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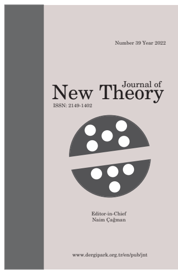

Distribution Formulae of the Solute in Transport of Advection-Dispersion of Air Pollution for Different Wind Velocities and Dispersion Coefficients

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Abstract — In this paper, we obtain certain distribution formulae of the solute in transport of the typical advection-dispersion of air pollution through separation in two dimensional space variables by introducing different wind velocities and dispersion coefficients. As a consequence, by introducing different values of the solute velocity and dispersion coefficients, we evaluate the solute distribution formulae of the air pollution in terms of various known and unknown special functions.

Keywords — *Transport of advection-dispersion problems, air pollution, distribution formulae of the solute, wind velocities, dispersion coefficients, special functions*

Mathematics Subject Classification (2020) — 35G61, 33C90

1. Introduction

The solute transport is described by the advection-dispersion equation (in short ADE) (see for example [1])

$$\frac{\partial C}{\partial t} + \mathcal{U}\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} \tag{1}$$

where, C is solute concentration distribution, the positive constants $\mathcal{U}$ represent the average fluid (wind) velocity; D, the dispersion coefficient; x, the spatial domain and t is time. The ADE is a deterministic equation describing a probability function for the location of particles in a continuum. The fundamental solutions of the ADE over time t have studied in the Gaussian densities with means and variances based on the values of the macroscopic transport coefficients $\mathcal{U}$ and D.

The extension of the Eqn. (1) is presented in the typical advection-dispersion vector equation as

$$\frac{\partial C}{\partial t} + div(C\mathcal{U}) = div(D\nabla C) + F \tag{2}$$

Here, the Eqn. (2) consists the scalar quantities C, D, and F, such that $D \neq 0$ and $\mathcal{U}$, a vector quantity.

We refer the principles of air pollution meteorology described in the researches [2–5]. Liu et al. [6,7] presented various computational methods for solute transport in the advection-dispersion problems.

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The study of wind speed conditions is of interest, partly because the simulation of airborne pollutant dispersion in certain conditions is rather difficult.

In our paper, we determine the distribution formulae of the solute transport by the typical advection-dispersion of air pollution problem (2) through separation in two dimensional space variables. We evaluate the solute distribution formulae of the air pollution in terms of Gauss and confluent hypergeometric functions by introducing different values of the solute velocity and dispersion coefficients.

2. Theory and Methods of Solute Distribution in Advection-dispersion Equation by Separate Variables

In this section, we plug the Eqn. (2) via the theory and methods of separation in two dimensional space variables stated on the basis of the researches done in [8–11].

We suppose that, $\forall x, y \in \mathbb{R}$, the solute concentration distribution $C = C(x, y, t)$, the wind velocity $U = u(x, y, t)i + v(x, y, t)j$; i and j are unit vectors; $u(x, y, t)$ and $v(x, y, t)$ are scalar quantities; the dispersion coefficient $D = D_1(x)D_2(y), D_1(x) \neq 0, D_2(y) \neq 0, \forall x \in \mathbb{R}, y \in \mathbb{R}$, and the scalar quantity

$$F = F(x, y, t), \quad \lim_{t\to 0^+} C(x, y, t) = f(x, y), \quad \lim_{t\to\infty} C(x, y, t) = h(x, y), \quad \nabla \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}$$

Also, the concentration distribution $C(x, y, t)$ exists and have non - zero values for $\forall x \in \mathbb{R}, y \in \mathbb{R}, t \geq 0$, and does not exist when $t < 0$.

By above assumptions, we convert the Eqn. (2) in the typical two variables advection-dispersion equation given by

$$\begin{aligned}\frac{\partial C(x,y,t)}{\partial t} &+ \frac{\partial}{\partial x}(C(x,y,t)u(x,y,t)) + \frac{\partial}{\partial y}(C(x,y,t)v(x,y,t)) \\ &= D_2(y)\frac{\partial}{\partial x}(D_1(x)\frac{\partial}{\partial x}C(x,y,t)) + D_1(x)\frac{\partial}{\partial y}(D_2(y)\frac{\partial}{\partial y}C(x,y,t)) + F(x,y,t) \quad (3)\end{aligned}$$

