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# On the challenge of identifying space dependent coefficient in space-time fractional diffusion equations by fractional scaling transformations method

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**Abstract.** In this study, we get over the challenge of recovering unknown space dependent coefficient in space-time fractional diffusion equations by means of fractional scaling transformations method. Fractional differential equation is given in the sense of the conformable fractional derivative having substantial properties. By these properties and fractional scaling transformations method the fractional problem is reduced into integer order problem which allows us to tackle the problem better. Then we establish the solution and unknown coefficient of the reduced problem. Later, by employing inverse transformation, the solution and unknown coefficient of the fractional problem are obtained. Finally, some examples are presented to illustrate the implementation and effectiveness of the method.

## 1. Introduction

Last couple of decades fractional differential equations play a significant role in modelling of various processes. As a result, they attract growing attention of many scientists in diverse branches of sciences such as engineering, mathematics, chemistry and physics [1–8]. Consequently, numerous analytical and numerical methods have been utilized to construct solutions of mathematical problems including fractional differential equations [9–15].

Therefore, identification of unknown coefficients in fractional differential equations with or without additional measured data becomes one of the trend challenges in inverse problems [16–18]. Hence, many researchers in various research areas have been developing new methods to tackle with this kind of inverse problems including fractional derivatives [16–20].

In this research, our focus is on establishing space dependent diffusivity coefficient and the solution of the mathematical problem including space-time fractional diffusion equation by means of fractional scaling transformation methods. The main advantage of this method is that it turns fractional order differential equations into integer order differential equations which makes the problem easier to tackle with. We remark that this method works out for the fractional differential equations in the sense of conformable fractional derivative. The main goal in this article is to reveal the unknown coefficient of the following

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governing space-time fractional diffusion equation:

$$D_t^{\alpha}u(x,t) = D_x^{\beta}(k(x)D_x^{\beta}u(x,t)) + f(x,t), 0 < x < x_1, 0 < t < t_1, 0 < \alpha, \beta \leqslant 1, \tag{1}$$

where $u(x,t)$ and $k(x) > 0$ represent the temperature and thermal diffusivity, respectively. Associated to (1) the prescribed initial condition is

$$u(x,0) = \varphi(x), 0 \leqslant x \leqslant x_1, \tag{2}$$

and the prescribed Dirichlet boundary conditions are

$$D_x^{\beta}u(0,t) = 0, 0 < t \leqslant t_1, \tag{3}$$

$$u(x_1,t) = g(t), 0 < t \leqslant t_1, \tag{4}$$

with additional condition

$$u(x,t_1) = E(x), 0 < x \leqslant x_1. \tag{5}$$

Having the condition $k(x) > 0$ makes the problem (1)-(5) well-posed.

## 2. Preliminaries

**Definition 1.** [21] Given a function $f : [0,\infty) \to R$. Then the conformable fractional derivative of $f$ of order $\alpha$ is defined by

$$T_t^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{6}$$

for all $t > 0$, $\alpha \epsilon (0,1)$.
**Theorem 2.** [21] Let $\alpha \epsilon (0,1]$ and $f$ be $\alpha$-differentiable at a point $t > 0$. Then $T_t^{\alpha} f(t) = t^{1-\alpha} f'(t)$.
**Theorem 3.** [21] Let $\alpha \epsilon (0,1]$ and $f, g$ be $\alpha$-differentiable at a point $t > 0$. Then,
$i)T^{\alpha}(af + bg) = aT^{\alpha}(f) + bT^{\alpha}(g)$ for all real constant $a, b$,
$ii)T^{\alpha}(t^p) = pt^{p-\alpha}$ for all $p \epsilon R$,
$iii)T^{\alpha}(fg) = fT^{\alpha}(g) + gT^{\alpha}(f)$,
$iv)T^{\alpha}(\frac{f}{g}) = \frac{gT^{\alpha}(f) - fT^{\alpha}(g)}{g^2}$,
$v)T^{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
**Definition 4.** [21] Let $\alpha \epsilon (n, n+1], n \epsilon N$ and $f$ be an $\alpha$-differentiable at $t$ where $t > 0$, then the conformable fractional derivative of $f$ of order $\alpha$ is defined as

$$T_t^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f^{(n)}(t+\varepsilon t^{n+1-\alpha}) - f^{(n)}(t)}{\varepsilon}, \tag{7}$$

where $f$ is $n$-differentiable at $t > 0$.
**Definition 5.** [21] The conformable $\alpha$-fractional integral of a function $f$ is defined by

$$I_t^{\alpha}(f)(t) = \int_0^t \frac{f(x)}{x^{1-\alpha}} dx, \alpha \epsilon (0,1). \tag{8}$$

