Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (3) (2024), 595 – 607
DOI : 10.15672/hujms.1130102

Research Article

Redundancy, weaving and Q-dual of K-g-frames in Hilbert spaces

Xiangchun Xiao*, Guoping Zhao, Guorong Zhou

School of Mathematics and Statistics, Xiamen University of Technology, Xiamen 361024, P.R.China

Abstract

In this paper we study exact K-g-frames, weaving of K-g-frames and Q-duals of g-frames in Hilbert spaces. We present a sufficient condition for a g-Bessel sequence to be an exact K-g-frame. Given two woven pairs (Λ, Γ) and (Θ, Δ) of K-g-frames, we investigate under what conditions Λ can be K-g-woven with Δ if Γ is K-g-woven with Θ. Given a K-g-frame Λ and its dual Γ on $\mathcal{U}$, we construct a new pair based on Λ and Γ so that they are woven on a subspace $R(K)$ of $\mathcal{U}$. Finally, we characterize the Q-dual of g-frames using their induced sequences.

Mathematics Subject Classification (2020). 42C15

Keywords. K-g-frame; exact K-g-frame; weaving; Q-dual

1. Introduction

In 2006 Sun [17] proposed the concept of g-frames, which generalizes frames [7], pseudo-frames [1], fusion frames [5, 6], and so on. Since then, g-frames have become a hot topic of research and have been studied intensively by many scholars. Recall that *a collection* $\{\Lambda_j : j \in J\}$ *is called a g-frame for* $\mathcal{U}$ *with respect to* $\{\mathcal{V}_j : j \in J\}$*, if there exist two positive constants* A, B *such that*

$$A\|f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{U}, \tag{1.1}$$

where $\mathcal{U}, \mathcal{V}_j$ are Hilbert spaces and $\Lambda_j, j \in J$ are bounded linear operators from $\mathcal{U}$ into $\mathcal{V}_j$. From the previous literature we know that although g-frames share many of the properties of the previously mentioned frames, there are still some different behaviours for g-frames, e.g. in Hilbert spaces an exact g-frame is not equivalent to a g-Riesz basis [15, 17]. For further information on g-frames, the reader can consult [11, 15, 17, 25] and the papers therein.

K-g-frames are proposed by Xiao et al. in [20] to combine the g-frames with a bounded linear operator K. The idea was from [10], in which the author used K-frames to study the atomic systems. From [20] we know that the properties between g-frames and K-g-frames are quite different, e.g., a g-Bessel sequence $\Lambda := \{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a

*Corresponding Author.
Email addresses: xxc570@163.com, (X. Xiao), 89246682@qq.com (G. Zhao), goonchow@xmut.edu.cn (G. Zhou)
Received: 13.06.2022; Accepted: 06.06.2023

g-frame for $\mathcal{U}$, iff its synthesis operator T_Λ is surjective on $\mathcal{U}$ (see [25]), but for K-g-frames, a g-Bessel sequence $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a K-g-frame for $\mathcal{U}$, is equivalent to the synthesis operator T_Λ being bounded and $R(K) \subseteq R(T_\Lambda)$ (see [20]). For more information on K-g-frame and its special case K-frame, readers can refer to the [10, 18–20]. In this paper we will give a sufficient condition for a g-Bessel sequence to be an exact K g-frame (see Theorem 3.1).

Due to the redundancy, frames provide a stable expansion of elements in the whole Hilbert space, which is very useful in practical applications. When expanding an element using a frame $\{f_i\}_{i\in I}$ in $\mathcal{U}$, the canonical dual $\{S_F^{-1} f_i\}_{i\in I}$ is often used, where S_F is the frame operator of $\{f_i\}_{i\in I}$. The disadvantage is that it is usually difficult to compute the inverse operator S_F^{-1} when the dimension of $\mathcal{U}$ is large. A feasible way is to use an alternate dual of $\{f_i\}_{i\in I}$ to reconstruct the element, that is $f = \sum_{i\in I}\langle f, g_i\rangle f_i$. Now types of duals of frames are suggested, such as alternate dual, oblique dual and Q-dual, etc. Note that Q-dual of fusion frames was first proposed by Heineken et al. in [12] to generalize the canonical dual, and recently Q-duals of frames and g-frames were further studied by Azandaryani in [2, 3]. For more information on duals of frames the reader can consult [2, 3, 12, 13]. In this paper we will characterize the Q-dual of g-frames in terms of their induced sequences.

In a wireless sensor network with M nodes, each node is regarded as a frame $\{f_{ij}\}_{i\in I}$, $j = 1, \cdots, M$, we measure a signal f either with f_{ij}, can the signal f be robustly recovered from these measurements $\{\langle f, f_{i1}\rangle\}_{i\in\sigma_1} \cup \cdots \cup \{\langle f, f_{iM}\rangle\}_{i\in\sigma_M}$, where $\{\sigma_i\}_{i=1}^M$ is an arbitrary partition of I. To simulate such a question in distributed signal processing, Bemrose, Casazza, Grochenig, et al. introduced a new concept *weaving* of frames in [4]. After that, the weaving of frames became a research hotspot studied by many scholars, we refer the readers to consult [4, 8, 9, 14, 16, 21–24] and the papers therein. Now the weaving principle is applied to other frames. In [9] the authors introduced the weaving of K-g-frames. In this paper, we will further study the properties of the weaving of K-g-frames. We are motivated by the following question.

Question: Suppose that $(\{\Lambda_j : j \in J\}, \{\Gamma_j : j \in J\})$, $(\{\Theta_j : j \in J\}, \{\Delta_j : j \in J\})$ are two K-g-woven pairs on $\mathcal{U}$. If $\{\Gamma_j : j \in J\}$ is K-g-woven with either $\{\Theta_j : j \in J\}$ or $\{\Delta_j : j \in J\}$ on $\mathcal{U}$, under what conditions can $\{\Lambda_j : j \in J\}$ be K-g-woven with $\{\Delta_j : j \in J\}$ or $\{\Theta_j : j \in J\}$ on $\mathcal{U}$?

In [8] the authors discussed that a g-frame and its dual g-frames are woven. Motivated by the work of [8], it is natural to consider whether a K-g-frame $\{\Lambda_j : j \in J\}$ on $\mathcal{U}$ and its dual are woven on $\mathcal{U}$? It does not hold in general (see Section 5). We then construct a new pair based on $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ so that they are woven on the subspace $R(K)$ of $\mathcal{U}$.

This paper is organized as follows. In Section 2 we recall some lemmas and preliminaries of K-g-frames in Hilbert spaces. In Section 3 we give a sufficient condition for a given g-Bessel sequence to be an exact K-g-frame. Given two K-g-woven pairs (Λ, Γ) and (Θ, Δ), we will show in Section 4 that any two g-Bessel sequences in these two K-g-woven pairs are possible K-g-woven. Given a K-g-frame $\{\Lambda_j : j \in J\}$ and its dual $\{\Gamma_j : j \in J\}$ on $\mathcal{U}$, in Section 5 we will construct a new pair based on $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ so that they are woven on a subspace $R(K)$ of $\mathcal{U}$. In Section 6 we characterize a Q-dual pair of g-frames in terms of their induced sequences.