Theorem 2.1. If $u(x, y, t)$ and $v(x, y, t)$ are velocity components along unit vectors i and j $\forall x \in \mathbb{R}, y \in \mathbb{R}, t \geq 0$, and $C(x, y, t) = C_1(x, t)C_2(y, t)$, where, $C_1(x, t) \neq 0, C_2(y, t) \neq 0$ and $F(x, y, t) = f_1(x, t)C_2(y, t) + f_2(y, t)C_1(x, t)$, $\forall x \in \mathbb{R}, y \in \mathbb{R}, t \geq 0$, then by the Eqn. (3), there exists following separate differential equations with variable coefficients

$$\begin{aligned}D_2(y)\frac{\partial^2}{\partial y^2}C_2(y,t) &+ \left\{\frac{\partial}{\partial y}D_2(y) - \frac{v(x,y,t)}{D_1(x)}\right\}\frac{\partial}{\partial y}C_2(y,t) + \frac{f_2(y,t)}{D_1(x)} \\ &- \frac{1}{D_1(x)}\frac{\partial C_2(y,t)}{\partial t} - \frac{\frac{\partial}{\partial y}v(x,y,t)}{D_1(x)}C_2(y,t) = 0 \quad (4)\end{aligned}$$

and

$$\begin{aligned}D_1(x)\frac{\partial^2}{\partial x^2}C_1(x,t) &+ \left\{\frac{\partial}{\partial x}D_1(x) - \frac{u(x,y,t)}{D_2(y)}\right\}\frac{\partial}{\partial x}C_1(x,t) + \frac{f_1(x,t)}{D_2(y)} \\ &- \frac{1}{D_2(y)}\frac{\partial C_1(x,t)}{\partial t} - \frac{\frac{\partial}{\partial x}u(x,y,t)}{D_2(y)}C_1(x,t) = 0 \quad (5)\end{aligned}$$

Proof. Consider the Eqn. (3) and set

$$u(x,y,t) = u_1(x,t)u_2(y,t), v(x,y,t) = u_3(x,t)u_4(y,t) \quad (6)$$

Then, under the conditions given in the Theorem 2.1 and in Eqn. (6), the Eqn. (3) becomes as

$$\begin{aligned}C_1(x,t)\frac{\partial C_2(y,t)}{\partial t} + C_2(y,t)\frac{\partial C_1(x,t)}{\partial t} &+ u_2(y,t)C_2(y,t)\frac{\partial}{\partial x}(C_1(x,t)u_1(x,t)) \\ &+ C_1(x,t)u_3(x,t)\frac{\partial}{\partial y}(C_2(y,t)u_4(y,t)) \\ = C_2(y,t)D_2(y)\frac{\partial}{\partial x}(D_1(x)\frac{\partial}{\partial x}C_1(x,t)) &+ C_1(x,t)D_1(x)\frac{\partial}{\partial y}(D_2(y)\frac{\partial}{\partial y}C_2(y,t)) \\ &+ f_1(x,t)C_2(y,t) + f_2(y,t)C_1(x,t)\end{aligned} \quad (7)$$

Again, we write the Eqn. (7) in the form

$$\begin{aligned}C_1(x,t)\Big[\frac{\partial C_2(y,t)}{\partial t} + u_3(x,t)\frac{\partial}{\partial y}(C_2(y,t)u_4(y,t)) - D_1(x)\frac{\partial}{\partial y}(D_2(y)\frac{\partial}{\partial y}C_2(y,t)) - f_2(y,t)\Big]+ \\ C_2(y,t)\Big[\frac{\partial C_1(x,t)}{\partial t} + u_2(y,t)\frac{\partial}{\partial x}(C_1(x,t)u_1(x,t)) - D_2(y)\frac{\partial}{\partial x}(D_1(x)\frac{\partial}{\partial x}C_1(x,t)) - f_1(x,t)\Big] = 0\end{aligned} \quad (8)$$

Since in Eqn. (8) $C_1(x,t) \neq 0$ and $C_2(y,t) \neq 0$, then $\forall\ x,y \in \mathbb{R}, t \geq 0$, here the equality holds if following equations satisfy

$$\frac{\partial C_2(y,t)}{\partial t} + u_3(x,t)\frac{\partial}{\partial y}(C_2(y,t)u_4(y,t)) - D_1(x)\frac{\partial}{\partial y}(D_2(y)\frac{\partial}{\partial y}C_2(y,t)) - f_2(y,t) = 0 \quad (9)$$

and

$$\frac{\partial C_1(x,t)}{\partial t} + u_2(y,t)\frac{\partial}{\partial x}(C_1(x,t)u_1(x,t)) - D_2(y)\frac{\partial}{\partial x}(D_1(x)\frac{\partial}{\partial x}C_1(x,t)) - f_1(x,t) = 0 \quad (10)$$