**Theorem 6.** [21, 22] Let $a \geqslant 0$ and $\alpha \in (0,1)$. Also, let $f : (a,b) \to R$ be a continuous function such that $I_{\alpha}$ exist, then for all $t > a$, we have

$$T_t^{\alpha}(I_t^{\alpha}(f)(t)) = f(t), t \geq 0, \tag{9}$$

$$I_t^{\alpha}(T_t^{\alpha}(f)(t)) = f(t) - f(a). \tag{10}$$

## 3. Analysis of the new fractional derivative

By means of the following fractional scaling transformations

$$X = \frac{x^\beta}{\beta}, T = \frac{t^\alpha}{\alpha}, u(x,t) = V(X,T), \tag{11}$$

the problem (1)-(5) is converted to into the following integer order problem

$$V_T = (k(X)V_X)_X + \overline{f}(X,T), 0 < X < \frac{x_1^\beta}{\beta}, 0 < T < \frac{t_1^\alpha}{\alpha}, \tag{12}$$

with initial conditions

$$V(X,0) = \overline{\varphi}(X), 0 < X \leqslant \frac{x_1^\beta}{\beta}, \tag{13}$$

and the prescribed Dirichlet boundary conditions are

$$V_X(0,T) = 0, 0 < T \leqslant \frac{t_1^\alpha}{\alpha}, \tag{14}$$

$$V(\frac{x_1^\beta}{\beta}, T) = \overline{g}(T), 0 < T \leqslant \frac{t_1^\alpha}{\alpha}, \tag{15}$$

and additional condition

$$V(X, \frac{t_1^\alpha}{\alpha}) = \overline{E}(X), 0 < X \leqslant \frac{x_1^\beta}{\beta}. \tag{16}$$

After establishing the solution and unknown coefficient of problem (12)-(16), by employing inverse transformation we obtain the solution $u(x,t)$ and an unknown diffusivity coefficient $k(x)$.

## 4. Illustrative Examples

In this section, we illustrate three examples of inverse problems about determination of unknown space dependent coefficient.

*Example 1.* Consider the inverse coefficient problem involving space-time fractional differential equations [23, 24]:

$$D_t^\alpha u(x,t) = D_x^\beta(k(x)D_x^\beta u(x,t)), 0 < x < \beta^{\frac{1}{\beta}}, 0 < t < \alpha^{\frac{1}{\alpha}}, \tag{17}$$

$$u(x,0) = \frac{x^{3\beta}}{\beta^3}, 0 \leqslant x \leqslant \beta^{\frac{1}{\beta}}, \tag{18}$$

$$D_x^\beta u(0,t) = 0, 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \tag{19}$$

$$u(\beta^{\frac{1}{\beta}}, t) = exp(t), 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \tag{20}$$

$$u(x, \alpha^{\frac{1}{\alpha}}) = \frac{x^{3\beta}}{\beta^3} exp(1), 0 \leqslant x \leqslant \beta^{\frac{1}{\beta}}. \tag{21}$$

By taking fractional scaling transformation methods into account the problem (17)-(21) turns into following integer order problem:

$$V_T = (k(X)V_X)_X, 0 < X < 1, 0 < T < 1, \tag{22}$$

with initial conditions

$$V(X, 0) = X^3, 0 < X \leqslant 1, \tag{23}$$

and the prescribed Dirichlet boundary conditions are

$$V_X(0, T) = 0, 0 < T \leqslant 1, \tag{24}$$

$$V(1, T) = exp(T), 0 < T \leqslant 1, \tag{25}$$

and additional condition

$$V(X, 1) = X^3 exp(1), 0 < X \leqslant 1. \tag{26}$$

This inverse problem have the solution $V(X, T) = X^3 exp(T)$ and unknown diffusivity coefficient becomes $k(X) = \frac{1}{12}X^2$ . As seen from Figs.1-4, by means of inverse transformation the solution of problem (17)-(21) and unknown diffusivity coefficient are obtained in the following form respectively.

$$u(x, t) = \frac{x^{3\beta}}{\beta^3} exp(\frac{t^\alpha}{\alpha}) \tag{27}$$

and

$$k(x) = \frac{1}{12}\frac{x^{2\beta}}{\beta^2}. \tag{28}$$

Moreover, the values of exact and approximate solutions of problem (17)-(21) at $t = 0.8$ for different values of orders of $\alpha$ and $\beta$ are presented in Table 1.