Throughout this paper, we adopt such notations: $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces, with inner product $\langle\cdot,\cdot\rangle$, and norm $\|\cdot\|$; the identity operator on $\mathcal{U}$ is denoted by $I_\mathcal{U}$; $L(\mathcal{U}, \mathcal{V})$ denotes by the collection of all linear bounded operators from $\mathcal{U}$ to $\mathcal{V}$, if $\mathcal{U} = \mathcal{V}$, then $L(\mathcal{U}, \mathcal{V})$ is abbreviated as $L(\mathcal{U})$; $0 \neq K \in L(\mathcal{U})$, K^* and K^+ denote the adjoint operator and pseudo-inverse of K, respectively; if $Q \in L(\mathcal{U}, \mathcal{V})$, $R(Q)$ and $N(Q)$ denote the range and null space of Q, respectively; $\{\mathcal{V}_j\}_{j\in J}$ is a sequence of closed subspaces of $\mathcal{V}$, where J

is a subset of the integer set $\mathbb{Z}$; $\mathcal{U} \subset \mathcal{V}$ means $\mathcal{U}$ is strictly contained in $\mathcal{V}$, $\mathcal{U} \subseteq \mathcal{V}$ includes two cases $\mathcal{U} \subset \mathcal{V}$ and $\mathcal{U} = \mathcal{V}$.

2. Preliminaries

In this section we mainly recall some preliminaries of K-g-frames in Hilbert spaces.

Definition 2.1 ([20]). A sequence $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is called a K-g-frame for $\mathcal{U}$ with respect to (w.r.t.) $\{\mathcal{V}_j : j \in J\}$, if there exist $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \quad \forall\, f \in \mathcal{U}. \tag{2.1}$$

We call A, B the lower and upper frame bound of K-g-frame $\{\Lambda_j : j \in J\}$, respectively. We call $\{\Lambda_j : j \in J\}$ a g-Bessel sequence if only the right side of (2.1) holds.

We call $\{\Lambda_j : j \in J\}$ an exact K-g-frame if it ceases to be a K-g-frame whenever any one of its elements is removed.

We also need to introduce a basic space $l^2(\{\mathcal{V}_j\}_{j\in J})$ as follows:

$$l^2(\{\mathcal{V}_j\}_{j\in J}) = \left\{\{g_j\}_{j\in J} : g_j \in \mathcal{V}_j, j \in J \text{ and} \sum_{j\in J} \|g_j\|^2 < +\infty\right\},$$

with the inner product

$$\langle\{f_j\}_{j\in J}, \{g_j\}_{j\in J}\rangle = \sum_{j\in J}\langle f_j, g_j\rangle.$$

In [25] it was shown that $l^2(\{\mathcal{V}_j\}_{j\in J})$ is a complex Hilbert space.

Definition 2.2 ([17]). Let $\{\Lambda_j : j \in J\}$ be a g-Bessel sequence in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$. For $\{g_j\}_{j\in J} \in l^2(\{\mathcal{V}_j\}_{j\in J})$, if $\sum_{j\in J} \Lambda_j^* g_j = 0$ implies that $g_j = 0$ for any $j \in J$, then $\{\Lambda_j : j \in J\}$ is called $l^2(\{\mathcal{V}_j\}_{j\in J})$-linear independent.

Remark 2.3. Note that, if a g-Bessel sequence $\{\Lambda_j : j \in J\}$ is $l^2(\{\mathcal{V}_j\}_{j\in J})$-linear independent, then $\Lambda_j \neq 0$ for any $j \in J$.

Assume that $\Lambda := \{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a g-Bessel sequence in $\mathcal{U}$, the synthesis operator T_Λ is defined in [25] as follows:

$$T_\Lambda : l^2(\{\mathcal{V}_j\}_{j\in J}) \to \mathcal{U}, \quad T_\Lambda(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j. \tag{2.2}$$

In order to characterize exact K-g-frames, for some $j_0 \in J$, we also need to define T_{j_0} as follows

$$T_{j_0} : l^2(\{\mathcal{V}_j\}_{j\in J\setminus\{j_0\}}) \to \mathcal{U}, \quad T(\{g_j\}_{j\in J\setminus\{j_0\}}) = \sum_{j\in J, j\neq j_0} \Lambda_j^* g_j. \tag{2.3}$$

Given a K-g-frame $\{\Lambda_j : j \in J\}$ in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$, if there exists a g-Bessel sequence $\{\Gamma_j : j \in J\}$ in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$, such that

$$Kf = \sum_{j\in J} \Lambda_j^* \Gamma_j f, \qquad \forall f \in \mathcal{U}, \tag{2.4}$$

then $\{\Gamma_j\}_{j\in J}$ is called a dual K-g-Bessel sequence of $\{\Lambda_j\}_{j\in J}$ on $\mathcal{U}$. Note that in general $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ in (2.4) are not interchangeable, i.e. in general $Kf \neq \sum_{j\in J} \Gamma_j^* \Lambda_j f$. If $K = I_{\mathcal{U}}$, (2.4) becomes $f = \sum_{j\in J} \Lambda_j^* \Gamma_j f, \forall f \in \mathcal{U}$, in this case $\{\Lambda_j : j \in J\}$ is a g-frame, and $\{\Gamma_j\}_{j\in J}$ is called an alternate dual g-frame of $\{\Lambda_j\}_{j\in J}$. Moreover, if we let $K = I_{\mathcal{U}}$ and $\mathcal{V}_j = \mathbb{C}, \Lambda_j f = \langle f, f_i\rangle, \Gamma_j f = \langle f, g_i\rangle, \forall j \in J$, then $\{\Lambda_j : j \in J\}$ is a g-frame for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$, iff $\{f_j\}_{j\in J}$ is a frame for $\mathcal{U}$. Then from (2.4) we get $f = \sum_{j\in J}\langle f, g_j\rangle f_j$, $\forall f \in \mathcal{U}$, and $\{g_j\}_{j\in J}$ is called an alternate dual of $\{f_j\}_{j\in J}$.

In order to generalize the canonical dual of frames Heineken et al. introduced Q-dual of fusion frames in [12]. Later the properties of Q-dual of g-frames and frames were further studied by Azandaryani in [2,3]. In this paper we will give an equivalent characterization of Q-dual of g-frames.

Definition 2.4 ([2])**.** Let $Q \in L(l^2(\{\mathcal{V}_j\}_{j\in J}))$, $\Lambda := \{\Lambda_j : j \in J\}$ and $\Gamma := \{\Gamma_j : j \in J\}$ be g-Bessel sequences in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$, with synthesis operators T_Λ and T_Γ, respectively. If $T_\Lambda Q T_\Gamma^* = I_\mathcal{U}$, then $\{\Gamma_j : j \in J\}$ is called a Q-dual of $\{\Lambda_j : j \in J\}$. In particular, if $Q = I_{l^2(\{\mathcal{V}_j\}_{j\in J})}$, then $\{\Gamma_j : j \in J\}$ is called the alternate dual of $\{\Lambda_j : j \in J\}$.

In [4] the authors wanted to simulate a question in distributed signal processing and introduced a new concept weaving of frames as follows.

Definition 2.5 ([4])**.** Let I be an index set, and let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be frames for $\mathcal{H}$. If there exist $A, B > 0$ such that for any partition $\{\sigma_j\}_{j=1}^2$ of I, $\{f_i\}_{i\in\sigma_1} \cup \{g_i\}_{i\in\sigma_2}$ is a frame for $\mathcal{H}$ with frame bounds A, B, then $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are said to be woven on $\mathcal{H}$ with universal frame bounds A, B, and $\{f_i\}_{i\in\sigma_1} \cup \{g_i\}_{i\in\sigma_2}$ is called a weaving.