By the Eqn. (9), we obtain

$$\begin{aligned}\frac{\partial C_2(y,t)}{\partial t} &+ u_3(x,t)\Big\{C_2(y,t)\frac{\partial}{\partial y}u_4(y,t) + u_4(y,t)\frac{\partial}{\partial y}C_2(y,t)\Big\} \\ &- D_1(x)\Big\{D_2(y)\frac{\partial^2}{\partial y^2}C_2(y,t) + \frac{\partial}{\partial y}D_2(y)\frac{\partial}{\partial y}C_2(y,t)\Big\} - f_2(y,t) = 0,\ x,y \in \mathbb{R}, t \geq 0\end{aligned} \quad (11)$$

Then, for $x,y \in \mathbb{R}, t \geq 0$, by Eqn. (11) we find

$$\begin{aligned}\frac{\partial C_2(y,t)}{\partial t} &= D_1(x)D_2(y)\frac{\partial^2}{\partial y^2}C_2(y,t) + \Big\{D_1(x)\frac{\partial}{\partial y}D_2(y) - u_3(x,t)u_4(y,t)\Big\}\frac{\partial}{\partial y}C_2(y,t) \\ &\quad - u_3(x,t)\frac{\partial}{\partial y}u_4(y,t)C_2(y,t) + f_2(y,t)\end{aligned} \quad (12)$$

Further in a similar manner, $\forall x,y \in \mathbb{R}, t \geq 0$, by Eqn. (10) we find

$$\begin{aligned}\frac{\partial C_1(x,t)}{\partial t} &= D_2(y)D_1(x)\frac{\partial^2}{\partial x^2}C_1(x,t) + \Big\{D_2(y)\frac{\partial}{\partial x}D_1(x) - u_1(x,t)u_2(y,t)\Big\}\frac{\partial}{\partial x}C_1(x,t) \\ &\quad - u_2(y,t)\frac{\partial}{\partial x}u_1(x,t)C_1(x,t) + f_1(x,t)\end{aligned} \quad (13)$$

Note that $\forall x,y \in \mathbb{R}, t \geq 0$ the Eqns. (12) and (13) may be written as

$$\begin{aligned}\frac{1}{D_1(x)}\frac{\partial C_2(y,t)}{\partial t} &= D_2(y)\frac{\partial^2}{\partial y^2}C_2(y,t) + \Big\{\frac{\partial}{\partial y}D_2(y) - \frac{v(x,y,t)}{D_1(x)}\Big\}\frac{\partial}{\partial y}C_2(y,t) \\ &\quad - \frac{\frac{\partial}{\partial y}v(x,y,t)}{D_1(x)}C_2(y,t) + \frac{f_2(y,t)}{D_1(x)}\end{aligned} \quad (14)$$

and

$$\frac{1}{D_2(y)}\frac{\partial C_1(x,t)}{\partial t} = D_1(x)\frac{\partial^2}{\partial x^2}C_1(x,t) + \left\{\frac{\partial}{\partial x}D_1(x) - \frac{u(x,y,t)}{D_2(y)}\right\}\frac{\partial}{\partial x}C_1(x,t) - \frac{\frac{\partial}{\partial x}u(x,y,t)}{D_2(y)}C_1(x,t) + \frac{f_1(x,t)}{D_2(y)} \quad (15)$$

Finally, by the Eqns. (14) and (15) we obtain the Eqns. (4) and (5), respectively. □

By the Eqns. (4) and (5), we may obtain various distribution formulae of the solute in the transport of advection-dispersion of air pollution on setting different wind velocities and dispersion coefficients.

3. Distribution Formulae of the Solute in Transport of Advection-dispersion of Air Pollution for Different Wind Velocities and Dispersion Coefficients Involving Special Functions

In this section, we determine the solute distribution formulae in terms of certain special functions whose contiguity and analytic properties are described in the literature of the authors [12, 13]. These special functions are then applied in computation process of the related formulae. We present following theorems for evaluation of our results:

Theorem 3.1. If $\forall x, y \in (0,1), t \geq 0, c_1, c_2 \neq 0, -1, -2, -3, \ldots,$, $D_1(x) = x(1-x)$, $D_2(y) = y(1-y)$, $v(x,y,t) = [1 - c_2 + (a_2 + b_2 - 1)y]\{x(1-x)\}$, and a partial differential equation is satisfied by

$$\frac{1}{C_2(y,t)}\left\{f_2(y,t) - \frac{\partial C_2(y,t)}{\partial t}\right\} = (a_2+b_2-1-a_2b_2)\{x(1-x)\},\ u(x,y,t) = [1-c_1+(a_1+b_1-1)x]\{y(1-y)\}$$

and another partial differential equation is satisfied by

$$\frac{1}{C_1(x,t)}\left\{f_1(x,t) - \frac{\partial C_1(x,t)}{\partial t}\right\} = (a_1 + b_1 - 1 - a_1b_1)\{y(1-y)\}$$

then, by the Eqns. (4) and (5) of the Theorem 2.1, they also satisfy the simultaneous differential equations

$$y(1-y)\frac{\partial^2}{\partial y^2}C_2(y,t) + \{c_2 - (a_2 + b_2 + 1)y\}\frac{\partial}{\partial y}C_2(y,t) - a_2b_2C_2(y,t) = 0 \quad (16)$$

and

$$x(1-x)\frac{\partial^2}{\partial x^2}C_1(x,t) + \{c_1 - (a_1 + b_1 + 1)x\}\frac{\partial}{\partial x}C_1(x,t) - a_1b_1C_1(x,t) = 0 \quad (17)$$

respectively.