Table 1: The table of exact and approximate solutions of Ex. 1 at $t = 0.8$ .

| $x$ | Exact | $\alpha = 1$ $\beta = 1$ | $\alpha = 0.9$ $\beta = 1$ | $\alpha = 0.8$ $\beta = 1$ | $\alpha = 1$ $\beta = 0.9$ | $\alpha = 0.9$ $\beta = 0.9$ | $\alpha = 0.8$ $\beta = 0.9$ |
|---|---|---|---|---|---|---|---|
| 0.2 | 0.01780 | 0.01780 | 0.01780 | 0.02276 | 0.03958 | 0.04414 | 0.05060 |
| 0.4 | 0.14243 | 0.14243 | 0.15883 | 0.18209 | 0.25720 | 0.28680 | 0.32881 |
| 0.6 | 0.48072 | 0.48072 | 0.53605 | 0.61457 | 0.76863 | 0.85710 | 0.98264 |
| 0.8 | 1.13948 | 1.13948 | 1.27063 | 1.45675 | 1.67129 | 1.86366 | 2.13664 |
| 1 | 2.22554 | 2.22554 | 2.48171 | 2.84522 | 3.05287 | 3.40426 | 3.90290 |

Figure 1: The graphics of exact and approximate solution for $k(x)$ in Ex. 1 .

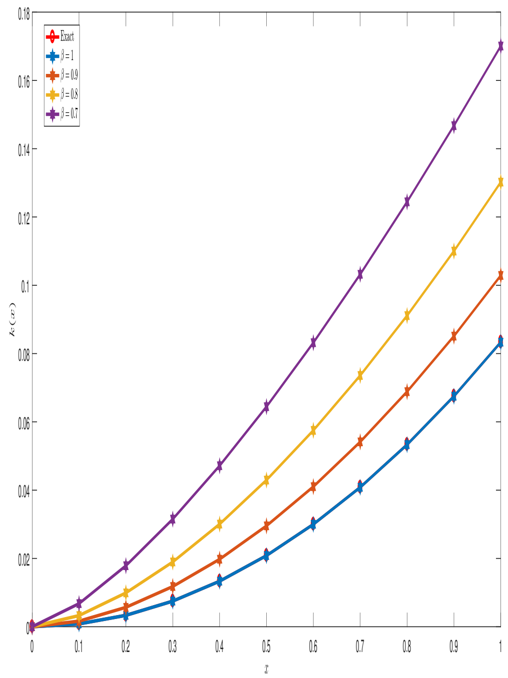



Figure 2: The graphics of exact solution for $u(x,t)$ in Ex. 1 .

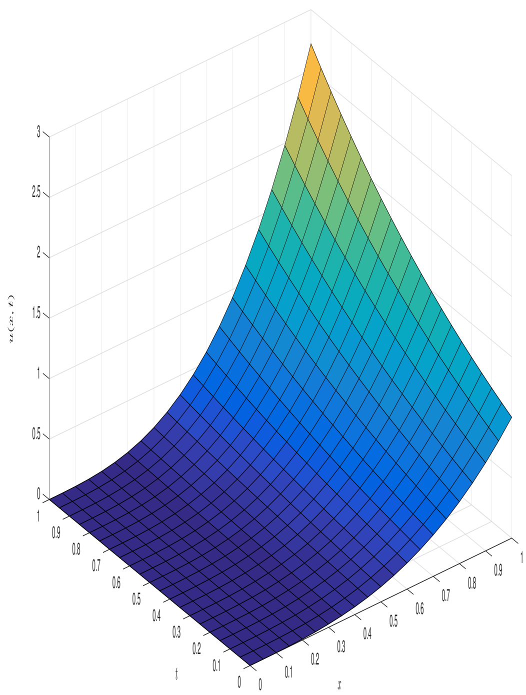


Figure 3: The graphics of approximate solution for $u(x,t)$ with $\alpha = 1$ and $\beta = 1$ in Ex. 1 .