Soon afterwards the weaving of frames was generalized to K-g-frames in [9].

Definition 2.6 ([9])**.** Let J be an index set, and let $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ be K-g-frames for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$. If there exist $A, B > 0$ such that for any partition $\{\sigma_j\}_{j=1}^2$ of J, $\{\Lambda_j\}_{i\in\sigma_1} \cup \{\Gamma_j\}_{i\in\sigma_2}$ is a K-g-frame for $\mathcal{U}$ with K-g-frame bounds A, B, then $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are said to be K-g-woven on $\mathcal{U}$ with universal K-g-frame bounds A, B, each $\{\Lambda_j\}_{j\in\sigma_1} \cup \{\Gamma_j\}_{j\in\sigma_2}$ is called a weaving.

If $K = I_\mathcal{U}$, then K-g-frame is just the g-frame. From Definition 2.6 we can get the weaving of g-frames as follows.

Definition 2.7 ([9,14])**.** Let J be an index set, and let $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ be g-frames in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$. If there exist $A, B > 0$ such that for any partition $\{\sigma_j\}_{j=1}^2$ of J, $\{\Lambda_j\}_{i\in\sigma_1} \cup \{\Gamma_j\}_{i\in\sigma_2}$ is a g-frame for $\mathcal{U}$ with g-frame bounds A, B, then $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ are said to be woven on $\mathcal{U}$ with universal g-frame bounds A, B.

In the rest of this section we recall some known lemmas which we need later.

Lemma 2.8 ([7])**.** *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces, and $Q \in L(\mathcal{H}_1, \mathcal{H}_2)$ with closed range. Then there exists a unique bounded operator $Q^+ : \mathcal{H}_2 \to \mathcal{H}_1$, called the pseudo-inverse operator of Q, satisfying*

$$N(Q^+) = R(Q)^\perp,\ R(Q^+) = N(Q)^\perp,\ QQ^+ = P_{R(Q)},\ Q^+Q = P_{R(Q^+)}, \tag{2.5}$$

where $P_{R(Q)}$ is the orthogonal projection from $\mathcal{H}_2$ onto $R(Q)$, $P_{R(Q^+)}$ is the orthogonal projection from $\mathcal{H}_1$ onto $R(Q^+)$.

If Q is a bounded invertible operator, then $Q^+ = Q^{-1}$.

Lemma 2.9 ([20])**.** *A sequence $\Lambda := \{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a K-g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j : j \in J\}$, if and only if the synthesis operator T_Λ defined by (2.2) is well defined and bounded, and $R(K) \subseteq R(T_\Lambda)$.*

Remark 2.10. In fact when $R(K) = R(T_\Lambda)$ Theorem 3.5 in [20] also holds, hence in Lemma 2.9 we use $R(K) \subseteq R(T_\Lambda)$.

3. Conditions of exact K-g-frames

In the following we give a sufficient condition for a given g-Bessel sequence to be an exact K-g-frame.

Theorem 3.1. *Suppose that $\Lambda := \{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ is a g-Bessel sequence in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$. If the following two conditions hold,*

(*i*) $R(K) = R(T_\Lambda)$, *where* T_Λ *is the synthesis operator for* $\{\Lambda_j : j \in J\}$*;*
(*ii*) *for any* $j \in J$*, we have* $R(T_j) \subsetneqq R(T_\Lambda)$*, where* T_j *is defined as in (2.3);*

then $\{\Lambda_j : j \in J\}$ *is an exact K-g-frame for* $\mathcal{U}$*.*

Proof. Suppose that the conditions $(i), (ii)$ hold. Λ is a g-Bessel sequence in $\mathcal{U}$, so T_Λ is bounded. By Lemma 2.9 we know that $\{\Lambda_j : j \in J\}$ is a K-g-frame for $\mathcal{U}$. Next we use the contradiction to prove that $\{\Lambda_j : j \in J\}$ is exact.

Assume that K-g-frame $\{\Lambda_j : j \in J\}$ is not exact. Then there at least exists some $j_0 \in J$ such that $\{\Lambda_j : j \in J \backslash \{j_0\}\}$ is a K-g-frame for $\mathcal{U}$. Again by Lemma 2.9 we get $R(K) \subseteq R(T_{j_0})$. Combining with the condition (ii) we have $R(K) \subseteq R(T_{j_0}) \subsetneqq R(T_\Lambda)$, which contradicts to $R(K) = R(T_\Lambda)$. Therefore $\{\Lambda_j : j \in J\}$ is an exact K-g-frame for $\mathcal{U}$. □

Note that the condition (ii) in Theorem 3.1 is necessary for an exact K-g-frame.

Theorem 3.2. *Suppose that* $\Lambda := \{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ *is a g-Bessel sequence in* $\mathcal{U}$ *w.r.t.* $\{\mathcal{V}_j : j \in J\}$*. If* $\{\Lambda_j : j \in J\}$ *is an exact K-g-frame for* $\mathcal{U}$*, then the condition* (ii) *in Theorem 3.1 holds.*

Proof. Let $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ be an exact K-g-frame for $\mathcal{U}$, with synthesis operator T_Λ. It is obvious that $R(T_j) \subseteq R(T_\Lambda), \forall j \in J$, where T_j is defined as in (2.3). We apply the proof by contradiction to prove $R(T_j) \subsetneqq R(T_\Lambda)$ for any $j \in J$. Assume that there exists some $j_0 \in J$ such that $R(T_{j_0}) = R(T_\Lambda)$. Since $\{\Lambda_j : j \in J\}$ is an exact K-g-frame, from Lemma 2.9 we get $R(K) \subseteq R(T_\Lambda)$, then we have $R(K) \subseteq R(T_{j_0})$, again by Lemma 2.9 it follows that $\{\Lambda_j\}_{j \in J \backslash \{j_0\}}$ is a K-g-frame for $\mathcal{U}$. This contradicts to that $\{\Lambda_j : j \in J\}$ is exact. Hence $R(T_j) \neq R(T_\Lambda)$, combining with $R(T_j) \subseteq R(T_\Lambda)$, therefore we have $R(T_j) \subsetneqq R(T_\Lambda)$ for any $j \in J$. □

In Theorem 3.10 in [23] the author got an equivalent characterization of an exact K-g-frame as follows.

Theorem 3.3 ([23])**.** *Suppose that* $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ *is a g-Bessel sequence in* $\mathcal{U}$ *w.r.t.* $\{\mathcal{V}_j : j \in J\}$*, and for any* $j \in J$*,* $\dim \mathcal{V}_j = 1$*. Then* $\{\Lambda_j : j \in J\}$ *is an exact K-g-frame for* $\mathcal{U}$*, if and only if the following two conditions hold:*

(*i*) $\{\Lambda_j : j \in J\}$ *is* $l^2(\{\mathcal{V}_j\}_{j \in J})$*-linear independent;*
(*ii*) *There exists a dual K-g-Bessel sequence* $\{\Gamma_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ *in* $\mathcal{U}$ *satisfying (2.4) such that for any* $j \in J$*,* $\Gamma_j \neq 0$*.*

Note that, given a g-Bessel sequence $\{\Lambda_j \in L(\mathcal{U}, \mathcal{V}_j) : j \in J\}$ in $\mathcal{U}$, if $\{\Lambda_j : j \in J\}$ only satisfies condition (i) in Theorem 3.1, or only satisfies condition (ii) in Theorem 3.3, we can't deduce that $\{\Lambda_j : j \in J\}$ is an exact K-g-frame for $\mathcal{U}$. Please see the following counterexamples.