Proof. Consider the Eqn. (4) in which by the statement of this Theorem 3.1, put $D_1(x) = x(1-x)$, $D_2(y) = y(1-y)$, $v(x,y,t) = [1 - c_2 + (a_2 + b_2 - 1)y]\{x(1-x)\}$ and set $\frac{1}{C_2(y,t)}\{f_2(y,t) - \frac{\partial C_2(y,t)}{\partial t}\} = (a_2 + b_2 - 1 - a_2b_2)\{x(1-x)\}$, we get the Eqn. (16).

Similarly, for the particular values $u(x,y,t) = [1 - c_1 + (a_1 + b_1 - 1)x]\{y(1-y)\}$, $\frac{1}{C_1(x,t)}\{f_1(x,t) - \frac{\partial C_1(x,t)}{\partial t}\} = (a_1 + b_1 - 1 - a_1b_1)\{y(1-y)\}$, from the Eqn. (5), we obtain the required Eqn. (17). □

Theorem 3.2. If $\forall x, y \in (0,1), t \geq 0$, in the relation $\frac{1}{C_1(x,t)}\left\{f_1(x,t) - \frac{\partial C_1(x,t)}{\partial t}\right\} = (a_1 + b_1 - 1 - a_1b_1)\{y(1-y)\}$, it is assumed that $\forall x, y$ such that $0 < x < 1, 0 < y < 1$, $C_1(x,t) = e^{-\alpha_1 t}H_1(x,y), \alpha_1 > 0$, then by Eqn. (17) of the Theorem 3.1, there exists a formula

$$C_1(x,t) = \exp[-(a_1 + b_1 - 1 - a_1b_1)\{y(1-y)\}t] \times \int_0^t \exp[(a_1 + b_1 - 1 - a_1b_1)\{y(1-y)\}\tau]f_1(x,\tau)d\tau + \mu_1\ {}_2F_1\left[\begin{matrix}a_1, b_1; \\ c_1;\end{matrix} x\right] \quad (18)$$

μ_1 is an arbitrary constant and ${}_2F_1$ is Gauss hypergeometric function (see [12,13]). Similarly, for the relation $\frac{1}{C_2(y,t)}\{f_2(y,t)-\frac{\partial C_2(y,t)}{\partial t}\} = (a_2+b_2-1-a_2b_2)\{x(1-x)\}$ and $C_2(y,t) = e^{-\beta_1 t}H_2(x,y), \beta_1 > 0$, there exists another formula

$$C_2(y,t) = \exp\Big[-(a_2+b_2-1-a_2b_2)\{x(1-x)\}t\Big] \times \int_0^t \exp\Big[(a_2+b_2-1-a_2b_2)\{x(1-x)\}\tau\Big] f_2(y,\tau)d\tau + \nu_1 \; {}_2F_1\left[\begin{matrix} a_2, b_2; \\ c_2; \end{matrix} y\right] \tag{19}$$

ν_1 is an arbitrary constant.

Proof. The relation of the Theorem 3.2 is written by the linear differential equation $\frac{\partial C_1(x,t)}{\partial t} + (a_1 + b_1 - 1 - a_1b_1)\{y(1-y)\}C_1(x,t) = f_1(x,t)$, so that its solution is found by

$$C_1(x,t) = \exp[-(a_1+b_1-1-a_1b_1)\{y(1-y)\}t] \times \int_0^t \exp[(a_1+b_1-1-a_1b_1)\{y(1-y)\}\tau] f_1(x,\tau)d\tau + \lambda_1(x,y) \tag{20}$$

Now in Eqn. (17) set $C_1(x,t) = e^{-\beta_1 t}H_1(x,y), \beta_1 > 0$, so that $C_1(x,0) = H_1(x,y)$, and then $\lambda_1(x,y) = H_1(x,y)$ and hence we get

$$C_1(x,t) = \exp[-(a_1+b_1-1-a_1b_1)\{y(1-y)\}t] \times \int_0^t \exp[(a_1+b_1-1-a_1b_1)\{y(1-y)\}\tau] f_1(x,\tau)d\tau + H_1(x,y) \tag{21}$$