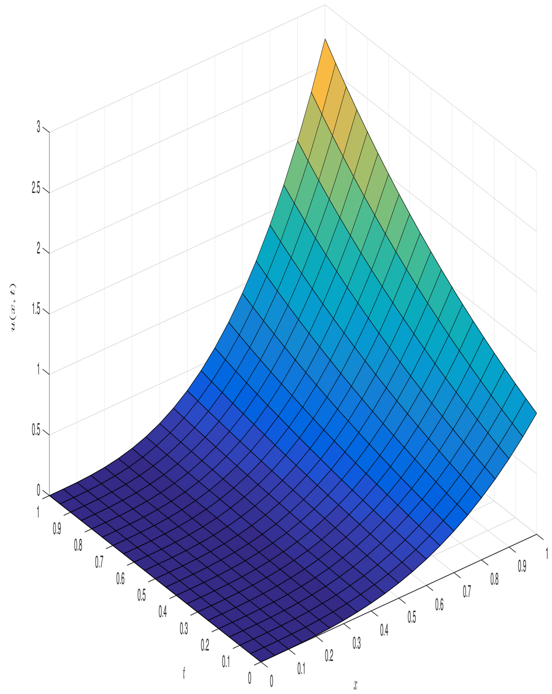
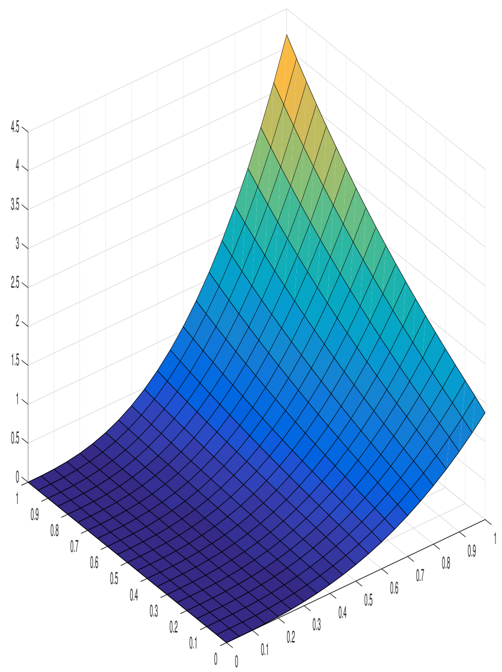



Figure 4: The graphics of approximate solution for $u(x,t)$ with $\alpha = 0.9$ and $\beta = 0.9$ in Ex. 1 .



*Example 2.* Consider the inverse coefficient problem involving space-time fractional differential equations [23, 24]:

$$D_t^{\alpha}u(x,t) = D_x^{\beta}(k(x)D_x^{\beta}u(x,t)), 0 < x < \beta^{\frac{1}{\beta}}, 0 < t < \alpha^{\frac{1}{\alpha}}, \tag{29}$$

$$u(x,0) = \frac{x^{2\beta}}{\beta^2}exp\Big(\frac{x^{\beta}}{\beta}\Big), 0 \leqslant x \leqslant \beta^{\frac{1}{\beta}}, \tag{30}$$

$$D_x^{\beta}u(0,t) = 0, 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \tag{31}$$

$$u(\beta^{\frac{1}{\beta}},t) = exp(1 + \frac{t^{\alpha}}{\alpha}), 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \tag{32}$$

$$u(x,\alpha^{\frac{1}{\alpha}}) = \frac{x^{2\beta}}{\beta^2}exp\Big(\frac{x^{\beta}}{\beta} + 1\Big), 0 \leqslant x \leqslant \beta^{\frac{1}{\beta}}. \tag{33}$$

