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Research Article

A Survey on Timelike-Spacelike Involute-Evolute Curve Pair

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Abstract

This paper concentrates on the requirements of being an integral curve for the geodesic spray of the natural lift curves (n.l.c) of spherical indicatrices of the timelike-spacelike involute-evolute curve pair in Lorentz 3-space. In addition, the obtained results were supported by one example.

Keywords: Involute-evolute curve pair, Lorentz space, geodesic spray, spherical indicatrix, natural lift curve.

Zamansı-Uzaysı İnvolüt-Evolüt Eğri Çifti Üzerine Bir Araştırma

Öz

Bu makale, Lorentz 3-uzayında zamansı-uzaysı involüt-evolüt eğri çiftinin küresel göstergelerinin tabii lift eğrilerinin jeodezik spray için bir integral eğri olma gerekliliklerine odaklanır. Ayrıca elde edilen sonuçlar bir örnekle desteklenmiştir.

Anahtar Kelimeler: İnvolüt-evolüt eğri çifti, Lorentz uzayı, jeodezik spray, küresel gösterge, tabii lift eğrisi

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1. Introduction

One of the exciting curves derived with the help of a curve is the involute curve. C. Huygens discovered the involute curve for the first time in 1673. The curve whose tangent vectors form right angles at each point of a given curve is called the involute curve, and these two curves are called the involute-evolute curve pair. For example, the involute of a circle is a spiral. In particular, a circle and its involute are significant in gear technology. For the characteristic properties and the more detailed information on the involute-evolute curve pair in classical differential geometry, see [1-3]. This article will focus more on n.l.c and geodesic sprays. See [4] for basic information on these concepts. In [5], authors have examined these concepts in the three dimensional space of Euclidean geometry. In [6], Bilici et al. generalized this problem to spherical indicatrices of the involute-evolute curve pair. Later, the authors adapted this problem for a non-null curves in the three dimensional Lorentz spaces [7, 8]. Moreover, there are also many studies on spherical indicatrix curves and the n.l.c in Euclidean space and Lorentz space [9-15]. Recently, authors have found the relations between the Frenet vectors of the curve pair in the Lorentz 3-space [16]. These relationships were the source of inspiration for this study. In this sense, in the present paper, firstly, we defined the spherical indicatrices of the involute curve on the hyperbolic unit sphere and Lorentzian unit sphere. Then, we investigated the n.l.c and geodesic sprays for these indicatrices in Lorentz 3-space, and obtained some significant results.

2. Preliminaries

The space E^3 with the metric tensor

$$\langle A,B\rangle = -a_1b_1 + a_2b_2 + a_3b_3$$

is called Lorentz 3-spaces and denoted by E_1^3. There can be three states for a vector $A \in E_1^3$:

$$\begin{cases} \textit{spacelike}, \langle A,A\rangle > 0 \ \textit{or}\ A = 0 \\ \textit{timelike}, \langle A,A\rangle < 0 \\ \textit{null(lightlike)}, \langle A,A\rangle = 0 \ \textit{and}\ A \neq 0 \end{cases}$$

Similarly, a curve $\gamma = \gamma(s)$ in E_1^3 can be spacelike, timelike or null, if all of its velocity vectors $\gamma'(s)$ are spacelike, timelike or null respectively, [17]. The cross product of two vectors can be defined as follows:

$$A \times B = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_1b_2 - a_2b_1).$$

Let denote the moving Frenet frame along the curve γ by $\{E_1, E_2, E_3\}$. In this trihedron, E_1 and E_3 are spacelike vectors, E_2 is timelike vector. From [18], definition of Lorentzian cross product we can write

$$E_1 \times E_2 = -E_3,\ E_2 \times E_3 = -E_1,\ E_3 \times E_1 = E_2.$$

Then the following Frenet formulas hold between these three vectors [19]:

$$\begin{cases} E_1' = p\,E_2, \\ E_2' = p\,E_1 + q\,E_3\,, \\ E_3' = q\,E_2. \end{cases}$$

The Darboux vector is expressed by

$$\omega = -q\,E_1 + pE_3\,,$$

where $p = \|\omega\| cos\,\theta$, $q = \|\omega\| sin\,\theta$.