Example 3.4. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for $\mathcal{U}$.

(i) Let $\mathcal{V}_j = \overline{span}\{e_j, e_{j+1}, e_{j+2}\}, j \in \mathbb{N} \backslash \{4,5,6\}$, $\mathcal{V}_j = \mathcal{V}_{j-3}, j = 4,5,6$. Define $K : \mathcal{U} \to \mathcal{U}$ and $\Lambda_j : \mathcal{U} \to \mathcal{V}_j$ as follows

$$Ke_i = e_i, i = 1,2,3, Ke_j = 0, j \geq 4; \tag{3.1}$$
$$\Lambda_j f = \langle f, e_j \rangle e_j, j = 1,2,3, \;\; \Lambda_j f = \Lambda_{j-3} f, j = 4,5,6, \;\; \Lambda_j f = 0, j \geq 7.$$

Obviously $\Lambda := \{\Lambda_j\}_{j=1}^{\infty}$ is a g-Bessel sequence in $\mathcal{U}$. For any $f \in \mathcal{U}$, and $g_j = c_j e_j + c_{j+1} e_{j+1} + c_{j+2} e_{j+2} \in \mathcal{V}_j$, where $c_j, c_{j+1}, c_{j+2} \in \mathbb{C}, j = 1,2,3$, we have

$$\langle \Lambda_j^* g_j, f \rangle = \langle g_j, \Lambda_j f \rangle = \langle c_j e_j + c_{j+1} e_{j+1} + c_{j+2} e_{j+2}, \langle f, e_j \rangle e_j \rangle = c_j \overline{\langle f, e_j \rangle} = \langle c_j e_j, f \rangle.$$

Hence we get $R(K) = R(T_\Lambda) = \overline{span}\{e_1, e_2, e_3\}$, by Lemma 2.9 $\{\Lambda_j\}_{j=1}^{\infty}$ is a K-g-frame for $\mathcal{U}$. It's easy to check that $\{\Lambda_j\}_{j \in \mathbb{N} \backslash \{j_0\}}$ is still a K-g-frame for $\mathcal{U}$ if we erase any $\Lambda_{j_0}, j_0 \geq 4$, since $R(K) = R(T_{j_0})$. Hence $\{\Lambda_j\}_{j=1}^{\infty}$ is not an exact K-g-frame for $\mathcal{U}$.

(ii) Let K be defined as in (3.1). Let $\mathcal{V}_j = \overline{span}\{e_j\}, j = 1, 2$, $\mathcal{V}_3 = \mathcal{V}_4 = \overline{span}\{e_3\}$, $\mathcal{V}_j = \overline{span}\{e_{j-1}\}, j \geq 5$. Define $\Lambda_j, \Gamma_j : \mathcal{U} \to \mathcal{V}_j$ as follows

$$\Lambda_j f = \langle f, e_j\rangle e_j, j = 1, 2, \Lambda_3 f = \Lambda_4 f = \langle f, e_3\rangle e_3, \Lambda_j f = 0, j \geq 5;$$

$$\Gamma_j f = \langle f, e_j\rangle e_j, j = 1, 2, \Gamma_3 f = \Gamma_4 f = \langle f, \frac{1}{2}e_3\rangle e_3, \Gamma_j f = \langle f, e_{j-1}\rangle e_{j-1}, j \geq 5.$$

It's easy to check that $\Lambda := \{\Lambda_j\}_{j=1}^{\infty}$ and $\Gamma := \{\Gamma_j\}_{j=1}^{\infty}$ are g-Bessel sequences in $\mathcal{U}$. By direct calculations we obtain $\Lambda_j^* g_j = g_j, j = 1, 2, 3, 4, \Lambda_j^* g_j = 0, j \geq 5$, where $g_j \in \mathcal{V}_j$. So for any $g_j = c_j e_j \in \mathcal{V}_j$, where $c_j \in \mathbb{C}$, we have $T_\Lambda(\{g_j\}_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \Lambda_j^* g_j = \sum_{j=1}^{4} \Lambda_j^* g_j = \sum_{j=1}^{3} c_j e_j + c_4 e_3 = c_1 e_1 + c_2 e_2 + (c_3 + c_4)e_3$. Hence $R(K) = R(T_\Lambda) = \overline{span}\{e_1, e_2, e_3\}$, by Lemma 2.9 $\{\Lambda_j\}_{j=1}^{\infty}$ is a K-g-frame for $\mathcal{U}$.

On the other hand, for any $f \in \mathcal{U}$, there exist $\{c_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ such that $f = \sum_{j=1}^{\infty} c_j e_j$. So we have

$$\begin{aligned} Kf &= \sum_{j=1}^{\infty} c_j K e_j = \sum_{j=1}^{3} c_j e_j, \\ \sum_{j=1}^{\infty} \Lambda_j^* \Gamma_j f &= \sum_{j=1}^{4} \Lambda_j^* \Gamma_j f = \sum_{j=1}^{4} \Gamma_j f \\ &= \langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_2 + \langle f, \frac{1}{2}e_3\rangle e_3 + \langle f, \frac{1}{2}e_3\rangle e_3 \\ &= \sum_{j=1}^{3} \langle f, e_j\rangle e_j = \sum_{j=1}^{3} c_j e_j = Kf. \end{aligned}$$

Hence $\{\Gamma_j\}_{j=1}^{\infty}$ is a dual K-g-Bessel sequence of $\{\Lambda_j\}_{j=1}^{\infty}$ satisfying (2.4). It is obvious $\Gamma_j \neq 0$, $j \in \mathbb{N}$.

Next we show that $\{\Lambda_j\}_{j=1}^{\infty}$ is not an exact K-g-frame for $\mathcal{U}$. If we erase $\Lambda_{j_0}, j_0 \geq 4$, we can verify $R(K) = R(T_{j_0}) = \overline{span}\{e_1, e_2, e_3\}$, hence $\{\Lambda_j\}_{j\in\mathbb{N}\setminus\{j_0\}}$ is a K-g-frame for $\mathcal{U}$ by Lemma 2.9.

4. Weaving of any two g-Bessel sequences in two K-g-woven pairs

In this section we will answer the question proposed in Section 1 on the weaving of K-g-frames. Then we get a result as follows.