By taking fractional scaling transformation methods into account the problem (29)-(33) turns into following integer order problem:

$$V_T = (k(X)V_X)_X, 0 < X < 1, 0 < T < 1, \tag{34}$$

with initial conditions

$$V(X,0) = X^2exp(X), 0 < X \leqslant 1, \tag{35}$$

and the prescribed Dirichlet boundary conditions are

$$V_X(0,T) = 0, 0 < T \leqslant 1, \tag{36}$$

$$V(1,T) = exp(1 + T), 0 < T \leqslant 1, \tag{37}$$

and additional condition

$$V(X,1) = X^2exp(X + 1), 0 < X \leqslant 1. \tag{38}$$

This inverse problem have the solution $V(X,T) = X^2exp(X+T)$ and unknown diffusivity coefficient becomes $k(X) = \frac{X^2-2X+2}{X^2+2X}$ . As seen from Figs.5-8, by means of inverse transformation the solution of problem (34)-(38) and unknown diffusivity coefficient are obtained in the following form respectively.

$$u(x,t) = \frac{x^{2\beta}}{\beta^2}exp\Big(\frac{x^{\beta}}{\beta} + \frac{t^{\alpha}}{\alpha}\Big) \tag{39}$$

and

$$k(x) = \frac{x^{2\beta} - 2\beta x^{\beta} + 2\beta^2}{x^{2\beta} + 2\beta x^{\beta}} \tag{40}$$

Moreover, the values of exact and approximate solutions of problem (28)-(32) at $t = 0.8$ for different values of orders of $\alpha$ and $\beta$ are presented in Table 2.

Table 2: The table of exact and approximate solutions of Ex. 2 at $t = 0.8$.

| $x$ | Exact | $\alpha = 1$ $\beta = 1$ | $\alpha = 0.9$ $\beta = 1$ | $\alpha = 0.8$ $\beta = 1$ | $\alpha = 1$ $\beta = 0.9$ | $\alpha = 0.9$ $\beta = 0.9$ | $\alpha = 0.8$ $\beta = 0.9$ |
|---|---|---|---|---|---|---|---|
| 0.2 | 0.10873 | 0.10873 | 0.12125 | 0.13901 | 0.19686 | 0.21952 | 0.25168 |
| 0.4 | 0.53122 | 0.53122 | 0.59236 | 0.67913 | 0.85941 | 0.95833 | 1.09870 |
| 0.6 | 1.45987 | 1.45987 | 1.62791 | 1.86636 | 2.20967 | 2.46401 | 2.82492 |
| 0.8 | 3.16994 | 3.16994 | 3.53481 | 4.05257 | 4.56313 | 5.08836 | 5.83369 |
| 1 | 6.04965 | 6.04965 | 6.74598 | 7.73410 | 8.34642 | 9.30711 | 10.67038 |

Figure 5: The graphics of exact and approximate solution for $k(x)$ in Ex. 2 .

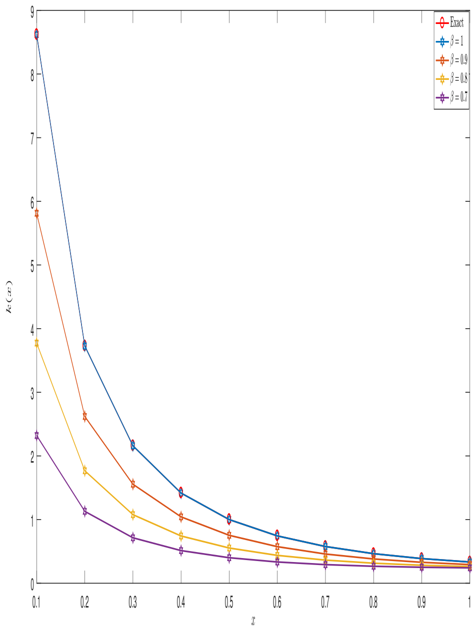


Figure 6: The graphics of exact solution for $u(x,t)$ in Ex. 2 .