On the other hand, from [16] the unit Darboux vector C can be given as follows

$$C = \frac{\omega}{\|\omega\|} = -sin\,\theta\,E_1 + cos\,\theta\,E_3\,.$$

Theorem 1. γ is a general helix if and only if $f = \dfrac{q}{p} = \tan\theta = \text{constant}.$

We can generalize the conception of integral curve to the hypersurface $M \subset E_1^3$ easily as follows:

A smooth curve $\gamma : I \to M$ is an integral curve of a smooth vector field $W \in \chi(M)$ if

$$\gamma'(t) = W(\gamma(t)) \tag{1}$$

for all $t \in I$, [18].

For a curve γ, the parametrized curve $\gamma : I \to TM$ is defined as the n.l.c given by the equation

$$\gamma(t) = (\gamma(t), \gamma'(t)) = \gamma'(t)\big|_{\gamma(t)}\,, \tag{2}$$

where $TM = \bigcup_{m \in M} T_m(M)$ is the set of all tangent vectors to *M*.

The Lorentzian and hyperbolic unit sphere of radius 1 and center 0 in E_1^3 are given by

$$S_1^2\left(H_0^2\right) = \left\{A = \left(a_1, a_2, a_3\right) \in E_1^3 : \langle A, A\rangle = 1\ \left(\langle A, A\rangle = -1\right)\right\}$$

respectively, [18].

For $u \in TM$, the smooth vector field $W \in \chi(TM)$ defined by

$$W(u) = \varepsilon\langle u, S(u)\rangle\,\xi\big|_{\gamma(t)} \tag{3}$$

is called the geodesic spray on *TM*.

We know from [7] that "The n.l.c γ of the curve γ is an integral curve of the geodesic spray *W* if and only if γ is a geodesic on *M* ".

Theorem 2. Let (η, γ) be the involute-evolute curve pair. The following equality [16] gives the Frenet vectors of the curve pair:

$$\begin{cases} T^* = E_2 \\ N^* = -cos\,\theta E_1 - sin\,\theta E_3\,. \\ B^* = -sin\,\theta E_1 - cos\,\theta E_3 \end{cases} \tag{4}$$

3. Main Theorems and Proofs

In this section, we investigate the n.l.c η_{T^*} , η_{N^*} , η_{B^*} derived from spherical indicatrices of the involute curve η in E_1^3. For this purpose, we seek an answer to the following question. What type of curve must the evolute curve γ be for the n.l.c of the spherical indicatrix curve to be an integral curve for geodesic spray?

3.1. The curve η_{T^*} for the first indicatrix of η

Let η_{T^*} be n.l.c of the tangent indicatrix $\eta_{T^*} = T^*$. If η_{T^*} is an integral curve of the geodesic spray, then by means of Lemma 1. we can write

$$\overline{\overline{D}}_{\eta'_{T^*}} \eta'_{T^*} = 0, \tag{5}$$

where $\overline{\overline{D}} = \dfrac{d}{ds_{T^*}}$ is the differential operator of H_0^2. Thus from Theorem 2. and (5) we have

$$-\frac{\theta'}{\|\omega\|} sin\theta\, E_1 + \frac{\theta'}{\|\omega\|} cos\theta\, E_3 = 0.$$

Because of $\{ E_1, E_2, E_3 \}$ are linear independent, we get

$$\theta = \text{constant.}$$

Thus, from Theorem 1. we obtain

$$f = \text{constant.}$$

Result 3.1.1. If the curve γ is a general helix, then η_{T^*} is a geodesic on H_0^2.