Again, by the relation $C_1(x,t) = e^{-\beta_1 t}H_1(x,y), \beta_1 > 0$ and the Eqn. (17), we get $H_1(x,y) = \mu_1 \; {}_2F_1\left[\begin{matrix} a_1, b_1; \\ c_1; \end{matrix} x\right]$. Therefore, we obtain

$$C_1(x,t) = \exp[-(a_1+b_1-1-a_1b_1)\{y(1-y)\}t] \times \int_0^t \exp[(a_1+b_1-1-a_1b_1)\{y(1-y)\}\tau] f_1(x,\tau)d\tau + \mu_1 \; {}_2F_1\left[\begin{matrix} a_1, b_1; \\ c_1; \end{matrix} x\right] \tag{22}$$

Similarly, we have for $C_2(y,t) = e^{-\alpha_1 t}H_2(x,y), \alpha_1 > 0$, then by Eqn. (16) we get $H_2(x,y) = \nu_1 \; {}_2F_1\left[\begin{matrix} a_2, b_2; \\ c_2; \end{matrix} y\right]$ and by the relation $\frac{1}{C_2(y,t)}\left\{f_2(y,t) - \frac{\partial C_2(y,t)}{\partial t}\right\} = (a_2+b_2-1-a_2b_2)\{x(1-x)\}$, we get

$$C_2(y,t) = \exp[-(a_2+b_2-1-a_2b_2)\{x(1-x)\}t] \times \int_0^t \exp[(a_2+b_2-1-a_2b_2)\{x(1-x)\}\tau] f_2(y,\tau)d\tau + \nu_1 \; {}_2F_1\left[\begin{matrix} a_2, b_2; \\ c_2; \end{matrix} y\right] \tag{23}$$

□

Theorem 3.3. If $\forall x, y \in (0,1), t \geq 0$, all conditions of the Theorem 3.2 and 3.3 are satisfied, then there exists following distribution formula of the solute as

$$C(x,y,t) = G_1(x,y,t)G_2(x,y,t) + \nu_1 G_1(x,y,t) \; {}_2F_1\left[\begin{matrix} a_2, b_2; \\ c_2; \end{matrix} y\right] + \mu_1 G_2(x,y,t) \; {}_2F_1\left[\begin{matrix} a_1, b_1; \\ c_1; \end{matrix} x\right] + \nu_1\mu_1 \; {}_2F_1\left[\begin{matrix} a_1, b_1; \\ c_1; \end{matrix} x\right] {}_2F_1\left[\begin{matrix} a_2, b_2; \\ c_2; \end{matrix} y\right] \tag{24}$$

Here in (24), it is given that

$$G_1(x,y,t) = \{\exp[-(a_1+b_1-1-a_1b_1)\{y(1-y)\}t] \times \int_0^t \exp[(a_1+b_1-1-a_1b_1)\{y(1-y)\}\tau]f_1(x,\tau)d\tau \tag{25}$$

and

$$G_2(x,y,t) = \{\exp[-(a_2+b_2-1-a_2b_2)\{x(1-x)\}t] \times \int_0^t \exp[(a_2+b_2-1-a_2b_2)\{x(1-x)\}\tau]f_2(y,\tau)d\tau \tag{26}$$

PROOF. Apply the results of the Theorems 3.1 and 3.2 in the result $C(x,y,t) = C_1(x,t)C_2(y,t)$ of the Theorem 2.1 to find the result (21). □

Theorem 3.4. If $\forall x, y \in (0,1), t \geq 0, c_1, c_2 \neq 0, -1, -2, -3, \dots,$, $D_1(x) = x$, $D_2(y) = y$

$$v(x,y,t) = [1-(y-c_2)]x, \quad \frac{1}{C_2(y,t)}\left\{\frac{\partial C_2(y,t)}{\partial t} - f_2(y,t)\right\} = (a_2+1)x$$

$$u(x,y,t) = [1-(x-c_1)]y, \quad \frac{1}{C_1(x,t)}\left\{\frac{\partial C_1(x,t)}{\partial t} - f_1(x,t)\right\} = (a_1+1)y$$

then, by the Eqns. (4) and (5) of the Theorem 2.1, they also satisfy following differential equations

$$y\frac{\partial^2}{\partial y^2}C_2(y,t) + (c_2-y)\frac{\partial}{\partial y}C_2(y,t) - a_2C_2(y,t) = 0 \tag{27}$$

and

$$x\frac{\partial^2}{\partial x^2}C_1(x,t) + (c_1-x)\frac{\partial}{\partial x}C_1(x,t) - a_1C_1(x,t) = 0 \tag{28}$$

respectively.