Theorem 4.1. *Let $K, Q \in L(\mathcal{U})$ be surjective operators on $\mathcal{U}$, $\{\Lambda_j : j \in J\}$, $\{\Gamma_j : j \in J\}$, $\{\Theta_j : j \in J\}$ and $\{\Delta_j : j \in J\}$ be g-Bessel sequences on $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$. Suppose that $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$, $\{\Theta_j : j \in J\}$ and $\{\Delta_j : j \in J\}$ are K-g-woven on $\mathcal{U}$, respectively with universal K-g-frames bounds A, B and C, D. If $\{\Gamma_j : j \in J\}$ is K-g-woven with $\{\Theta_j : j \in J\}$ with universal K-g-frame bounds A_1, B_1, and satisfies $QK = KQ$, $A + C > B_1\|Q\|^2\|K^+\|^2\|Q^+\|^2$ and $(B + D)\|Q\|^2\|Q^+\|^2\|K^+\|^2 > A_1$, then $\{\Lambda_j Q^* : j \in J\}$ and $\{\Delta_j Q^* : j \in J\}$ are K-g-woven on $\mathcal{U}$, with universal K-g-frame bounds*

$$\frac{A + C - B_1\|Q\|^2\|K^+\|^2\|Q^+\|^2}{\|Q^+\|^2}, \ \frac{(B + D)\|Q\|^2\|Q^+\|^2\|K^+\|^2 - A_1}{\|Q^+\|^2\|K^+\|^2}.$$

Proof. Since $Q \in L(\mathcal{U})$ is surjective on $\mathcal{U}$, by Lemma 2.8 there exists a pseudo-inverse operator Q^+ such that $QQ^+ = P_{R(Q)} = P_{\mathcal{U}}$. It follows that $P_{\mathcal{U}} = P_{\mathcal{U}}^* = (Q^+)^* Q^*$. Hence for any $f \in \mathcal{U}$, we have

$$\|f\| = \|(Q^+)^* Q^* f\| \leq \|(Q^+)^*\|\|Q^* f\| = \|Q^+\|\|Q^* f\|, \tag{4.1}$$

and consequently

$$\|Q^* f\| \geq \frac{1}{\|Q^+\|}\|f\|, \quad \forall f \in \mathcal{U}. \tag{4.2}$$

$K \in L(\mathcal{U})$ is also surjective on $\mathcal{U}$, similarly we can get, for any $f \in \mathcal{U}$,

$$\|f\| \leq \|K^+\| \|K^* f\|, \tag{4.3}$$

$$\|K^* f\| \geq \frac{1}{\|K^+\|} \|f\|. \tag{4.4}$$

Since $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$, $\{\Theta_j : j \in J\}$ and $\{\Delta_j : j \in J\}$, $\{\Gamma_j : j \in J\}$ and $\{\Theta_j : j \in J\}$ are K-g-woven on $\mathcal{U}$, respectively with universal K-g-frame bounds A, B, C, D, and A_1, B_1, for any partition $\{\sigma_i\}_{i=1}^2$ of J and any $f \in \mathcal{U}$, we have

$$A\|K^* f\|^2 \leq \sum_{j\in\sigma_1} \|\Lambda_j f\|^2 + \sum_{j\in\sigma_2} \|\Gamma_j f\|^2 \leq B\|f\|^2, \tag{4.5}$$

$$C\|K^* f\|^2 \leq \sum_{j\in\sigma_1} \|\Theta_j f\|^2 + \sum_{j\in\sigma_2} \|\Delta_j f\|^2 \leq D\|f\|^2, \tag{4.6}$$

$$A_1\|K^* f\|^2 \leq \sum_{j\in\sigma_1} \|\Gamma_j f\|^2 + \sum_{j\in\sigma_2} \|\Theta_j f\|^2 \leq B_1\|f\|^2. \tag{4.7}$$

Combining with (4.2) and (4.5), and $KQ = QK$, it follows that, for any $f \in \mathcal{U}$,

$$\begin{aligned} \frac{A}{\|Q^+\|^2}\|K^* f\|^2 &\leq A\|Q^* K^* f\|^2 = A\|K^* Q^* f\|^2 \\ &\leq \sum_{j\in\sigma_1} \|\Lambda_j Q^* f\|^2 + \sum_{j\in\sigma_2} \|\Gamma_j Q^* f\|^2 \\ &\leq B\|Q^* f\|^2 \leq B\|Q\|^2\|f\|^2. \end{aligned} \tag{4.8}$$

Similarly we get

$$\frac{C}{\|Q^+\|^2}\|K^* f\|^2 \leq \sum_{j\in\sigma_1} \|\Theta_j Q^* f\|^2 + \sum_{j\in\sigma_2} \|\Delta_j Q^* f\|^2 \leq D\|Q\|^2\|f\|^2, \tag{4.9}$$

$$\frac{A_1}{\|Q^+\|^2}\|K^* f\|^2 \leq \sum_{j\in\sigma_1} \|\Gamma_j Q^* f\|^2 + \sum_{j\in\sigma_2} \|\Theta_j Q^* f\|^2 \leq B_1\|Q\|^2\|f\|^2. \tag{4.10}$$

Next we prove that $\{\Lambda_j Q^* : j \in J\}$ and $\{\Delta_j Q^* : j \in J\}$ are K-g-woven on $\mathcal{U}$. In fact, for any $f \in \mathcal{U}$ and any partition $\{\sigma_i\}_{i=1}^2$ of J, by (4.8), (4.9) and (4.10), we obtain

$$\begin{aligned} & \sum_{j\in\sigma_1} \|\Lambda_j Q^* f\|^2 + \sum_{j\in\sigma_2} \|\Delta_j Q^* f\|^2 \\ = & \sum_{j\in\sigma_1} \|\Lambda_j Q^* f\|^2 + \sum_{j\in\sigma_2} \|\Gamma_j Q^* f\|^2 + \sum_{j\in\sigma_1} \|\Theta_j Q^* f\|^2 + \sum_{j\in\sigma_2} \|\Delta_j Q^* f\|^2 \\ - & \left(\sum_{j\in\sigma_2} \|\Gamma_j Q^* f\|^2 + \sum_{j\in\sigma_1} \|\Theta_j Q^* f\|^2 \right) \\ \geq & \frac{A}{\|Q^+\|^2}\|K^* f\|^2 + \frac{C}{\|Q^+\|^2}\|K^* f\|^2 - B_1\|Q\|^2\|f\|^2 \\ \geq & \frac{A+C}{\|Q^+\|^2}\|K^* f\|^2 - B_1\|Q\|^2\|K^+\|^2\|K^* f\|^2 \\ = & \frac{A + C - B_1\|Q\|^2\|K^+\|^2\|Q^+\|^2}{\|Q^+\|^2}\|K^* f\|^2, \end{aligned} \tag{4.11}$$

where the last inequality is deduced by (4.2).

On the other hand, from (4.4), (4.8), (4.9), (4.10) and (4.11) we have

$$\begin{aligned}
&\sum_{j\in\sigma_1}\|\Lambda_j Q^* f\|^2+\sum_{j\in\sigma_2}\|\Delta_j Q^* f\|^2\\
\leq\ & B\|Q\|^2\|f\|^2+D\|Q\|^2\|f\|^2-\frac{A_1}{\|Q^+\|^2}\|K^* f\|^2\\
\leq\ & (B+D)\|Q\|^2\|f\|^2-\frac{A_1}{\|Q^+\|^2}\frac{1}{\|K^+\|^2}\|f\|^2\\
=\ & \frac{(B+D)\|Q\|^2\|Q^+\|^2\|K^+\|^2-A_1}{\|Q^+\|^2\|K^+\|^2}\|f\|^2.
\end{aligned}$$

Hence $\{\Lambda_j Q^* : j\in J\}$ and $\{\Delta_j Q^* : j\in J\}$ are K-g-woven on $\mathcal{U}$. □

If $K=I_{\mathcal{U}}$ or $Q=I_{\mathcal{U}}$ in Theorem 4.1, we can respectively obtain the following corollaries.