3.2. The curve η_{N^*} for the second indicatrix of η

Let η_{N^*} be the n.l.c of the principal normal indicatrix $\eta_{N^*} = N^*$. If η_{N^*} is an integral curve of the geodesic spray, then from Lemma1. we can write

$$\overline{D}_{\eta'_{N^*}} \eta'_{N^*} = 0. \tag{6}$$

From the Theorem 2. and (6) we get,

$$\overline{D}_{\eta'_{N^*}} \eta'_{N^*} = \Bigg[(-\sigma' sin\theta - \sigma\theta' cos\theta + \frac{\kappa}{k_N} + \|\omega\| k_N \, cos\theta) E_1 + (-\frac{k_N'}{{k_N}^2}) E_2$$
$$+ (\sigma' cos\theta - \theta'\sigma\, sin\theta + \frac{\tau}{k_N} - \|\omega\| k_N \, sinh\theta) E_3 \Bigg] \frac{1}{\|\omega\| k_N},$$

where $\overline{D} = \frac{d}{ds_{N^*}}$ is the differential operator of S_1^2 and $\sigma = \frac{g_N}{k_N}$. From [20], $g_N = \frac{\theta'}{\|\omega\|}$ and $k_N = \frac{1}{\|\omega\|}\sqrt{\left|\theta'^2 - \|\omega\|^2\right|}$ are the geodesic curvatures of the principal normal indicatrix $\alpha_N = N$ with respect to S_1^2 and E_1^3, respectively. Since $\{E_1, E_2, E_3\}$ are linear independent, we can write

$$\begin{cases} \sigma' \sin\theta + \theta'\sigma \cos\theta - \dfrac{\kappa}{k_N} - \|\omega\| k_N \cos\theta = 0, \\ \dfrac{k'_N}{k_N{}^2} = 0, \\ \sigma' \cos\theta - \theta'\sigma \sin\theta + \dfrac{\tau}{k_N} + \|\omega\| k_N \sin\theta = 0, \end{cases}$$

then we have

$$k_N = \text{constant}, \ g_N = \text{constant}.$$

Result 3.2.1. If $k_N = \text{constant}$, $g_N = \text{constant}$, then η_{N^*} is a geodesic line on S_1^2.

3.3. The curve η_{B^*} for the third indicatrix of η

Let η_{B^*} be the n.l.c of the binormal indicatrix $\eta_{B^*} = B^*$. If η_{B^*} is an integral curve of the geodesic spray, then we can write

$$\overline{D}_{\eta'_{B^*}} \eta'_{B^*} = 0,$$

that is,

$$\frac{\|\omega\|}{\theta'} E_2 = 0.$$

Since $\{E_1, E_2, E_3\}$ are linear independent, we have

$$\|\omega\| = 0.$$

Thus we get

$$\kappa = 0, \ \tau = 0.$$

Thus, we can say that γ is a line and know its involute is a circle segment. Thus we can give the following result.

Result 3.3.1. If the curve γ is a line, then η_{B^*} is a geodesic on S_1^2.

Example 3.3.2. Let $\gamma(s)=\left(\frac{\sqrt{3}}{2}\cosh s,\frac{\sqrt{3}}{2}\sinh s,\frac{s}{2}\right)$ be a unit speed spacelike helix. For the curve γ we obtain

$$\begin{cases} E_1=\left(\frac{\sqrt{3}}{2}\sinh s,\frac{\sqrt{3}}{2}\cosh s,\frac{1}{2}\right) \\ E_2=\left(\cosh s,\sinh s,0\right) \\ E_3=\left(\frac{1}{2}\sinh s,\frac{1}{2}\cosh s,\frac{\sqrt{3}}{2}\right) \end{cases}, \quad \kappa=\frac{\sqrt{3}}{2},\ \tau=\frac{1}{2}.$$

Then we get the timelike involute curve η of γ such that

$$\eta(s)=\left(\frac{\sqrt{3}}{2}\left(\cosh s+(\sigma-s)\sinh s\right),\frac{\sqrt{3}}{2}\left(\sinh s+(\sigma-s)\cosh s\right),\frac{\sigma}{2}\right),$$

The following figures show the spacelike evolute curve γ (Fig. 1) and timelike involute curve η (Fig. 2) for $\sigma=2$ and $s\in[-5,5]$.