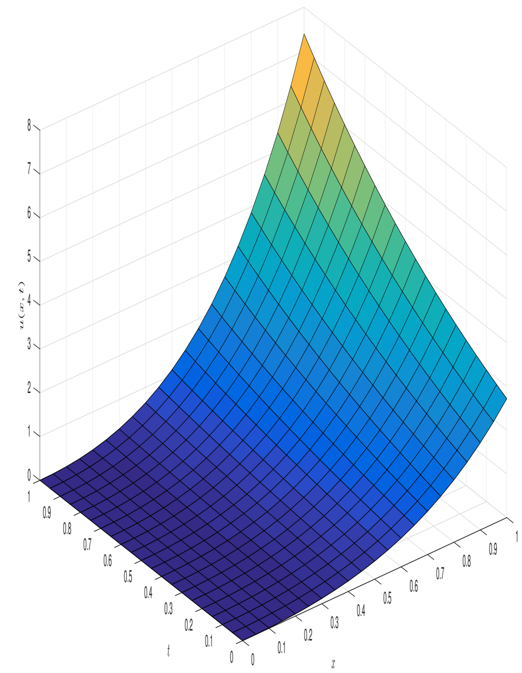



Figure 7: The graphics of approximate solution for $u(x,t)$ with $\alpha = 1$ and $\beta = 1$ in Ex. 2.

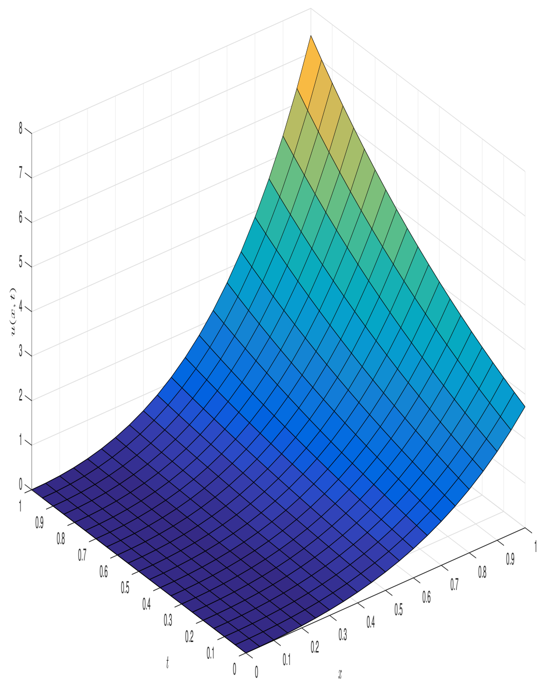
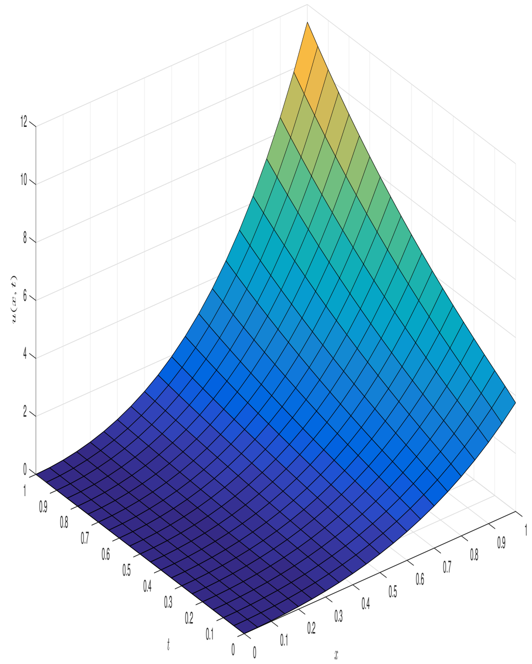


Figure 8: The graphics of approximate solution for $u(x,t)$ with $\alpha = 0.9$ and $\beta = 0.9$ in Ex. 2 .




*Example 3.* Consider the inverse coefficient problem involving space-time fractional differential equations [23, 24]:

$$D_t^{\alpha}u(x,t) = D_x^{\beta}(k(x)D_x^{\beta}u(x,t)) + (x^2 - 4x)exp(t),$$
$$0 < x < \beta^{\frac{1}{\beta}}, 0 < t < \alpha^{\frac{1}{\alpha}}, \tag{41}$$

$$u(x,0) = \frac{x^{2\beta}}{\beta^2}, 0 \leqslant x \leqslant \beta^{\frac{1}{\beta}}, \tag{42}$$

$$D_x^{\beta}u(0,t) = 0, 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \tag{43}$$

$$u(\beta^{\frac{1}{\beta}}, t) = exp(t), 0 < t \leqslant \alpha^{\frac{1}{\alpha}}, \tag{44}$$

$$u(x, \alpha^{\frac{1}{\alpha}}) = \frac{x^{2\beta}}{\beta^2}exp(1), 0 < x \leqslant \beta^{\frac{1}{\beta}}. \tag{45}$$