Corollary 4.2. *Let $Q\in L(\mathcal{U})$ be a surjective operator on $\mathcal{U}$, $\{\Lambda_j : j\in J\}$, $\{\Gamma_j : j\in J\}$, $\{\Theta_j : j\in J\}$ and $\{\Delta_j : j\in J\}$ be g-Bessel sequences in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j\in J\}$. Suppose that $\{\Lambda_j : j\in J\}$ and $\{\Gamma_j : j\in J\}$, $\{\Theta_j : j\in J\}$ and $\{\Delta_j : j\in J\}$ are woven on $\mathcal{U}$, respectively with universal g-frames bounds A,B and C,D. If $\{\Gamma_j : j\in J\}$ is woven with $\{\Theta_j : j\in J\}$ with universal g-frame bounds A_1,B_1, and satisfies $A+C>B_1\|Q\|^2\|Q^+\|^2$ and $(B+D)\|Q\|^2\|Q^+\|^2>A_1$, then $\{\Lambda_j Q^* : j\in J\}$ and $\{\Delta_j Q^* : j\in J\}$ are woven on $\mathcal{U}$, with universal g-frame bounds*

$$\frac{A+C-B_1\|Q\|^2\|Q^+\|^2}{\|Q^+\|^2},\ \frac{(B+D)\|Q\|^2\|Q^+\|^2-A_1}{\|Q^+\|^2}.$$

Corollary 4.3. *Let $K\in L(\mathcal{U})$ be a surjective operator on $\mathcal{U}$, $\{\Lambda_j : j\in J\}$, $\{\Gamma_j : j\in J\}$, $\{\Theta_j : j\in J\}$ and $\{\Delta_j : j\in J\}$ be g-Bessel sequences in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j\in J\}$. Suppose that $\{\Lambda_j : j\in J\}$ and $\{\Gamma_j : j\in J\}$, $\{\Theta_j : j\in J\}$ and $\{\Delta_j : j\in J\}$ are K-g-woven on $\mathcal{U}$, respectively with universal K-g-frames bounds A,B and C,D. If $\{\Gamma_j : j\in J\}$ is K-g-woven with $\{\Theta_j : j\in J\}$ with universal K-g-frame bounds A_1,B_1, and satisfies $A+C>B_1\|K^+\|^2$ and $(B+D)\|K^+\|^2>A_1$, then $\{\Lambda_j : j\in J\}$ and $\{\Delta_j : j\in J\}$ are K-g-woven on $\mathcal{U}$, with universal K-g-frame bounds*

$$A+C-B_1\|K^+\|^2,\ \frac{(B+D)\|K^+\|^2-A_1}{\|K^+\|^2}.$$

Moreover, in Corollary 4.3 if $\Gamma_j=\Theta_j$, $\forall j\in J$, then we have the transitivity of weaving for K-g-frames.

Corollary 4.4. *Let $K\in L(\mathcal{U})$ be a surjective operator on $\mathcal{U}$, $\{\Lambda_j : j\in J\}$, $\{\Gamma_j : j\in J\}$ and $\{\Delta_j : j\in J\}$ be g-Bessel sequences in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j\in J\}$. Suppose that $\{\Lambda_j : j\in J\}$ and $\{\Gamma_j : j\in J\}$, $\{\Gamma_j : j\in J\}$ and $\{\Delta_j : j\in J\}$ are K-g-woven on $\mathcal{U}$, respectively with universal K-g-frames bounds A,B and C,D. If $A+C>B\|K^+\|^2$ and $(B+D)\|K^+\|^2>A$, then $\{\Lambda_j : j\in J\}$ and $\{\Delta_j : j\in J\}$ are K-g-woven on $\mathcal{U}$, with universal K-g-frame bounds*

$$A+C-B\|K^+\|^2,\ \frac{(B+D)\|K^+\|^2-A}{\|K^+\|^2}.$$

5. Weaving of a pair of dual of K-g-frames

In [8] the authors studied that a g-frame and its dual g-frame which are weaving. Motivated by this, we will study the case of K-g-frames. Given a K-g-frame $\{\Lambda_j : j\in J\}$ on $\mathcal{U}$ and its dual K-g-Bessel sequence $\{\Gamma_j : j\in J\}$ (see (2.4)), in general $\{\Lambda_j : j\in J\}$ and $\{\Gamma_j : j\in J\}$ are not woven on $\mathcal{U}$. In fact, in Example 3.4 (ii) we know that $\{\Gamma_j\}_{j=1}^{\infty}$ is a dual K-g-Bessel sequence of $\{\Lambda_j\}_{j=1}^{\infty}$ on $\mathcal{U}$, if we take $\sigma=\mathbb{N}\backslash\{1,2,3,4\}$, $\sigma^c=\{1,2,3,4\}$, then we obtain a weaving $\{\Lambda_j : j \in \sigma\} \cup \{\Gamma_j : j \in \sigma^c\} = \{\Gamma_j\}_{j=1}^4$, which is obviously not a g-frame for $\mathcal{U}$.

Although in general a K-g-frame $\{\Lambda_j : j \in J\}$ on $\mathcal{U}$ and its dual K-g-Bessel sequence $\{\Gamma_j : j \in J\}$ are not woven on $\mathcal{U}$, next we construct a new pair based on $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ so that they are woven on $R(K)$.

Theorem 5.1. *Suppose that $K \in L(\mathcal{U})$ has a closed range and $\Lambda := \{\Lambda_j : j \in J\}$ is a K-g-frame for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$, with K-g-frame bounds A_Λ, B_Λ. $\Gamma := \{\Gamma_j : j \in J\}$ is a dual K-g-Bessel sequence of $\{\Lambda_j\}_{j\in J}$ on $\mathcal{U}$, with g-Bessel bound B_Γ. Then $\{\Gamma_j K^* : j \in J\}$ and $\{\Lambda_j : j \in J\}$ are woven on $R(K)$, with universal g-frame bounds*

$$\frac{1}{2\max\{B_\Lambda\|K^+\|^2, B_\Gamma\}\|K^+\|^2}, \quad B_\Gamma\|K\|^2 + B_\Lambda.$$

Proof. Since $K \in L(\mathcal{U})$ has a closed range, by Lemma 2.8 there exists a pseudo-inverse operator K^+ such that $KK^+ = P_{R(K)}$, and consequently $P_{R(K)} = (P_{R(K)})^* = (KK^+)^* = (K^+)^*K^*$. For any $f \in R(K)$, we have

$$\|f\| = \|(K^+)^*K^*f\| \le \|(K^+)^*\| \cdot \|K^*f\| = \|K^+\| \cdot \|K^*f\|. \tag{5.1}$$

It follows from (5.1) that

$$\|K^*f\| \ge \frac{1}{\|K^+\|}\|f\|, \quad \forall f \in R(K). \tag{5.2}$$

Since $\{\Gamma_j : j \in J\}$ is a dual K-g-Bessel sequence of $\{\Lambda_j\}_{j\in J}$ on $\mathcal{U}$, from (2.4) we get

$$Kf = \sum_{j\in J} P_{R(K)}\Lambda_j^*\Gamma_j f, \qquad \forall f \in \mathcal{U}. \tag{5.3}$$