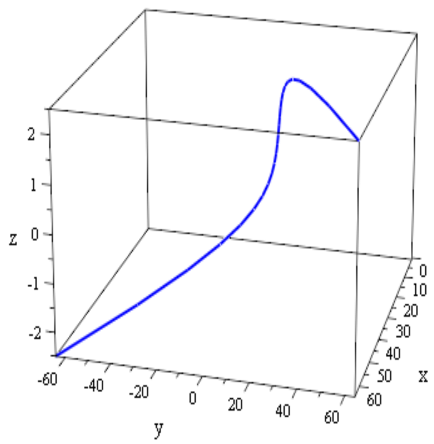

Figure 1. Evolute curve γ

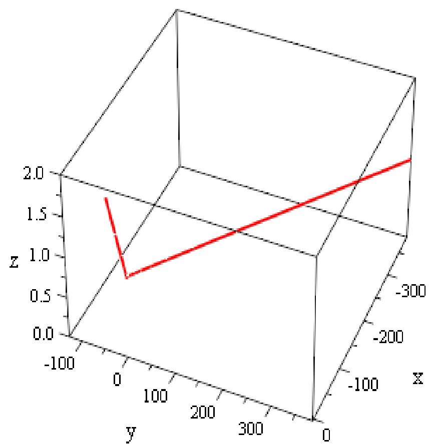

Figure 2. Involute curve η

Then we can give the spherical indicatrices and the n.l.c of η as follow:

$$\begin{cases} \eta_{T^*}=\left(\cosh s,\sinh s,0\right), \\ \eta_{N^*}=\left(\sinh s,\cosh s,0\right), \\ \eta_{B^*}=\left(0,0,1\right), \end{cases} \qquad \begin{cases} \eta_{T^*}=\left(\sinh s,\cosh s,0\right), \\ \eta_{N^*}=\left(\cosh s,\sinh s,0\right), \\ \eta_{B^*}=\left(0,0,0\right). \end{cases}$$

Since $\left\langle \eta_{T^*}{}', \eta_{T^*}{}' \right\rangle = 1$, η_{T^*} is a spacelike curve. For being η_{T^*} is a spacelike, its tangent indicatrix which is a geodesic line on H_0^2 and the n.l.c of this spherical curve are as Fig.3. On the other hand, the normal indicatrix which is a geodesic on S_1^2 and its n.l.c are as Fig.4. It is also obvious that the third spherical indicatrix (binormal indicatrix) and its n.l.c is a point. Furthermore, we can say that (η, γ) is timelike-spacelike involute-evolute curve pair, and the condition $g(E_1, T^*) = 0$ is satisfied.

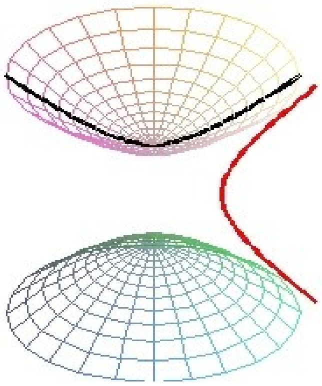

Figure 3. Tangent indicatrix curve η_{T^*}

and its n.l.c η_{T^*} (red color)

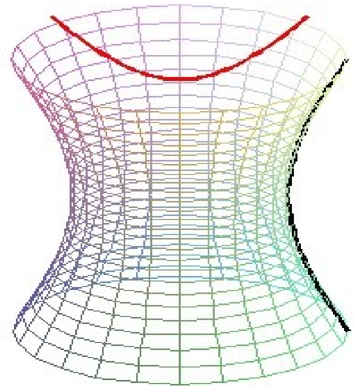

Figure 4. Normal indicatrix curve η_{N^*}

and its n.l.c η_{N^*} (red color)

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

References

[1] Millman, R. S., Parker, G. D., (1977) Elements of Differential Geometry, Prentice-Hall Inc., Englewood Cliffs Press, New Jersey.