By taking fractional scaling transformation methods into account the problem (41)-(45) turns into following integer order problem:

$$V_T = (k(X)V_X)_X, 0 < X < 1, 0 < T < 1, \tag{46}$$

with initial conditions

$$V(X,0) = X^2, 0 < X \leqslant 1, \tag{47}$$

and the prescribed Dirichlet boundary conditions are

$$V_X(0,T) = 0, 0 < T \leqslant 1, \tag{48}$$

$$V(1,T) = exp(T), 0 < T \leqslant 1, \tag{49}$$

$$V(X,1) = X^2 exp(1), 0 < X \leqslant 1, \tag{50}$$

This inverse problem have the solution $V(X,T) = X^2 exp(T)$ and unknown diffusivity coefficient becomes $k(X) = X$ . As seen from Figs.9-12, by means of inverse transformation the solution of problem (34)-(38) and unknown diffusivity coefficient are obtained in the following form respectively

$$u(x,t) = \frac{x^{2\beta}}{\beta^2} exp(\frac{t^\alpha}{\alpha}) \tag{51}$$

and

$$k(x) = \frac{x^\beta}{\beta}. \tag{52}$$

Moreover, the values of exact and approximate solutions of problem (41)-(45) at $t = 0.8$ for different values of orders of $\alpha$ and $\beta$ are presented in Table 3.

Table 3: The table of exact and approximate solutions of Ex. 3 at $t = 0.8$.

| $x$ | Exact | $\alpha = 1$ $\beta = 1$ | $\alpha = 0.9$ $\beta = 1$ | $\alpha = 0.8$ $\beta = 1$ | $\alpha = 1$ $\beta = 0.9$ | $\alpha = 0.9$ $\beta = 0.9$ | $\alpha = 0.8$ $\beta = 0.9$ |
|---|---|---|---|---|---|---|---|
| 0.2 | 0.08902 | 0.08902 | 0.09927 | 0.11381 | 0.15164 | 0.16909 | 0.19386 |
| 0.4 | 0.35609 | 0.35609 | 0.39707 | 0.45523 | 0.52803 | 0.58881 | 0.67505 |
| 0.6 | 0.80119 | 0.80119 | 0.89341 | 1.02428 | 1.09553 | 1.22162 | 1.40056 |
| 0.8 | 1.42435 | 1.42435 | 1.58829 | 1.82094 | 1.83871 | 2.05035 | 2.35067 |
| 1 | 2.22554 | 2.22554 | 2.48171 | 2.84522 | 2.74758 | 2.05035 | 3.51261 |

Figure 9: The graphics of exact and approximate solution for $k(x)$ in Ex. 3 .

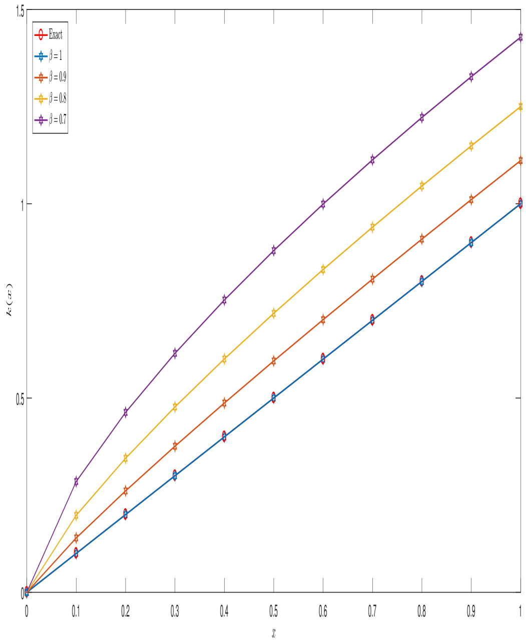



Figure 10: The graphics of exact solution for $u(x,t)$ in Ex. 3 .