PROOF. Consider the Eqn. (4) in which by the statement of this Theorem, put $D_1(x) = x$, $D_2(y) = y$, $v(x,y,t) = [1-(y-c_2)]x$, then $\frac{\frac{\partial}{\partial y}v(x,y,t)}{x} = -1$, and $\frac{1}{C_2(y,t)}\left\{\frac{\partial C_2(y,t)}{\partial t} - f_2(y,t)\right\} = (a_2+1)x$ to get the Eqn. (27) as

$$y\frac{\partial^2}{\partial y^2}C_2(y,t) + (c_2-y)\frac{\partial}{\partial y}C_2(y,t) - a_2C_2(y,t) = 0$$

Similarly, by the Eqn. (5) in which on putting $u(x,y,t) = [1-(x-c_1)]y$, to get $\frac{\frac{\partial}{\partial x}u(x,y,t)}{y} = -1$, $\frac{1}{C_1(x,t)}\left\{\frac{\partial C_1(x,t)}{\partial t} - f_1(x,t)\right\} = (a_1+1)y$, gives us the Eqn. (28). □

Theorem 3.5. If all the conditions of the Theorem 3.4 are satisfied and $\forall t \geq 0$, let

$$C_1(x,t) = e^{-\alpha_2 t}K_1(x,y) = e^{-\alpha_2 t}K_1(x)K_1(y) = e^{-\alpha_2 t}K_1(x)\,(\text{ for } K_1(y)=1)\,, \alpha_2 > 0;$$

$$C_2(y,t) = e^{-\beta_2 t}K_2(x,y) = e^{-\beta_2 t}K_2(x)K_2(y) = e^{-\beta_2 t}K_2(y)\,(\textbf{ for } K_2(x)=1)\,, \beta_2 > 0.$$

Then, there exists the formulae

$$C_1(x,t) = \exp\left[(a_1+1)\,yt\right]\int_0^t \exp\left[-(a_1+1)\,y\tau\right]f_1(x,\tau)d\tau + \mu_2\;{}_1F_1\left[\begin{array}{c} a_1; \\ c_1; \end{array}\; x\right] \tag{29}$$

and

$$C_2(y,t) = \exp\left[(a_2+1)\,xt\right]\int_0^t \exp\left[-(a_2+1)\,x\tau\right]f_2(y,\tau)d\tau + v_2\;{}_1F_1\left[\begin{array}{c} a_2; \\ c_2; \end{array}\; y\right] \tag{30}$$

Proof. Consider the assumptions of the Theorem 3.5 and make an appeal to the Eqns. (27) and (28) to get the confluent differential equations (see [12,13])

$$x\frac{d^2}{dx^2}K_1(x)+(c_1-x)\frac{d}{dx}K_1(x)-a_1K_1(x)=0 \text{ and } y\frac{d^2}{dy^2}K_2(y)+(c_2-y)\frac{d}{dy}K_2(y)-a_2K_2(y)=0$$

respectively. Then we have their respective solutions

$$K_1(x)=\mu_2\ {}_1F_1\left[\begin{matrix}a_1;\\c_1;\end{matrix}\ x\right] \text{ and } K_2(y)=\upsilon_2\ {}_1F_1\left[\begin{matrix}a_2;\\c_2;\end{matrix}\ y\right]$$

Again due to the conditions of the Theorem 3.4, we get the linear partial differential equations

$$\frac{\partial C_1(x,t)}{\partial t}-(a_1+1)\,yC_1(x,t)=f_1(x,t) \text{ and } \frac{\partial C_2(y,t)}{\partial t}-(a_2+1)\,xC_2(y,t)=f_2(y,t),$$

respectively. We obtain the solutions of these linear partial differential equations

$$\begin{aligned}C_1(x,t)&=\exp\left[(a_1+1)\,yt\right]\int_0^t\exp\left[-(a_1+1)\,y\tau\right]f_1(x,\tau)d\tau+K_1(x,y)\\&=\exp\left[(a_1+1)\,yt\right]\int_0^t\exp\left[-(a_1+1)\,y\tau\right]f_1(x,\tau)d\tau+K_1(x)\end{aligned}$$

and

$$\begin{aligned}C_2(y,t)&=\exp\left[(a_2+1)\,xt\right]\int_0^t\exp\left[-(a_2+1)\,x\tau\right]f_2(y,\tau)d\tau+K_2(x,y)\\&=\exp\left[(a_2+1)\,xt\right]\int_0^t\exp\left[-(a_2+1)\,x\tau\right]f_2(y,\tau)d\tau+K_2(y)\end{aligned}$$

respectively.

