{\left(\theta^2(1+x_i)\right)}{(1+\theta+\theta x_i)}\right] * \left[\left\{\frac{-log\left(\frac{\exp(-\theta x_i)(1+\theta+\theta x_i)}{(\theta+1)}\right)}{\beta}\right\}^{c-1}\right] * \left\{\exp\left[-\left\{\frac{-log\left[\frac{(\theta+1+\theta x_i)\exp(-\theta x_i)}{(1+\theta)}\right]}{\beta}\right\}^{c}\right]\right\}\right\},$$

where $\psi = (\theta, \beta, c)$.

We use partial derivatives with respect to $\psi = (\theta, \beta, c)$, respectively, to derive the normal equations, which result in the following equations after simplifications.

$$\frac{dl}{d\theta} = 0 \rightarrow \frac{2n}{\theta} + \sum_{i=1}^{n}\left(\theta - \frac{x_i}{(1+\theta+x_i)}\right)$$
$$+(c-1)\sum_{i=1}^{n}\left(\frac{x_i}{(\theta+1+\theta x_i)} - \frac{1}{(\theta+1)} - x_i\right) - \frac{1}{\beta^c}\sum_{i=1}^{n}\left(\frac{cx_i}{(\theta+1+\theta x_i)} - cx_i - \frac{c}{(\theta+1)}\right) = 0 \quad (22)$$

$$\frac{dl}{dc} = 0 \rightarrow \frac{n}{c} + ln\left\{-\sum_{i=1}^{n}\frac{log\left(\frac{(\theta+1+\theta x_i)}{(\theta+1)}\exp(-\theta x_i)\right)}{\beta}\right\} - nln(\beta) +$$
$$-ln\left\{-\sum_{i=1}^{n}\frac{log\left(\frac{(\theta+1+\theta x_i)}{(\theta+1)}\exp(-\theta x_i)\right)}{\beta}\right\} = 0 \quad (23)$$

$$\frac{dl}{d\beta} = 0$$

$$\frac{-nc}{\beta} - (c-1)\beta^{-2}ln\left\{-\sum_{i=1}^{n}\left(log\left(\frac{(\theta+1+\theta x_i)}{(\theta+1)}\exp(-\theta x_i)\right)\right)\right\}$$
$$\frac{c}{\beta^{(c+1)}}\left\{-\sum_{i=1}^{n}\left(log\left(\frac{(\theta+1+\theta x_i)}{(\theta+1)}\exp(-\theta x_i)\right)\right)\right\}^{c} = 0 \quad (24)$$

The usual computation methods may make it challenging to derive analytical solutions to the above normal equations. To obtain numerical solutions for the parameters, we recommend using computer software such as MATLAB or R.

5. Application

In this section, we use two real-world survival datasets for newly generated distribution. These datasets are introduced by Gross and Clark [16] and Lee and Wang [17], respectively. The first data set is about relief times (in a minute) for 20 analgesic individuals and the second data set is about the remission times (in months) for 128 cancer patients.

For comprising, Weibull-Lindley (WL), Akash distribution, Lindley distribution [11], Two Parameter Lindley (TPL) [18], and New Weibull-F are also fitted as an additional model. The minus log likelihood, AIC, HQIC, CAIC, and BIC values are used for these comparisons and the smaller values are selected. The results are given in Tables 2 and 3, respectively.

Table 2. Results of the first data set

Distribution	Parameters	$-2logL$	AIC	BIC	CAIC	HQIC	K-S
NWL	**β =0.869** **C =1.972** **θ =0.696**	**21.381**	**48.763**	**51.750**	**50.263**	**49.346**	**D = 0.999**
Akash		59.5	61.7	62.5	61.7		D=0.320
Lindley	θ =0.813	30.249	62.499	63.495	62.721	62.693	D = 4.371
TPL	θ =1.109 α =0.003	26.258	56.517	58.509	57.223	56.906	D = 1.091
WL	α =1.887 λ =0.964 β =0.023	30.838	67.676	70.663	69.176	68.259	D = 5.133

Table 3. Results of the second data set

Distribution	Parameters	$-2logL$	AIC	BIC	CAIC	HQIC	K-S
NWL	**β =1.489** **C =0.837** **θ =0.278**	**412.654**	**831.309**	**839.865**	**831.502**	**834.785**	**D = 0.521**
NW-F			831.52	840.08	831.72	835.00	D=0.070
Lindley	θ =0.200	448.225	841.201	844.053	841.232	842.359	D = 79.857
TPL	θ =0.203 α =1.466	418.993	841.986	847.690	842.082	844.303	D = 117.17
WL	α =0.407 λ =0.143 β =1.470	521.822	1049.644	1058.2	1049.838	1053.121	D = 79.201

6. Conclusion

The new lifetime distribution is introduced based on the Weibull-X family, which is proposed by Alzaatreh et al. [10]. Some of the statistical properties of the NWL distribution are derived and obtained, respectively. After theoretical properties, we use two different real-world survival datasets related to relief times for analgesic individuals and the remission times for cancer patients. Using these data sets, the results shows that the introduced NWL distribution is performing well among compared Weibull-Lindley (WL), Akash, Lindley, Two Parameter Lindley, and New Weibull-F distributions. In future studies, the more properties of the NWL distribution can be studied and compared for generating different forms of NWL distribution.

7. References

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