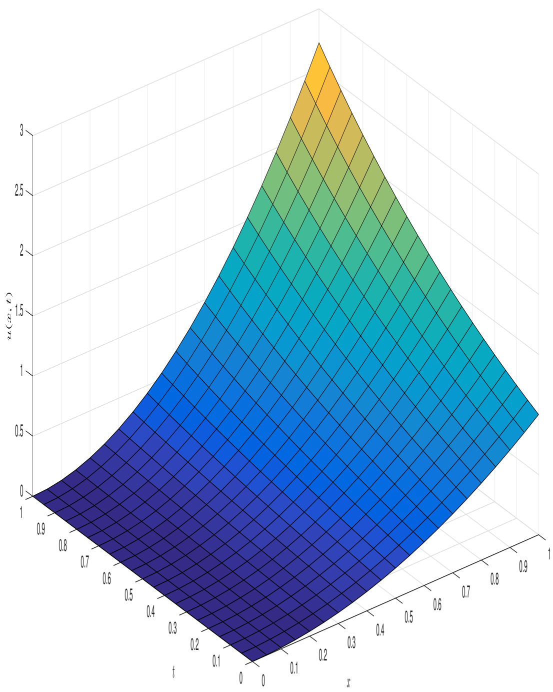


Figure 11: The graphics of approximate solution for $u(x,t)$ with $\alpha = 1$ and $\beta = 1$ in Ex. 3 .

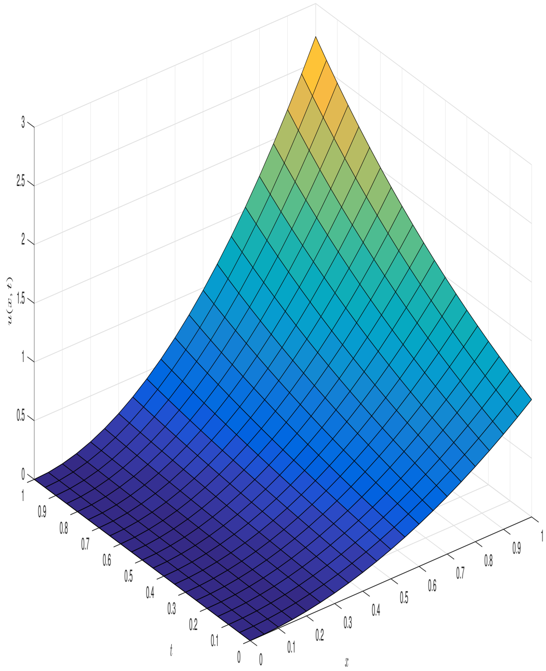
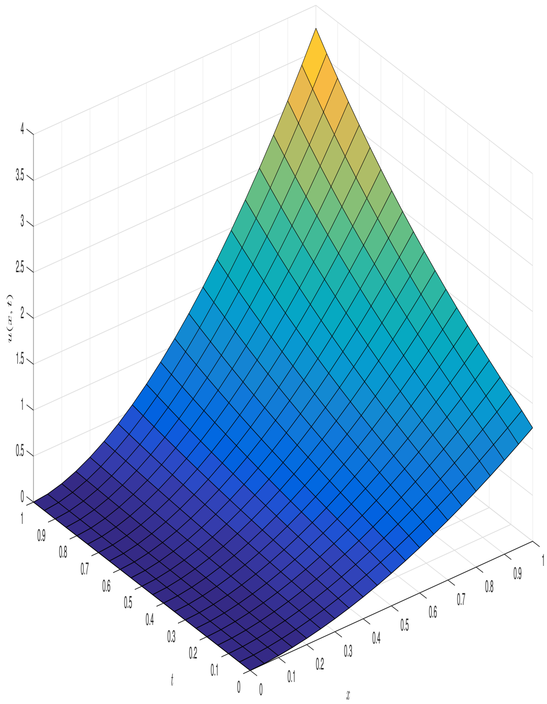



Figure 12: The graphics of approximate solution for $u(x,t)$ with $\beta = 0.9$ in Ex. 3 .



## 5. Conclusion

In this study, we tackle with the challenge of constructing the solution and unknown space dependent coefficient of space-time fractional diffusion equations by utilizing fractional scaling transformation method. This method enable us to reduce the problem into integer order inverse problem which gives us the opportunity to cope with easier problem. Then, taking the inverse transformation into account, the solution and the unknown coefficient are recovered. The outcomes illustrate that this method works better for the fractional problems in the sense of conformable fractional derivative. Future work will be concerned with the construction of the unknown parameter in space-time fractional differential equations with various boundary conditions.

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