For any $f \in \mathcal{U}$ and any $\sigma \subset J$, from (5.1) we have

$$\begin{aligned}\Big|\sum_{j\in\sigma}\langle\Gamma_j K^*f, \Lambda_j P_{R(K)}f\rangle\Big| &\le \sum_{j\in\sigma}\|\Gamma_j K^*f\| \cdot \|\Lambda_j P_{R(K)}f\| \\ &\le \Big(\sum_{j\in\sigma}\|\Gamma_j K^*f\|^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{j\in\sigma}\|\Lambda_j P_{R(K)}f\|^2\Big)^{\frac{1}{2}} \\ &\le \sqrt{B_\Lambda}\|P_{R(K)}f\|\Big(\sum_{j\in\sigma}\|\Gamma_j K^*f\|^2\Big)^{\frac{1}{2}} \\ &\le \sqrt{B_\Lambda}\|K^+\| \cdot \|K^*P_{R(K)}f\|\Big(\sum_{j\in\sigma}\|\Gamma_j K^*f\|^2\Big)^{\frac{1}{2}},\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}\Big|\sum_{j\in J\setminus\sigma}\langle\Gamma_j K^*f, \Lambda_j P_{R(K)}f\rangle\Big| &\le \Big(\sum_{j\in J\setminus\sigma}\|\Gamma_j K^*f\|^2\Big)^{\frac{1}{2}} \cdot \Big(\sum_{j\in J\setminus\sigma}\|\Lambda_j P_{R(K)}f\|^2\Big)^{\frac{1}{2}} \\ &\le \sqrt{B_\Gamma}\|K^*f\|\Big(\sum_{j\in J\setminus\sigma}\|\Lambda_j P_{R(K)}f\|^2\Big)^{\frac{1}{2}}.\end{aligned} \tag{5.5}$$

Combining (5.3), (5.4) and (5.5) we obtain, for any $f \in R(K)$ and any $\sigma \subset J$,

$$\begin{aligned}\|K^*f\|^4 &= |\langle K^*f, K^*f\rangle|^2 \\ &= |\langle KK^*f, f\rangle|^2 \\ &= \Big|\Big\langle \sum_{j\in J} P_{R(K)}\Lambda_j^*\Gamma_j K^*f, f\Big\rangle\Big|^2 \\ &= \Big|\sum_{j\in J}\langle \Gamma_j K^*f, \Lambda_j P_{R(K)}f\rangle\Big|^2 \\ &= \Big|\sum_{j\in \sigma}\langle \Gamma_j K^*f, \Lambda_j P_{R(K)}f\rangle + \sum_{j\in J\setminus\sigma}\langle \Gamma_j K^*f, \Lambda_j P_{R(K)}f\rangle\Big|^2 \\ &\le 2\Big|\sum_{j\in \sigma}\langle \Gamma_j K^*f, \Lambda_j P_{R(K)}f\rangle\Big|^2 + 2\Big|\sum_{j\in J\setminus\sigma}\langle \Gamma_j K^*f, \Lambda_j P_{R(K)}f\rangle\Big|^2 \\ &\le 2B_\Lambda\|K^+\|^2\|K^*f\|^2\sum_{j\in\sigma}\|\Gamma_jK^*f\|^2 + 2B_\Gamma\|K^*f\|^2\sum_{j\in J\setminus\sigma}\|\Lambda_jf\|^2 \\ &\le 2\max\{B_\Lambda\|K^+\|^2, B_\Gamma\}\|K^*f\|^2\Big(\sum_{j\in\sigma}\|\Gamma_jK^*f\|^2 + \sum_{j\in J\setminus\sigma}\|\Lambda_jf\|^2\Big). \quad (5.6)\end{aligned}$$

For any $f \in R(K)$ and any $\sigma \subset J$, it follows from (5.2) and (5.6) that

$$\begin{aligned}\sum_{j\in\sigma}\|\Gamma_jK^*f\|^2 + \sum_{j\in J\setminus\sigma}\|\Lambda_jf\|^2 &\ge \frac{1}{2\max\{B_\Lambda\|K^+\|^2, B_\Gamma\}}\|K^*f\|^2 \\ &\ge \frac{1}{2\max\{B_\Lambda\|K^+\|^2, B_\Gamma\}\|K^+\|^2}\|f\|^2. \quad (5.7)\end{aligned}$$

On the other hand, it's easy to check that

$$\sum_{j\in\sigma}\|\Gamma_jK^*f\|^2 + \sum_{j\in J\setminus\sigma}\|\Lambda_jf\|^2 \le (B_\Gamma\|K\|^2 + B_\Lambda)\|f\|^2. \quad (5.8)$$

If we let $\sigma = J$ and $\sigma = \emptyset$ in (5.7) and (5.8), we know that $\{\Gamma_jK^* : j \in J\}$ and $\{\Lambda_j : j \in J\}$ are g-frames on $R(K)$. Hence $\{\Gamma_jK^* : j \in J\}$ and $\{\Lambda_j : j \in J\}$ are woven on $R(K)$. □

If $K = I_{\mathcal{U}}$ in Theorem 5.1, we can get Theorem 3.4 in [8] as a corollary as follows.

Corollary 5.2. *Suppose that $\Lambda := \{\Lambda_j : j \in J\}$ is a g-frame for $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$, with g-frame bounds A_Λ, B_Λ. $\Gamma := \{\Gamma_j : j \in J\}$ is a dual g-frame of $\{\Lambda_j\}_{j\in J}$ on $\mathcal{U}$, with g-Bessel bound B_Γ. Then $\{\Gamma_j : j \in J\}$ and $\{\Lambda_j : j \in J\}$ are woven on $\mathcal{U}$, with universal g-frame bounds*

$$\frac{1}{2\max\{B_\Lambda, B_\Gamma\}}, \quad B_\Gamma + B_\Lambda.$$

6. A Characterization of Q-duals of g-frames

In this section we characterize a Q-dual pair of g-frames in terms of their induced sequences.

Theorem 6.1. *Suppose that $\Lambda := \{\Lambda_j : j \in J\}$ and $\Gamma := \{\Gamma_j : j \in J\}$ are g-Bessel sequences in $\mathcal{U}$ w.r.t. $\{\mathcal{V}_j : j \in J\}$, with upper bounds B_Λ and B_Γ, respectively. For any $j \in J$, let $\{\varphi_{jk}\}_{k\in K_j}$ and $\{\phi_{jk}\}_{k\in K_j}$ be frames on $\mathcal{V}_j$, with frame bounds C_φ^j, D_φ^j and*

C^j_ϕ, D^j_ϕ, *and satisfy* $\inf_{j\in J}\{C^j_\phi, C^j_\phi\} = C > 0$, $\sup_{j\in J}\{D^j_\varphi, D^j_\phi\} = D < \infty$. *Define* Q *as follows*

$$Q : l^2(\{\mathcal{V}_j\}_{j\in J}) \to l^2(\{\mathcal{V}_j\}_{j\in J}),\ Q(\{h_j\}_{j\in J}) = \Big\{ \sum_{k\in K_j} \langle h_j, \phi_{jk}\rangle \varphi_{jk} \Big\}_{j\in J}. \tag{6.1}$$

Then the following conditions are equivalent.

(*i*) $\{\Gamma_j : j \in J\}$ *is a Q-dual of* $\{\Lambda_j : j \in J\}$ *on* $\mathcal{U}$;

(*ii*) $\{\Gamma_j^*\phi_{jk}\}_{j\in J,k\in K_j}$ *is an alternate dual of* $\{\Lambda_j^*\varphi_{jk}\}_{j\in J,k\in K_j}$ *on* $\mathcal{U}$.