Finally introduce the values of $K_1(x)$ and $K_2(y)$ in above solutions, we evaluate the required results (29) and (30). □

Theorem 3.6. If $\forall x,y\in(0,1), t\geq 0, c_1,c_2\neq 0,-1,-2,-3,\ldots,$ all conditions of the Theorems 3.4 and 3.5 are satisfied. Then, by the relation of the Theorem 3.4 there exists then solute distribution in the form

$$\begin{aligned}C(x,y,t)=G_1'(x,y,t)G_2'(x,y,t)+\nu_2G_1'(x,y,t)\ {}_1F_1\left[\begin{matrix}a_2;\\c_2;\end{matrix}y\right]+\mu_2G_2'(x,y,t)\ {}_1F_1\left[\begin{matrix}a_1;\\c_1;\end{matrix}x\right]\\+\nu_2\mu_2\ {}_1F_1\left[\begin{matrix}a_1;\\c_1;\end{matrix}x\right]\ {}_1F_1\left[\begin{matrix}a_2;\\c_2;\end{matrix}y\right]\end{aligned}\tag{31}$$

where

$$G_1'(x,y,t)=\exp[(a_1+1)yt]\int_0^t\exp[-(a_1+1)y\tau]f_1(x,\tau)d\tau+\mu_2\ {}_1F_1\left[\begin{matrix}a_1;\\c_1;\end{matrix}x\right]$$

and

$$G_2'(x,y,t)=\exp[(a_2+1)xt]\int_0^t\exp[-(a_2+1)x\tau]f_2(y,\tau)d\tau+\nu_2\ {}_1F_1\left[\begin{matrix}a_2;\\c_2;\end{matrix}y\right]$$

Proof. Consider the relation of the Theorem 2.1 that $C(x,y,t)=C_1(x,t)C_2(y,t)$, in which by making an appeal to the Theorems 3.4 and 3.5, we find the results of the Theorem 3.6. □

4. Special Cases

Example 4.1. In the Theorem 3.3, $\forall x, y \in (0,1), t \geq 0, c_1, c_2 \neq 0, -1, -2, -3, \ldots$, set $f_1(x,\tau) = e^{\sigma_1 x\tau}$ and $f_2(y,\tau) = e^{\sigma_2 y\tau}$, $\sigma_1 < 0, \sigma_2 < 0$, $a_1 + b_1 > (1 + a_1 b_1)$. Thus we get

$$C(x,y,t) = G_1(x,y,t)G_2(x,y,t) + \nu_1 G_1(x,y,t)\ {}_2F_1\left[\begin{matrix} a_2, b_2; \\ c_2; \end{matrix} y\right] + \mu_1 G_2(x,y,t)\ {}_2F_1\left[\begin{matrix} a_1, b_1; \\ c_1; \end{matrix} x\right] + \nu_1\mu_1\ {}_2F_1\left[\begin{matrix} a_1, b_1; \\ c_1; \end{matrix} x\right]\ {}_2F_1\left[\begin{matrix} a_2, b_2; \\ c_2; \end{matrix} y\right]. \tag{32}$$

Here in (32), it is given that

$$G_1(x,y,t) = \frac{1}{\{(a_1 + b_1 - 1 - a_1 b_1)\{y(1-y)\} + \sigma_1 x\}} \times \{\exp[\sigma_1 x t] - \exp[-(a_1 + b_1 - 1 - a_1 b_1)\{y(1-y)\}t]\} \tag{33}$$

and

$$G_2(x,y,t) = \frac{1}{\{(a_2 + b_2 - 1 - a_2 b_2)\{x(1-x)\} + \sigma_2 y\}} \times \{\exp[\sigma_2 y t] - \exp[-(a_2 + b_2 - 1 - a_2 b_2)\{x(1-x)\}t]\} \tag{34}$$

On making an application of the results (32)-(34), and by conditions of Example 4.1, we find that

$$G_1(x,y,0) = 0 = G_2(x,y,0) \text{ and } \lim_{t\to\infty} G_1(x,y,t) = \lim_{t\to\infty} G_2(x,y,t) = 0,$$

hence by Section 2 we get

$$\lim_{t\to 0^+} C(x,y,t) = \lim_{t\to\infty} C(x,y,t) = f(x,y) = h(x,y) = \nu_1\mu_1\ {}_2F_1\left[\begin{matrix} a_1, b_1; \\ c_1; \end{matrix} x\right]\ {}_2F_1\left[\begin{matrix} a_2, b_2; \\ c_2; \end{matrix} y\right] \tag{35}$$