[2] Hacısalihoğlu H. H., (2000) Differantial Geometry, Ankara University Faculty of Science Press, Ankara.

[3] Çalışkan, M., Bilici, M., (2002) Some characterizations for the pair of involute-evolute curves in Euclidean space E^3, *Bulletin of Pure and Applied Sciences,* 21(2), pp. 289-294.

[4] Thorpe, J. A., (1979) Elemantary Topics In Differantial Geometry, Springer-Verlag, New York, Heidelberg-Berlin.

[5] Çalışkan, M., Sivridağ, A.İ., (1984) Hacısalihoğlu. H. H., Some characterizations for the natural lift curves and the geodesic spray, *Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics,* 33, pp. 235-242.

[6] Bilici, M., Çalışkan, M., Aydemir, İ., (2003) The natural lift curves and the geodesic sprays for the spherical indicatrices of the pair of evolute-involute curves, *International Journal of Applied Mathematics,* 11(4), pp. 415-420.

[7] Bilici, M., (2011) Natural lift curves and the geodesic sprays for the spherical indicatrices of the involutes of a timelike curve in Minkowski 3- space. *International Journal of the Physical Sciences,* 6(20), pp. 4706-4711.

[8] Bilici, M., Ahmad, T. A., (2017) On the natural lift curves for the involute spherical indicatrices in Minkowski 3-space. *Malaya Journal of Matematik,* 5(2), pp. 407-415.

[9] Şenyurt, S., (2012) Natural lifts and the geodesic sprays for the spherical indicatrices of the Mannheim partner curves in E^3, *International Journal of the Physical Sciences,* 7(16), pp. 2414-2421.

[10] Ali, A.T., (2012) New special curves and their spherical indicatrix, *Global Journal of Advanced Research Clasical and Modern Geometries,* 1(2), pp. 28-38.

[11] İyigün, E, (2013) The tangent spherical image and ccr-curve of a time-like curve in L^3, *Jornal of Inequalities and Applications,* 55, pp. 1-5.

[12] Bilici, M., Ergün, E., Çalişkan, M., (2014) The natural lift curves for the spherical indicatrices of spacelike Bertrand couple in Minkowski 3-space, *Journal of Mathematical and Computational Science* 4(1), pp. 85-104.

[13] Bilici, M., Çalışkan, M., (2014) Some geometrical calculations for the spherical indicatrices of involutes of a timelike curve in Minkowski 3-Space, *Journal of Advances in Mathematıcs* 5(2), pp. 668-677.

[14] Bilici, M., Ergün, E., Çalıskan, M., (2015) A new approach to natural lift curves of the spherical indicatrices of timelike Bertrand mate of a spacelike curve in Minkowski 3-space, *International Journal of Mathematical Combinatorics,* 1, pp. 35-48.

[15] Bilici, M., Çalışkan, M., (2019) Some new results on the curvatures of the spherical indicatrices of the involutes of a spacelike curve with a spacelike binormal in Minkowski 3-space, *MathLAB Journal,* 2(1), pp. 110-119.

[16] Bilici, M., Çalışkan, (2018) M., A new perspective on the involutes of the spacelike curve with a spacelıke binormal in Minkowskı 3-space**,** *Journal of Science and Arts,* 18(3), 573-582.

[17] Ali, A. T., Lopez, R., (2011) Slant helices in Minkowski space E_1^3. *Journal of the Korean Mathematical Society*, 48(1), pp. 159–167.

[18] O'neill, B., (1983) Semi Riemann Geometry, Academic Press, New York, London.

[19] Woestijne, V. D. I., (1990) Minimal surfaces of the 3-dimensional Minkowski space, Word Scientific Publishing. Singapore, pp 344-369.

[20] Bilici, M., (2009) On the timelike or spacelike involute-evolute curve couples, Ph.D. thesis, Ondokuz Mayıs University, Samsun, Turkey.