Proof. We first show that Q is well defined and is bounded on $l^2(\{\mathcal{V}_j\}_{j\in J})$. In fact, for any $\{h_j\}_{j\in J} \in l^2(\{\mathcal{V}_j\}_{j\in J})$, we have

$$\begin{aligned}
\|Q(\{h_j\}_{j\in J})\| &= \Big\| \Big\{ \sum_{k\in K_j} \langle h_j, \phi_{jk}\rangle \varphi_{jk} \Big\}_{j\in J} \Big\| \\
&= \sup_{g=\{g_j\}_{j\in J}\in l^2(\{\mathcal{V}_j\}_{j\in J}),\|g\|=1} \Big| \Big\langle \Big\{ \sum_{k\in K_j} \langle h_j, \phi_{jk}\rangle \varphi_{jk} \Big\}_{j\in J}, \{g_j\}_{j\in J} \Big\rangle \Big| \\
&= \sup_{g\in l^2(\{\mathcal{V}_j\}_{j\in J}),\|g\|=1} \Big| \sum_{j\in J}\sum_{k\in K_j} \langle h_j, \phi_{jk}\rangle \langle \varphi_{jk}, g_j\rangle \Big| \\
&\le \sup_{g\in l^2(\{\mathcal{V}_j\}_{j\in J}),\|g\|=1} \sum_{j\in J}\sum_{k\in K_j} |\langle h_j, \phi_{jk}\rangle| \cdot |\langle \varphi_{jk}, g_j\rangle| \\
&\le \sup_{g\in l^2(\{\mathcal{V}_j\}_{j\in J}),\|g\|=1} \sum_{j\in J} \Big(\sum_{k\in K_j} |\langle h_j, \phi_{jk}\rangle|^2 \Big)^{\frac12} \cdot \Big(\sum_{k\in K_j} |\langle \varphi_{jk}, g_j\rangle|^2 \Big)^{\frac12} \\
&\le \sup_{g\in l^2(\{\mathcal{V}_j\}_{j\in J}),\|g\|=1} \sum_{j\in J} \sqrt{D^j_\phi}\|h_j\| \sqrt{D^j_\varphi}\|g_j\| \\
&\le \sup_{g\in l^2(\{\mathcal{V}_j\}_{j\in J}),\|g\|=1} D \sum_{j\in J} \|h_j\| \cdot \|g_j\| \\
&\le \sup_{g\in l^2(\{\mathcal{V}_j\}_{j\in J}),\|g\|=1} D \Big(\sum_{j\in J} \|h_j\|^2 \Big)^{\frac12} \cdot \Big(\sum_{j\in J} \|g_j\|^2 \Big)^{\frac12} \\
&= D\|\{h_j\}_{j\in J}\|.
\end{aligned}$$

It follows that Q is well defined on $l^2(\{\mathcal{V}_j\}_{j\in J})$ and $\|Q\| \le D$ since $\{h_j\}_{j\in J} \in l^2(\{\mathcal{V}_j\}_{j\in J})$ is arbitrary.

It is easy to check that $\{\Gamma_j^*\phi_{jk}\}_{j\in J,k\in K_j}$ and $\{\Lambda_j^*\varphi_{jk}\}_{j\in J,k\in K_j}$ are Bessel sequences in $\mathcal{U}$, under the conditions $\{\Lambda_j : j \in J\}$ and $\{\Gamma_j : j \in J\}$ being g-Bessel sequences in $\mathcal{U}$.

For any $f \in \mathcal{U}$, we obtain

$$\begin{aligned}
T_\Lambda Q T_\Gamma^* f &= T_\Lambda Q(\{\Gamma_j f\}_{j\in J}) \\
&= T_\Lambda \Big(\Big\{ \sum_{k\in K_j} \langle \Gamma_j f, \phi_{jk}\rangle \varphi_{jk} \Big\}_{j\in J} \Big) \\
&= T_\Lambda \Big(\Big\{ \sum_{k\in K_j} \langle f, \Gamma_j^*\phi_{jk}\rangle \varphi_{jk} \Big\}_{j\in J} \Big) \\
&= \sum_{j\in J} \Lambda_j^* \sum_{k\in K_j} \langle f, \Gamma_j^*\phi_{jk}\rangle \varphi_{jk} \\
&= \sum_{j\in J}\sum_{k\in K_j} \langle f, \Gamma_j^*\phi_{jk}\rangle \Lambda_j^*\varphi_{jk}.
\end{aligned}$$

Therefore $\{\Gamma_j : j \in J\}$ is a Q-dual of $\{\Lambda_j : j \in J\}$ on $\mathcal{U}$, iff $\{\Gamma_j^*\phi_{jk}\}_{j\in J,k\in K_j}$ is an alternate dual of $\{\Lambda_j^*\varphi_{jk}\}_{j\in J,k\in K_j}$ on $\mathcal{U}$. □

If for any $j \in J$, $\{\varphi_{jk}\}_{k\in K_j}$ and $\{\phi_{jk}\}_{k\in K_j}$ are a pair of alternate dual frames on $\mathcal{V}_j$, then Q defined in (6.1) is an identity operator on $l^2(\{\mathcal{V}_j\}_{j\in J})$. We can get a corollary from Theorem 6.1 as follows.

Corollary 6.2. *Suppose that* $\{\Lambda_j : j \in J\}$ *and* $\{\Gamma_j : j \in J\}$ *are g-Bessel sequences in* $\mathcal{U}$ *w.r.t.* $\{\mathcal{V}_j : j \in J\}$. *Suppose that for any* $j \in J$, $\{\varphi_{jk}\}_{k\in K_j}$ *and* $\{\phi_{jk}\}_{k\in K_j}$ *are a pair of alternate dual frames on* $\mathcal{V}_j$. *Then the following statements are equivalent.*

(*i*) $\{\Gamma_j : j \in J\}$ *is an alternate dual of* $\{\Lambda_j : j \in J\}$ *on* $\mathcal{U}$;

(*ii*) $\{\Gamma_j^*\phi_{jk}\}_{j\in J,k\in K_j}$ *is an alternate dual of* $\{\Lambda_j^*\varphi_{jk}\}_{j\in J,k\in K_j}$ *on* $\mathcal{U}$.

Moreover, if $\{e_{jk}\}_{k\in K_j}$ is an orthonormal basis on $\mathcal{V}_j$, $j \in J$, then $\{e_{jk}\}_{k\in K_j}$ and itself are a pair of alternate dual frames. We can get Theorem 2.5 (i) in [13] as follows.

Corollary 6.3. *Suppose that* $\{\Lambda_j : j \in J\}$ *and* $\{\Gamma_j : j \in J\}$ *are g-Bessel sequences in* $\mathcal{U}$ *w.r.t.* $\{\mathcal{V}_j : j \in J\}$. *Suppose that for any* $j \in J$, $\{e_{jk}\}_{k\in K_j}$ *is an orthonormal basis on* $\mathcal{V}_j$. *Then the following statements are equivalent.*

(*i*) $\{\Gamma_j : j \in J\}$ *is an alternate dual of* $\{\Lambda_j : j \in J\}$ *on* $\mathcal{U}$;

(*ii*) $\{\Gamma_j^*e_{jk}\}_{j\in J,k\in K_j}$ *is an alternate dual of* $\{\Lambda_j^*e_{jk}\}_{j\in J,k\in K_j}$ *on* $\mathcal{U}$.

Acknowledgment. We would like to thank the anonymous referees for their careful reading and helpful suggestions which led to an improvement of the original manuscript. This work is partly supported by the National Natural Science Foundation of China (Grant No. 12361028), the Natural Science Foundation of Fujian Province, China (Grant Nos. 2020J01267 and 2021J011192) and the projects of Xiamen University of Technology (Grant Nos. 40199071 and 50419004).

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