Example 4.2. In the Theorem 3.6, $\forall x, y \in (0,1), t \geq 0, c_1, c_2 \neq 0, -1, -2, -3, \ldots$, set $f_1(x,\tau) = e^{-\rho_1 x\tau}$ and $f_2(y,\tau) = e^{-\rho_2 y\tau}$, $\rho_1 > 0, \rho_2 > 0, (a_1 + 1) < 0$ and get

$$C(x,y,t) = G'_1(x,y,t)G'_2(x,y,t) + \nu_2 G'_1(x,y,t)\ {}_1F_1\left[\begin{matrix} a_2; \\ c_2; \end{matrix} y\right] + \mu_2 G'_2(x,y,t)\ {}_1F_1\left[\begin{matrix} a_1; \\ c_1; \end{matrix} x\right] + \nu_2\mu_2\ {}_1F_1\left[\begin{matrix} a_1; \\ c_1; \end{matrix} x\right]\ {}_1F_1\left[\begin{matrix} a_2; \\ c_2; \end{matrix} y\right] \tag{36}$$

Here in (36), it is given that

$$G'_1(x,y,t) = \frac{1}{\{(a_1 + 1)y + \rho_1 x\}}\{\exp[(a_1 + 1)yt] - \exp[-\rho_1 x t]\} \tag{37}$$

and

$$G'_2(x,y,t) = \frac{1}{\{(a_2 + 1)x + \rho_2 y\}}\{\exp[(a_2 + 1)xt] - \exp[-\rho_2 y t]\} \tag{38}$$

On applying the results (36)-(39), and by conditions of the Example 4.2, we find that $G'_1(x,y,0) = 0 = G'_2(x,y,0)$, and $\lim_{t\to\infty} G'_1(x,y,t) = \lim_{t\to\infty} G'_2(x,y,t) = 0$ and hence by Section 2 we get

$$\lim_{t\to 0^+} C(x,y,t) = \lim_{t\to\infty} C(x,y,t) = f(x,y) = h(x,y) = \nu_2\mu_2\ {}_1F_1\left[\begin{matrix} a_1; \\ c_1; \end{matrix} x\right]\ {}_1F_1\left[\begin{matrix} a_2; \\ c_2; \end{matrix} y\right] \tag{39}$$

Remark 4.3. Various elementary functions for example $(1 - z)^{-a} = {}_2F_1(a, b; b; z), \ln(1 + z) = z\ {}_2F_1(1, 1; 2; -z)$, Legendre functions of the first and second kinds, incomplete Beta function, complete elliptic integrals of the first and second kinds, Jacobi polynomials, Gegenbauer polynomials, Legendre polynomials, Tchebycheff polynomials of the first and second kinds are generally represented in terms of the hypergeometric function ${}_2F_1(.)$. By the Theorem 3.3 and Example 4.1, the solute distribution may be expressed in the form of these known hypergeometric functions,(also see [8, 10, 14]).

Remark 4.4. Various special functions like Bessel functions, Whittaker functions, incomplete Gamma functions, Hermite polynomials and Leguerre functions etc. are represented in terms of the confluent hypergeometric function ${}_1F_1(.)$. By the Theorem 3.6 and Example 4.2, the solute distribution may be expressed in the form of these known hypergeometric functions,(also see [9, 15, 16]).

5. Conclusion and Discussion

Air pollution meteorology, atmospheric diffusion models for regulatory applications, volume method for transient simulation of time- and scale-dependent transport in heterogeneous aquifer systems are other related topics which can be connected with our present study. A recent work [10,14-16] on obtaining Voigt functions via Quadrature formula for the fractional in time diffusion and wave problem, on a bi-dimensional basis involving Special Functions for partial in space and the time fractional wave mechanical problems and approximation, are such examples. The study of wind speed conditions is of interest, partly because the simulation of airborne pollutant dispersion in certain conditions is rather difficult. We have determined the distribution formulae of the solute transport by the typical advection-dispersion of air pollution problem through separation in two dimensional space variables. Several other methods are available. We have evaluated the solute distribution formulae of the air pollution in terms of Gauss and confluent hypergeometric functions by introducing different values of the solute velocity and dispersion coefficients.

We can determine the solute distribution formulae in terms of certain special functions whose contiguity and analytic properties are described in the literature of the authors [12, 13].The equation (2) via the theory and methods of separation in two dimensional space variables stated on the basis of the researches done in [8-11] may be useful by simply connecting relevant special functions in computation process of the related formulae. By the Theorem 3.6 and Example 4.2, the solute distribution may be expressed in the form of known special functions,(also see [9, 15, 16]). As a consequence, by introducing different values of the solute velocity and dispersion coefficients, we can evaluate the solute distribution formulae of the air pollution in terms of various known and unknown special functions.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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